Exterior control problem of strong damped nonlocal wave equation and nonlocal heat equation[1]

Sebastián Zamorano

Departamento de Matemática y Ciencia de la Computación
Universidad de Santiago de Chile

A joint work with Mahamadi Warma – University of Puerto Rico, Rio Piedras

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Plan of the Talk

1. Introduction
2. Strong damped nonlocal wave equation
3. Main results
4. Nonlocal Heat Equation

Strong damped nonlocal wave equation

Let $\Omega \subset \mathbb{R}^N$ ($N \geq 1$) be a bounded open set with a Lipschitz continuous boundary $\partial\Omega$. We consider the control problem of the **strong damped nonlocal wave equation**:

$$
(1) \qquad \begin{cases} u_{tt} + (-\Delta)^s u + \delta(-\Delta)^s u_t = 0 & \text{in } \Omega \times (0,T), \\ u = g\chi_{\mathcal{O}\times(0,T)} & \text{in } (\mathbb{R}^N \setminus \Omega) \times (0,T), \\ u(\cdot,0) = u_0, \quad u_t(\cdot,0) = u_1 & \text{in } \Omega, \end{cases}
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where

- $u = u(x,t)$ is the state to be controlled.
- $g = g(x,t)$ is the control function which is localized on a subset $\mathcal{O}$ of $\mathbb{R}^N \setminus \Omega$.
- $\delta \geq 0$ and $0 < s < 1$ are real numbers.
- $(-\Delta)^s$ denotes the fractional Laplace operator.

Main Goal

- Our first main result says that if $\delta > 0$, then the system is not exact or null controllable at any time $T > 0$.
- We also obtain that the adjoint system associated with (1) satisfies the unique continuous property for evolution equations.
- The third main result states that the system (1) is approximately controllable

Fractional Laplacian

- The *fractional Laplacian* $(-\Delta)^s$ is defined by the following singular integral

$$(-\Delta)^s u(x) := C_{N,s}\,\mathrm{P.V.}\int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}}\,dy, \quad x \in \mathbb{R}^N.$$

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- The following Dirichlet problem

$$\begin{cases} (-\Delta)^s u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

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- Then, the EXTERIOR CONTROL is the right notion that replaces the classical boundary control problems associated with local operators.

- Let $(-\Delta)^s_D$ be the selfadjoint operator in $L^2(\Omega)$ with domain

$$D((-\Delta)^s_D) := \left\{ u \in W_0^{s,2}(\overline{\Omega}),\ (-\Delta)^s u \in L^2(\Omega) \right\}, \quad (-\Delta)^s_D u := (-\Delta)^s u.$$

- Let $(-\Delta)_D^s$ be the selfadjoint operator in $L^2(\Omega)$ with domain

$$D((-\Delta)_D^s) := \left\{u \in W_0^{s,2}(\overline{\Omega}),\ (-\Delta)^s u \in L^2(\Omega)\right\}, \quad (-\Delta)_D^s u := (-\Delta)^s u.$$

- $(-\Delta)_D^s$ has a compact resolvent and its eigenvalues form a non-decreasing sequence of real numbers $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots$ satisfying $\lim_{n\to\infty} \lambda_n = \infty$. In addition, the eigenvalues are of finite multiplicity.

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- Let $(\varphi_n)_{n\in\mathbb{N}}$ be the orthonormal basis of eigenfunctions associated with $(\lambda_n)_{n\in\mathbb{N}}$. Then $\varphi_n \in D((-\Delta)_D^s)$ for every $n \in \mathbb{N}$, $(\varphi_n)_{n\in\mathbb{N}}$ is total in $L^2(\Omega)$ and satisfies

$$\begin{cases} (-\Delta)^s \varphi_n = \lambda_n \varphi_n & \text{in } \Omega, \\ \varphi_n = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

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Nonlocal normal derivative

- We introduce the *nonlocal normal derivative* $\mathcal{N}_s$ given by

$$\mathcal{N}_s u(x) := C_{N,s} \int_\Omega \frac{u(x) - u(y)}{|x-y|^{N+2s}}\, dy, \quad x \in \mathbb{R}^N \setminus \overline{\Omega},$$

where $C_{N,s}$ is the constant given in the definition of the fractional Laplacian.

[1] M. Warma. Approximate controllability from the exterior of space-time fractional diffusion equations with the fractional Laplacian. *Applied Mathematics & Optimization*, 2018

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where $C_{N,s}$ is the constant given in the definition of the fractional Laplacian.

- Unique Continuation Property for the fractional Laplacian[1]:

Lemma (Warma 2018)

Let $\lambda > 0$ be a real number and $\mathcal{O} \subset \mathbb{R}^N \setminus \overline{\Omega}$ a non-empty open set. If $\varphi \in D((-\Delta)^s_D)$ satisfies

$$(-\Delta)^s_D \varphi = \lambda \varphi \ \text{ in } \ \Omega \ \text{ and } \ \mathcal{N}_s \varphi = 0 \ \text{ in } \ \mathcal{O}, \quad \Rightarrow \quad \varphi = 0 \text{ in } \mathbb{R}^N.$$

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$s \to 1^-$

Lemma

Let $u \in W_0^{1,2}(\Omega) \hookrightarrow W_0^{s,2}(\overline{\Omega})$ be such that $(-\Delta)^s u, \Delta u \in L^2(\Omega)$. Then the following assertions hold.

1. *For every $v \in W_0^{1,2}(\Omega)^a$,*

$$\lim_{s \uparrow 1^-} \int_\Omega v(-\Delta)^s u \, dx = -\int_\Omega v \Delta u \, dx.$$

2. *For every $v \in W^{1,2}(\mathbb{R}^N)^b$,*

$$\lim_{s \uparrow 1^-} \int_{\mathbb{R}^N \setminus \Omega} v \mathcal{N}_s u \, dx = \int_{\partial \Omega} v \partial_\nu u \, d\sigma,$$

where $\partial_\nu u$ is the normal derivative of u in direction of the outer normal vector $\vec{\nu}$.

---

[a] L. Brasco, E. Parini, and M. Squassina. Stability of variational eigenvalues for the fractional p–Laplacian. *Discrete Contin. Dyn. Systm.*, 36(4)1813–1845, 2016.

[b] S. Dipierro, X. Ros–Oton, and E. Valdinoci. Nonlocal problems with Neumann boundary conditions. *Rev. Mat. Iberoam.*, 33(2):377–416, 2017.

Strong damped nonlocal wave equation: Series representation

We consider the control problem of the **strong damped nonlocal wave equation**:

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We shall denote by $(\varphi_n)_{n\in\mathbb{N}}$ the orthornormal basis of eigenfunctions of the operator $(-\Delta)^s_D$ associated with the eigenvalues $(\lambda_n)_{n\in\mathbb{N}}$.

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We shall denote by $(\varphi_n)_{n\in\mathbb{N}}$ the orthornormal basis of eigenfunctions of the operator $(-\Delta)_D^s$ associated with the eigenvalues $(\lambda_n)_{n\in\mathbb{N}}$.

Let $\delta \geq 0$ and set

$$\mathbf{D}_\mathbf{n}^\delta := \delta^2\lambda_n^2 - 4\lambda_n.$$

We have the following two situations.

- If $\delta > 0$, since $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots$ and $\lim_{n\to\infty} \lambda_n = +\infty$, it follows that there is a number $N_0 \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ such that $\delta^2\lambda_n < 4$ for all $n \leq N_0$. In that case we shall use the following notations.
  - If $\mathbf{D}_\mathbf{n}^\delta \geq 0$, that is, if $\delta^2\lambda_n - 4 \geq 0$, then we shall let

$$\lambda_n^\pm := \frac{-\delta\lambda_n \pm \sqrt{\mathbf{D}_\mathbf{n}^\delta}}{2}.$$

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- if $\mathbf{D_n^\delta} < 0$, that is, if $\delta^2\lambda_n - 4 < 0$, then we shall let

$$\widetilde{\lambda}_n^{\pm} := \frac{-\delta\lambda_n \pm i\sqrt{-\mathbf{D_n^\delta}}}{2},$$
$$\alpha_n := \mathrm{Re}(\widetilde{\lambda}_n^{+}) = \frac{-\delta\lambda_n}{2} \quad \beta_n = \mathrm{Im}(\widetilde{\lambda}_n^{+}) = \frac{\sqrt{-\mathbf{D_n^\delta}}}{2}.$$

- If $\delta = 0$, then $\mathbf{D_n^0} := -4\lambda_n < 0$ for all $n \in \mathbb{N}$. In that case we shall let

$$\widetilde{\lambda}_n^{\pm} := \pm i\sqrt{\lambda_n}, \ \ \alpha_n = 0 \ \text{ and } \ \beta_n = \sqrt{\lambda_n}.$$

- if $\mathbf{D_n^\delta} < 0$, that is, if $\delta^2\lambda_n - 4 < 0$, then we shall let

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- If $\delta = 0$, then $\mathbf{D_n^0} := -4\lambda_n < 0$ for all $n \in \mathbb{N}$. In that case we shall let

$$\widetilde{\lambda}_n^\pm := \pm i\sqrt{\lambda_n}, \;\; \alpha_n = 0 \;\text{ and }\; \beta_n = \sqrt{\lambda_n}.$$

- An immediate and important consequence is the following. If $\mathbf{D_n^\delta} \geq 0$, then we have that $\lambda_n^\pm < 0$ for all $n > N_0$, and

$$\lambda_n^+ \to -\delta, \quad \lambda_n^- \to -\infty, \quad \text{as } \; n \to \infty. \tag{3}$$

Theorem

For every $(u_0, u_1) \in W_0^{s,2}(\overline{\Omega}) \times L^2(\Omega)$ and $g \in L^2((\mathbb{R}^N \setminus \Omega) \times (0, T))$, the system (2) has a unique solution (u, u_t) given by

$$
\begin{aligned}
u(x,t) = & \sum_{n=1}^{\infty} \Big(A_n(t)(u_0, \varphi_n)_{L^2(\Omega)} + B_n(t)(u_1, \varphi_n)_{L^2(\Omega)} \Big) \varphi_n(x) \\
& + \sum_{n=1}^{\infty} \left(\int_0^t \Big(g(\cdot, \tau), \mathcal{N}_s \varphi_n \Big)_{L^2(\mathbb{R}^N \setminus \Omega)} \frac{1}{\lambda_n} B_n''(t-\tau) d\tau \right) \varphi_n(x).
\end{aligned}
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$$u(x,t) = \sum_{n=1}^{\infty} \Big(A_n(t)(u_0, \varphi_n)_{L^2(\Omega)} + B_n(t)(u_1, \varphi_n)_{L^2(\Omega)} \Big) \varphi_n(x)$$
$$+ \sum_{n=1}^{\infty} \left(\int_0^t \Big(g(\cdot, \tau), \mathcal{N}_s \varphi_n \Big)_{L^2(\mathbb{R}^N \setminus \Omega)} \frac{1}{\lambda_n} B_n''(t-\tau) d\tau \right) \varphi_n(x).$$

where

$$A_n(t) = \begin{cases} \left(\cos(\beta_n t) - \dfrac{\alpha_n}{\beta_n} \sin(\beta_n t) \right) e^{\alpha_n t} & \text{if } n \leq N_0, \\ \dfrac{\lambda_n^- e^{\lambda_n^+ t} - \lambda_n^+ e^{\lambda_n^- t}}{\lambda_n^- - \lambda_n^+} & \text{if } n > N_0, \end{cases}$$

and

$$B_n(t) = \begin{cases} \dfrac{\sin(\beta_n t)}{\beta_n} e^{\alpha_n t} & \text{if } n \leq N_0, \\ \dfrac{e^{\lambda_n^- t} - e^{\lambda_n^+ t}}{\lambda_n^- - \lambda_n^+} & \text{if } n > N_0. \end{cases}$$

Adjoint problem

Now we consider the dual system. That is, the backward system

$$
\text{(4)} \qquad \begin{cases} \psi_{tt} + (-\Delta)^s \psi - \delta(-\Delta)^s \psi_t = 0 & \text{in } \ \Omega \times (0, T), \\ \psi = 0 & \text{in } \ (\mathbb{R}^N \setminus \Omega) \times (0, T), \\ \psi(\cdot, T) = \psi_0, \quad -\psi_t(\cdot, T) = \psi_1 & \text{in } \ \Omega, \end{cases}
$$

Let

$$
\psi_{0,n} := (\psi_0, \varphi_n)_{L^2(\Omega)} \ \text{ and } \ \psi_{1,n} := (\psi_1, \varphi_n)_{L^2(\Omega)}.
$$

Theorem

For every $(\psi_0, \psi_1) \in W_0^{s,2}(\overline{\Omega}) \times L^2(\Omega)$, the dual system (4) has a unique weak solution (ψ, ψ_t) given by

$$\psi(x,t) = \sum_{n=1}^{\infty} \Big(\psi_{0,n} A_n(T-t) - \psi_{1,n} B_n(T-t) \Big) \varphi_n(x), \tag{5}$$

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1. There is a constant $C > 0$ such that for all $t \in [0, T]$,

   $$\|\psi(\cdot,t)\|^2_{W_0^{s,2}(\overline{\Omega})} + \|\psi_t(\cdot,t)\|^2_{L^2(\Omega)} \le C \left(\|\psi_0\|^2_{W_0^{s,2}(\overline{\Omega})} + \|\psi_1\|^2_{L^2(\Omega)} \right), \tag{6}$$

   and

   $$\|\psi_{tt}(\cdot,t)\|^2_{W^{-s,2}(\overline{\Omega})} \le \left(\|\psi_0\|^2_{W_0^{s,2}(\overline{\Omega})} + \|\psi_1\|^2_{L^2(\Omega)} \right). \tag{7}$$

2. We have that $\psi \in C([0,T]; D((-\Delta)^s_D)) \cap L^\infty((0,T); L^2(\Omega))$.
3. The mapping $[0,T) \ni t \mapsto \mathcal{N}_s \psi(\cdot,t) \in L^2(\mathbb{R}^N \setminus \Omega)$, can be analytically extended to the half-plane $\Sigma_T := \{z \in \mathbb{C} : \ Re(z) < T\}$.

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Controllability problems

The set of reachable states is given by

$$\mathcal{R}((u_0, u_1), T) = \Big\{(u(\cdot, T), u_t(\cdot, T)) : \ g \in L^2((0, T); W^{s,2}(\mathbb{R}^N \setminus \Omega))\Big\}.$$

We shall consider the following three notions of controllability.

- The system is said to be null controllable at $T > 0$, if

$$(0, 0) \in \mathcal{R}((u_0, u_1), T).$$

- The system is said to be exact controllable at $T > 0$, if

$$\mathcal{R}((u_0, u_1), T) = L^2(\Omega) \times W^{-s,2}(\overline{\Omega}).$$

- The system is said to be approximately controllable at $T > 0$, if

$$\mathcal{R}((u_0, u_1), T) \ \text{ is dense in } \ L^2(\Omega) \times W^{-s,2}(\overline{\Omega}),$$

Lemma

The following assertions hold.

1. The system (2) is null controllable if and only if for each initial condition $(u_0, u_1) \in W_0^{s,2}(\overline{\Omega}) \times L^2(\Omega)$, there exists a control function g such that the solution (ψ, ψ_t) of the dual system (4) satisfies

$$(u_1, \psi(\cdot, 0))_{L^2(\Omega)} - \langle u_0, \psi_t(\cdot, 0)\rangle_{\frac{1}{2},-\frac{1}{2}} + \langle u_0, \delta(-\Delta)^s\psi(\cdot, 0)\rangle_{\frac{1}{2},-\frac{1}{2}} = \int_0^T \int_{\mathbb{R}^N\setminus\Omega} \Big(g(x, t) + \delta g_t(x, t)\Big)\mathcal{N}_s\psi(x, t)dxdt,$$

   for each $(\psi_0, \psi_1) \in L^2(\Omega) \times W^{-s,2}(\overline{\Omega})$.

2. The system (2) is exact controllable at time $T > 0$, if and only if there exists a control function g such that the solution (ψ, ψ_t) of (4) satisfies

$$-(u_t(\cdot, T), \psi_0)_{L^2(\Omega)} + \langle u(\cdot, T), \psi_1\rangle_{\frac{1}{2},-\frac{1}{2}} - \langle u(\cdot, T), \delta(-\Delta)^s\psi_0\rangle_{\frac{1}{2},-\frac{1}{2}} = \int_0^T \int_{\mathbb{R}^N\setminus\Omega} \Big(g(x, t) + \delta g_t(x, t)\Big)\mathcal{N}_s\psi(x, t)dxdt,$$

   for each $(\psi_0, \psi_1) \in L^2(\Omega) \times W^{-s,2}(\overline{\Omega})$.

Lack of controllability

Theorem 1

Let $\delta > 0$. Then the system (2) is not exact or null controllable at time $T > 0$.

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Definition

The system (2) is said to be *spectrally controllable* if any finite linear combination of eigenvectors, that is,

$$u_0 = \sum_{n=1}^{M} u_{0,n}\varphi_n, \quad u_1 = \sum_{n=1}^{M} u_{1,n}\varphi_n, \quad M \geq 1 \text{ arbitrary},$$

can be steered to zero by a control function g.

Remark

Taking the limit in the Lemma as $s \uparrow 1^-$, we can deduce that

$$\begin{aligned}(u_1, \psi(\cdot,0))_{L^2(\Omega)} - \langle u_0, \psi_t(\cdot,0)\rangle_{1,-1} - \langle u_0, \delta\Delta\psi(\cdot,0)\rangle_{1,-1} \\ = \int_0^T \int_{\partial\Omega} \Big(g(x,t) + \delta g_t(x,t)\Big) \frac{\partial\psi(x,t)}{\partial\nu}\, d\sigma dt,\end{aligned}$$

for every $(\psi_0, \psi_1) \in L^2(\Omega) \times (W_0^{1,2}(\Omega))^\star$, and

$$\begin{aligned}-(u_t(\cdot,T), \psi_0)_{L^2(\Omega)} + \langle u(\cdot,T), \psi_1\rangle_{1,-1} - \langle u(\cdot,T), \delta\Delta\psi_0\rangle_{1,-1} \\ = \int_0^T \int_{\partial\Omega} \Big(g(x,t) + \delta g_t(x,t)\Big) \frac{\partial\psi(x,t)}{\partial\nu}\, d\sigma dt,\end{aligned}$$

respectively.

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These are the notions of null and exact controllabilities, respectively, of the following (possible) strong damping local wave equation:

$$
\begin{cases}
u_{tt} - \Delta u - \delta \Delta u_t = 0 & \text{in } \ \Omega \times (0, T), \\
u = g\chi_{\omega \times (0,T)} & \text{on } \ \partial\Omega \times (0, T); \\
u(\cdot, 0) = u_0, \ u_t(\cdot, 0) = u_1 & \text{in } \ \Omega.
\end{cases} \tag{8}
$$

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These are the notions of null and exact controllabilities, respectively, of the following (possible) strong damping local wave equation:

$$(8) \qquad \begin{cases} u_{tt} - \Delta u - \delta \Delta u_t = 0 & \text{in } \Omega \times (0, T), \\ u = g\chi_{\omega \times (0,T)} & \text{on } \partial\Omega \times (0, T); \\ u(\cdot, 0) = u_0, \ u_t(\cdot, 0) = u_1 & \text{in } \Omega. \end{cases}$$

Remark:

Following the techniques we developed, we anticipate that the lack of exact/null controllability of the system (8) proved by Rosier and Rouchon[a] for one space dimension, is also valid for any dimension $N \geq 1$.

[a] L. Rosier and P. Rouchon. On the controllability of a wave equation with structural damping. *Int. J. Tomogr. Stat*, 5(W07):79–84, 2007.

Unique Continuation Property

Theorem 2

Let $(\psi_0, \psi_1) \in W_0^{s,2}(\overline{\Omega}) \times L^2(\Omega)$ and let (ψ, ψ_t) be the unique weak solution of (4). Let $\mathcal{O} \subset \mathbb{R}^N \setminus \Omega$ be an arbitrary non-empty open set. If

$$\mathcal{N}_s\psi = 0 \text{ in } \mathcal{O} \times (0, T), \text{ then } \psi = 0 \text{ in } \Omega \times (0, T).$$

Here, $\mathcal{N}_s\psi$ is the nonlocal normal derivative of ψ.

Proof: By hand, using the series representation of the nonlocal normal derivative

$$\mathcal{N}_s\psi(x, t) = \sum_{n=1}^{\infty} \Big(\psi_{0,n} A_n(T - t) - \psi_{1,n} B_n(T - t)\Big)\mathcal{N}_s\varphi_n(x),$$

and complex analysis (residue Theorem).

Approximate controllability

Theorem 3

The system (2) is approximately controllable for any $T > 0$ and $g \in L^2(\mathcal{O} \times (0, T))$, where $\mathcal{O} \subset \mathbb{R}^N \setminus \Omega$ is an arbitrary non-empty open set.

Remark

Using similar ideas we can also prove that the following nonlocal Sobolev–Galpern type equation, known as nonlocal Barenblatt–Zheltov–Kochina equation,

$$\begin{cases} y_t + (-\Delta)^s y + \delta(-\Delta)^s y_t = 0 & \text{in } \Omega \times (0,T), \\ y = g\chi_{\mathcal{O}\times(0,T)} & \text{in } (\mathbb{R}^N \setminus \Omega) \times (0,T), \\ y(\cdot,0) = y_0 & \text{in } \Omega, \end{cases} \tag{9}$$

satisfy the following controllability properties:

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satisfy the following controllability properties:

- The system (9) is not exact or null controllable at time $T > 0$.
- The system (9) is approximately controllable for any $T > 0$ and $g \in L^2(\mathcal{O} \times (0,T))$, where $\mathcal{O} \subset \mathbb{R}^N \setminus \Omega$ is an arbitrary non-empty open set.

Nonlocal Heat Equation

We are interested in the null controllability of the fractional heat equation in the interval $(-1,1)$. That is,

$$\begin{cases} \partial_t u + (-\partial_x^2)^s u = 0 & \text{in } (-1,1)\times(0,T), \\ u = g\chi_{\mathcal{O}\times(0,T)} & \text{in } (\mathbb{R}\setminus(-1,1))\times(0,T), \\ u(\cdot,0) = u_0 & \text{in } (-1,1). \end{cases} \tag{10}$$

Nonlocal Heat Equation

We are interested in the null controllability of the fractional heat equation in the interval $(-1,1)$. That is,

$$(10) \qquad \begin{cases} \partial_t u + (-\partial_x^2)^s u = 0 & \text{in } (-1,1)\times(0,T), \\ u = g\chi_{\mathcal{O}\times(0,T)} & \text{in } (\mathbb{R}\setminus(-1,1))\times(0,T), \\ u(\cdot,0) = u_0 & \text{in } (-1,1). \end{cases}$$

More precisely, given u_0, find g such that the solution of (10) satisfies:

$$u(\cdot,T) = 0, \text{ in } (-1,1).$$

Here $\mathcal{O} \subset (\mathbb{R}\setminus(-1,1))$.

Main result

Theorem 4

Let $0 < s < 1$ and let $\mathcal{O} \subset (\mathbb{R} \setminus (-1, 1))$ be an arbitrary nonempty open set. Then the following assertions hold.

(a) If $\frac{1}{2} < s < 1$, then the system (10) is null controllable at any time $T > 0$.

(b) If $0 < s \leq \frac{1}{2}$, then the system (10) is not null controllable at time $T > 0$.

(b) If $\frac{1}{2} < s < 1$, then the system (10) is exactly controllable to the trajectories at any time $T > 0$.

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Proof of Theorem

The system (10) is null controllable if and only if the following observability inequality holds for the dual system: there exists a constant $C > 0$ such that

$$\|\psi(\cdot,0)\|^2_{L^2(-1,1)} \leq C \int_0^T \int_{\mathcal{O}} |\mathcal{N}_s \psi(x,t)|^2 \; dxdt. \tag{11}$$

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Using the representations of ψ and $\mathcal{N}_s\psi$, and employing the orthonormality of the eigenfunctions in $L^2(-1,1)$, then the observability inequality (11) becomes

$$\sum_{n=1}^{\infty} |\psi_{0,n}|^2 e^{-2\lambda_n T} \leq C \int_0^T \int_{\mathcal{O}} \left| \sum_{n=1}^{\infty} \psi_{0,n} e^{-\lambda_n (T-t)} \mathcal{N}_s \varphi_n(x) \right|^2 dxdt.$$

Proof of Theorem...

IF $\|\mathcal{N}_s\varphi_n\|_{L^2(\mathcal{O})}$ IS UNIFORMLY BOUNDED FROM BELOW BY $\eta > 0$, we can deduce that

$$\int_0^T \int_{\mathcal{O}} \left| \sum_{n=1}^{\infty} \psi_{0,n} e^{-\lambda_n(T-t)} \mathcal{N}_s\varphi_n(x) \right|^2 dxdt \geq \eta^2 \int_0^T \left| \sum_{n=1}^{\infty} \psi_{0,n} e^{-\lambda_n(T-t)} \right|^2 dt.$$

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Thus, the observability inequality (11) holds if the following estimate is proved:

$$\left\| \sum_{n=1}^{\infty} \psi_{0,n} e^{-\lambda_n t} \right\|^2_{L^2(0,T)} \geq C \sum_{n=1}^{\infty} |\psi_{0,n}|^2 e^{-2\lambda_n T}. \tag{12}$$

It is a well known result for parabolic equations, that an inequality of the type (12) holds if and only if the eigenvalues $\{\lambda_n\}_{n\in\mathbb{N}}$ satisfy the following Müntz condition. That is, the series

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n} \tag{13}$$

is convergent.

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Proof of Theorem...

The eigenvalues $\{\lambda_n\}_{n\geq 1}$ satisfy

$$\lambda_n = \left(\frac{n\pi}{2} - \frac{(2-2s)\pi}{8}\right)^{2s} + O\left(\frac{1}{n}\right) \text{ as } n \to \infty. \tag{14}$$

Therefore, we have the following two situations.

- If $0 < s \leq \frac{1}{2}$, then the series (13) will have the behavior of the harmonic series, which implies that it is divergent.
- On the other hand, if $\frac{1}{2} < s < 1$, hence, $2s > 1$, then using (14) we can deduce that the series (13) is convergent.

The proof of Parts (a) and (b) is complete.

Lemma

Let $\{\varphi_k\}_{k\in\mathbb{N}}$ be the orthogonal basis of normalized eigenfunctions associated with the eigenvalues $\{\lambda_k\}_{k\in\mathbb{N}}$. Then, for every nonempty open set $\mathcal{O}\subset(\mathbb{R}\setminus(-1,1))$, there exists a scalar $\eta>0$ such that for every $k\in\mathbb{N}$, the function $\mathcal{N}_s\varphi_k$ is uniformly bounded from below by η in $L^2(\mathcal{O})$. Namely,

$$\exists\,\eta>0,\ \forall\,k\in\mathbb{N},\ \|\mathcal{N}_s\varphi_k\|_{L^2(\mathcal{O})}\geq\eta.$$

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