On convergence of nonlocal conservation laws towards local conservation laws and nonlocal delay conservation laws

VIII Partial differential equations, optimal design and numerics

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1. Problems considered in this talk
2. Convergence nonlocal to local (nonlocal reach tending to zero), monotonical case
3. Nonlocal delay conservation laws
4. Open Problems
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Problems Considered in this Talk

Assumptions and Notations (throughout this talk)

- **Assumptions** on the **input datum**: $\lambda \in W^{1,\infty}_{\text{loc}}(\mathbb{R}),\ \gamma \in W^{1,\infty}((-1,1);\mathbb{R}_{\geq 0}),\ \eta \in \mathbb{R}_{>0}$
- **Definition nonlocal operator**: $W[q](t,x) := \frac{1}{\eta}\int_x^{x+\eta} \gamma(\frac{y-x}{\eta}) q(t,y)\,\mathrm{d}y,\ (t,x) \in (0,T) \times \mathbb{R}$

Nonlocal conservation laws

For $q_0 \in L^\infty(\mathbb{R})$ we consider the nonlocal conservation law

$$\begin{aligned} \partial_t q(t,x) &= -\partial_x\big(\lambda\big(W[q](t,x)\big)q(t,x)\big) && (t,x) \in (0,T) \times \mathbb{R} \\ q(0,x) &= q_0(x) && x \in \mathbb{R} \end{aligned}$$

and discussfor specific cases whether and in which sense the solution **converges** to the corresponding solution of the **local conservation law** when $\eta \to 0$.

Nonlocal delay conservation laws

Given $\delta \in \mathbb{R}_{>0}$ and $q_0 \in C\left([-\delta,0];L^1(\mathbb{R})\right) \cap L^\infty((-\delta,0);L^\infty(\mathbb{R}))$ we investigate whether

$$\begin{aligned} q_t(t,x) + \partial_x\big(\lambda(W[q](t-\delta,x))q(t,x)\big) &= 0 && (t,x) \in (0,T) \times \mathbb{R} \\ q(t,x) &= q_0(t,x) && (t,x) \in (-\delta,0] \times \mathbb{R} \end{aligned}$$

possesses a solution and **converges** to the **non-delayed** solution for $\delta \to 0$.

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Convergence Nonlocal – Local

- Justifies the broad application of nonlocal modelling also as a reasonable approximation of local and well-studied/well-known models.
- Provides another way for defining the proper (Entropy) solutions for local conservation laws as limits of weak solutions to nonlocal conservation laws (which are unique without any Entropy condition).

Literature – the Ups and Downs of Convergence

- **Numerical convergence observed**:

  P. Amorim, R.M. Colombo, and A. Teixeira. On the numerical integration of scalar nonlocal conservation laws. *ESAIM: Mathematical Modelling and Numerical Analysis*, 49(1):19–37, 2015.

  P. Goatin and S. Scialanga, Well-posedness and finite volume approximations of the LWR traffic flow model with non-local velocity, *Networks and Hetereogeneous Media,* **11** *(2016), 107–121.*

- **Generally, no convergence of $q_\eta \to q$ when q the local Entropy solution:**

  M. Colombo, G. Crippa, and L.V. Spinolo. On the singular local limit for conservation laws with nonlocal fluxes. *Archive for Rational Mechanics and Analysis*, 233(3):1131–1167, 2019.

- **No total variation bound on q_η**:

  M. Colombo, G. Crippa, and L.V. Spinolo. Blow-up of the total variation in the local limit of a nonlocal traffic model. *arXiv preprint arXiv:1808.03529*, 2018.

- **Convergence for monotone datum, etc.:**

  A. Keimer and L. Pflug. On approximation of local conservation laws by nonlocal conservation laws. *Journal of Mathematical Analysis and Applications*, 475(2):1927 – 1955, 2019.

Convergence for Monotone Initial Datum and Proper Velocity

Nonlocal traffic flow PDE on $\mathbb{R}$

Recall for $\eta \in \mathbb{R}_{>0}$ the weak solution q_η of the nonlocal conservation law (for traffic flow) on $\mathbb{R}$

$$
\begin{aligned}
q_t(t,x) &= -\partial_x \left(\lambda\Big(W[q,\gamma_\eta]\Big)(t,x)q(t,x)\right)\\
q(0,x) &= q_0(x)\\
W[q,\gamma_\eta](t,x) &:= \tfrac{1}{\eta}\int_x^{x+\eta} \gamma(\tfrac{x-y}{\eta})q(t,y)\,\mathrm{d}y
\end{aligned}
$$

and its local counter-part q as weak Entropy solution of

$$
\begin{aligned}
q_t(t,x) &= -\partial_x \left(\lambda(q(t,x))q(t,x)\right)\\
q(0,x) &= q_0(x)
\end{aligned}
$$

Theorem (Convergence Nonlocal – Local)

Given monotone initial datum q_0 and monotone decreasing λ (and assumptions on γ), we obtain

$$
\lim_{\eta\to 0} \|q - q_\eta\|_{L^1_{loc}((0,T)\times\mathbb{R})} = 0.
$$

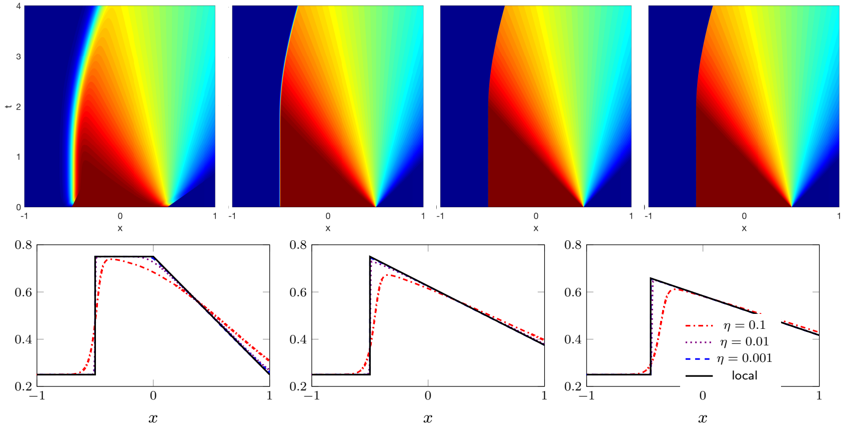

Figure 1: $q_t(t,x) + \left(\left(1 - \frac{1}{\eta}\int_x^{x+\eta} \gamma(\frac{y-x}{\eta}) q(t,y)\,\mathrm{d}y\right) q(t,x)\right)_x = 0,\ \ q(0,x) = \frac{1}{4} + \frac{1}{2}\chi_{(-0.5,0.5)}(x)$

Top: Solution q_η for $\eta \in \{0.1, 0.01, 0.001\}$ from left to right where the rightmost figure represents the analytical entropy solution of the local conservation law (right). **Bottom**: Solutions at time $t \in \{1, 2, 3\}$ from left to right.

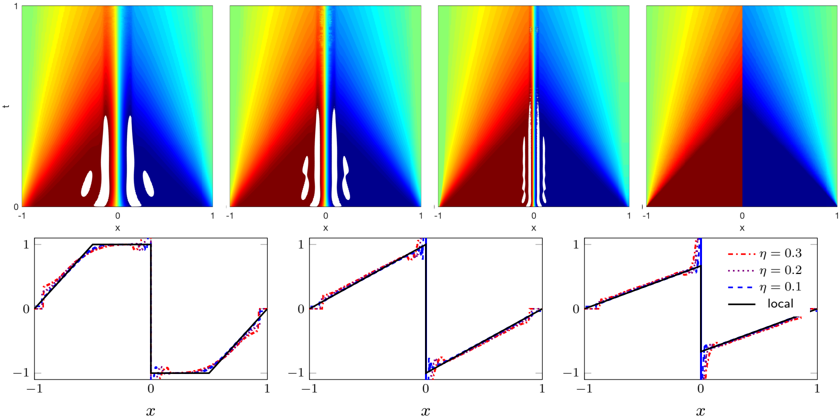

Figure 2: $q_t(t,x) + \left(\frac{1}{\eta}\int_{x-\eta}^{x+\eta} \gamma\left(\frac{y-x}{2\eta}\right) q(t,y)\,\mathrm{d}y\; q(t,x)\right)_x = 0,\;\; q(0,x) = \chi_{(-1,0)}(x) - \chi_{(0,1)}(x)$

Top: Nonlocal impact $W[q_\eta]$ of Burger's solution for $\eta \in \{0.3, 0.2, 0.1\}$ from left to right where the rightmost figure represents the analytical entropy solution of the local Burger's equation. The white regions in the figures denote locations in space-time where the absolute value of W is above 1.1, **Bottom**: Nonlocal impact $W[q_\eta]$ at time $t \in \{0.25, 0.5, 0.75\}$ from left to right.

Expected Convergence for Different Setups

What we hope for ☺

- In case a maximum principle holds: $W[q_\eta] \to q$ in L^1 when q is the local entropy solution.
- Not necessarily convergence of the nonlocal solution q_η but of $W[q_\eta] \to q$.
- Avoiding initial datum with zeros: Strong convergence of $q_\eta \to q$ in L^1?
- In case no maximum principle holds: $W[q_\eta] \overset{*}{\rightharpoonup} q$ in measure.

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Motivation

- Macroscopic traffic flow models taking into account the reaction time (delay).
- Stability of nonlocal models with regard to delay $\implies$ "superiority" of nonlocal models to local models.
- A "real" delay in the equation and not terms like in the ARZ model which "only" account for a delay.
- Fundamental results for delayed conservation laws not existent.

Literature and More

Literature for delay conservation laws:

- **Local delayed conservation laws**:

  M. Burger, S. Göttlich, and T. Jung. Derivation of a first order traffic flow model of Lighthill-Whitham-Richards type. *IFAC-PapersOnLine*, 51(9):49–54, 2018.

- **Nonlocal delay conservation laws**:

  A. Keimer, L. Pflug. Nonlocal conservation laws with time delay. *under review in NoDEA*, 2019.

Remark: Local delay conservation laws: No solution for specific initial datum

For $\delta \in \mathbb{R}_{>0}$ consider the delayed local conservation law

$$q_t(t,x) + \Big(\big(1 - q(t-\delta, x)\big) q(t,x)\Big)_x = 0 \qquad (t,x) \in (0,T) \times \mathbb{R}$$

$$q_0(t,x) = \tfrac{1}{2} + \tfrac{1}{2}\chi_{\mathbb{R}_{\geq 0}}(x) \qquad x \in \mathbb{R}.$$

Then, there exists no $\delta \in \mathbb{R}_{>0}$ and no time horizon $T \in \mathbb{R}_{>0}$ so that a solution exists.

The Considered Nonlocal Delay Conservation Laws

Nonlocal classical delay conservation law

For delay $\delta \in \mathbb{R}_{>0}$ and initial datum $q_0 \in L^\infty((-\delta,0);L^\infty(\mathbb{R})) \cap C([-\delta,0];L^1(\mathbb{R}))$

$$\begin{aligned} q_t(t,x) + \partial_x\big(\lambda(W[q,\gamma](t-\delta,x))q(t,x)\big) &= 0 && (t,x) \in (0,T)\times\mathbb{R} \\ q(t,x) &= q_0(t,x) && (t,x) \in (-\delta,0]\times\mathbb{R} \end{aligned}$$

Nonlocal delay conservation law

For delay $\delta \in \mathbb{R}_{>0}$ and initial datum $q_0 \in L^\infty((-\delta,0);L^\infty(\mathbb{R})) \cap C([-\delta,0];L^1(\mathbb{R}))$

$$\begin{aligned} q_t(t,x) &= -\partial_x\big(\lambda(W[q,\gamma](t-\delta,\xi[t,x](t-\delta)))q(t,x)\big) && (t,x) \in \Omega_T \\ q(t,x) &= q_0(t,x) && (t,x) \in (-\delta,0]\times\mathbb{R} \\ \partial_3\xi[t,x](\tau) &= \lambda\big(W[q,\gamma](\tau-\delta,\xi[t,x](\max\{\tau-\delta,0\}))\big) && \\ \xi[t,x](t) &= x && (t,x,\tau) \in \Omega_T\times[0,T] \end{aligned}$$

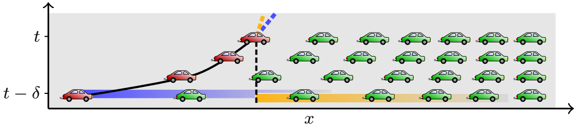

Figure 3: The different nonlocal delay models, classical delay vs. delay.

Existence and Uniqueness of Solutions

Theorem (Existence/uniqueness: Solutions for the classical delay model)

On every finite time horizon $(0,T) \subset \mathbb{R}_{>0}$, the **nonlocal classical delayed conservation law** *admits a unique weak solution*

$$q \in C([0,T]; L^1(\mathbb{R})) \cap L^\infty((0,T); L^\infty(\mathbb{R})).$$

Theorem (Existence/uniqueness: Solutions for the nonlocal delay model)

On a sufficient small time horizon $(0,T^*) \subset \mathbb{R}$ the nonlocal conservation law with delay admits a unique weak solution

$$q \in C([0,T]; L^1(\mathbb{R})) \cap L^\infty((0,T); L^\infty(\mathbb{R})).$$

Convergence when Delay Tends to Zero

Theorem (Convergence for delay tending to zero)

Let $q_\delta \in C\big([0,T]; L^1(\mathbb{R})\big)$ *denote one of the solutions of one of the nonlocal delay differential equations for* $\delta \in \mathbb{R}_{>0}$, *and let* $q \in C\big([0,T]; L^1(\mathbb{R})\big)$ *be the solution of the non-delay nonlocal conservation law. Then, we obtain on a sufficiently small time horizon* $T^* \in (0,T]$

$$\forall t \in [0,T^*]:\ q_\delta(t,\cdot) \underset{\delta\to 0}{\overset{*}{\rightharpoonup}} q(t,\cdot) \text{ in } L^\infty(\mathbb{R}).$$

Given in addition $q_0 \in L^\infty((-\delta,0); BV(\mathbb{R}))$ *and some technical assumptions on* γ, λ, *etc., we obtain even strong convergence in* L^1, *i.e.*

$$\lim_{\delta\to 0} \|q_\delta - q\|_{C([0,T^*];L^1(\mathbb{R}))}.$$

Remarks

- Delay equations present another way for defining (nonlocal, non-delayed) solutions and finding properties for nonlocal **non-delayed** conservation laws by the limit process $\delta \to 0$.
- Surprisingly, we cannot show the convergence result on the entire time horizon but only on a small time horizon, even though there is no reason why this should not hold on the full time horizon on which the nonlocal non-delayed solution exists.

Numerical Example 1: Nonlocal Classical Delay Model

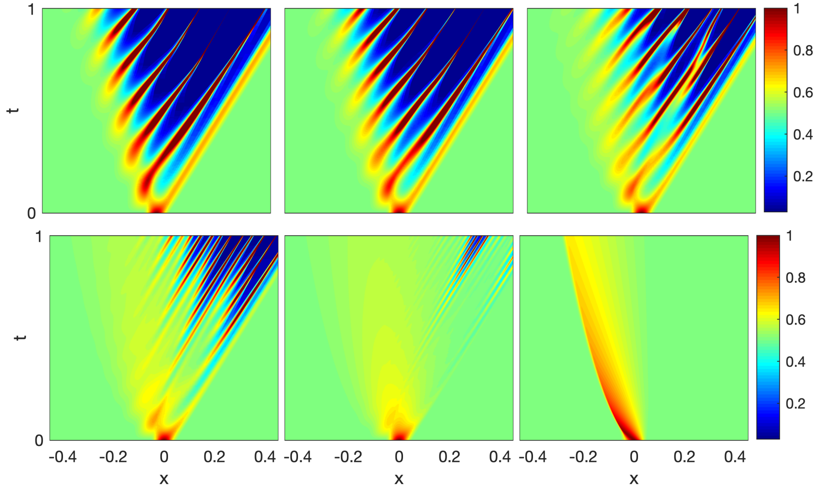

Figure 4: **Nonlocal classical delayed LWR model,** $\eta = 0.1$

$q_0(t,x) := \frac{1}{2} + \chi_{(-0.0625,0.0625)}(x)\left(\frac{3}{8}\cos(8\pi x) + \frac{1}{8}\cos(24\pi x)\right)$, $\delta \in \left\{\frac{10}{100}, \frac{8}{100}, \frac{6}{100}, \frac{4}{100}, \frac{2}{100}\right\}$,

Bottom rightmost figure: Local solution without delay, i.e. $\delta = \eta = 0$

Numerical Example 2: Nonlocal Delay Model

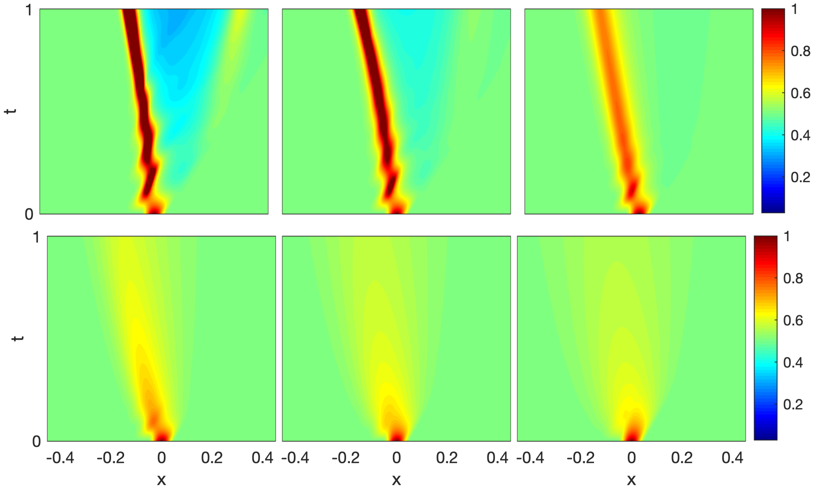

Figure 5: **Nonlocal delayed LWR model**, $\eta = 0.1$

$\boxed{q_0(t,x) := \frac{1}{2} + \chi_{(-0.0625,0.0625)}(x)\left(\frac{3}{8}\cos(8\pi x) + \frac{1}{8}\cos(24\pi x)\right)}$, $\delta \in \left\{\frac{10}{100}, \frac{8}{100}, \frac{6}{100}, \frac{4}{100}, \frac{2}{100}, 0\right\}$

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Open Problems

Convergence nonlocal – local

- General and consistent theory when nonlocal impact converges to zero.
- Avoiding restriction of the considered flux function (so far, we had $f(q) = q \cdot \lambda(q)$):

$$\begin{cases} & q_t(t,x) = -f(q(t,x))_x \\ \iff & q_t(t,x) = -f'(q(t,x))q_x(t,x) \end{cases} \overset{\text{nonlocal}}{\longrightarrow} q_t(t,x) = -\tfrac{1}{2\eta}\int_{x-\eta}^{x+\eta} f'(q(t,y))\,\mathrm{d}y \cdot q_x(t,x)$$

- Systems of (nonlocal) conservation laws.

Nonlocal delay equations

- For delay equations: Convergence on the entire time-horizon when delay approaches zero.
- Validation of the class of models.
- Better interpretation of the numerical results.

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Some Advertisement

A. Keimer and L. Pflug and M. Spinola.
Nonlocal scalar conservation laws on bounded domains and applications in traffic flow
SIAM Journal on Mathematical Analysis, 2018.

A. Keimer, N. Laurent-Brouty, F. Farokhi, H. Signargout, V. Cvetkovic, A. M. Bayen, and K. H. Johansson.
Information patterns in the modeling and design of mobility management services
Proceedings of the IEEE, 2018

A. Keimer and L. Pflug.
Existence, uniqueness and regularity results on nonlocal balance laws.
Journal of Differential Equations, 2017.

A. Keimer and L. Pflug and M. Spinola.
Existence, uniqueness and regularity of multi-dimensional nonlocal balance laws with damping.
Journal of Applied Mathematics and Application, 2018.

M. Gugat, A. Keimer, G. Leugering, and Z. Wang.
Analysis of a system of nonlocal conservation laws for multi-commodity flow on networks,
Networks and Heterogeneous Media, 2015.

M. Gröschel, A. Keimer, G. Leugering, and Z. Wang.
Regularity Theory and Adjoint Based Optimality Conditions for a Nonlinear Transport Equation with Nonlocal Velocity.
SIAM Journal on Control and Optimization, 2014.

A. Keimer and G. Leugering and T. Sarkar.
Analysis of a system of nonlocal balance laws with weighted work in progress.
Journal of Hyperbolic Differential Equations, 2018.

A. Keimer and L. Pflug.
On approximation of local conservation laws by nonlocal conservation laws.
Journal of Mathematical Analysis and Applications, 2019.

A. Keimer and L. Pflug.
Nonlocal conservation laws with time delay.
under review in NoDEA, 2019.

The End?

Thank you very much!

The End?

Thank you very much!

The Issue with Proving Convergence Exemplary for Burgers' Equation

The issue with proving the Entropy condition

Consider the nonlocal Burgers' equation for nonnegative initial datum q_0

$$q_t(t,x) + \tfrac{1}{\eta}\int_{x-\eta}^{x} q(t,y)\,\mathrm{d}y\; q_x(t,x) = 0 \qquad (t,x)\in(0,T)\times\mathbb{R} \tag{1}$$

$$q(0,x) = q_0(x) \geq 0 \qquad x\in\mathbb{R}. \tag{2}$$

Assume that q_0 is smooth (in the nonlocal setting, the solution will remain smooth), we can compute by taking advantage of Equation (1)

$$q_{t,x}(t,x) = -\tfrac{1}{\eta}\left(q(t,x)q_x(t,x) - q(t,x-\eta)q_x(t,x)\right) - \tfrac{1}{\eta}\int_{x-\eta}^{x} q(t,y)\,\mathrm{d}y\; q_{xx}(t,x)$$

Oleinik's Entropy condition would be satisfied if we have an upper bound on q_x. So assume that $x\in\mathbb{R}$ is chosen so that $q_x(t,x)$ is maximal ($\implies q_{xx}(t,x)=0$), we obtain

$$\begin{aligned} q_{t,x}(t,x) &= -q_x(t,x)\tfrac{q(t,x)-q(t,x-\eta)}{\eta} = -q_x(t,x)\tfrac{\int_{x-\eta}^{x} q_y(t,y)\,\mathrm{d}y}{\eta} \\ &\overset{?}{\leq} -q_x(t,x)^2 \end{aligned}$$

Recall that $q\equiv q_\eta$ so that the upper limit for $\eta\to 0$ cannot so easily be carried out.

Does not work without knowing how q_{xx} will change with regard to η.