Minimal null-control time issues for parabolic systems

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PDEs, Optimal design and Numerics
Benasque

(mostly) joint work with A. Benabdallah and M. Morancey

Our controlled parabolic problem is $$(S) \quad \begin{cases} \partial_t y + \mathcal{A}y = \mathcal{B}u & \text{in }]0,T[, \\ y(0) = y_0 \in X, \end{cases}$$

- $\mathcal{A} : D(\mathcal{A}) \subset X \mapsto X$ is a suitable unbounded operator.
- $\mathcal{B} : U \mapsto D(\mathcal{A}^\star)'$ a control operator, $\mathcal{B}^\star$ its adjoint. We assume

$$\left\| \mathcal{B}^\star e^{-\cdot \mathcal{A}^\star} \psi \right\|_{L^2(0,T;U)} \leq C \|\psi\|_X, \quad \forall \psi \in D(\mathcal{A}^\star).$$

THEOREM (WELL-POSEDNESS OF (S) IN A DUAL SENSE)

For any $y_0 \in X$ and $u \in L^2(0,T;U)$, there exists a unique $y = y_{u,y_0} \in \mathcal{C}^0([0,T], X)$ such that

$$\langle y(t), \psi \rangle_X - \left\langle y_0, e^{-t\mathcal{A}^\star} \psi \right\rangle_X = \int_0^t \left(u(s), \mathcal{B}^\star e^{-(t-s)\mathcal{A}^\star} \psi \right)_U ds, \quad \forall t \in [0,T], \forall \psi \in X.$$

NULL-CONTROLLABILITY (NC) PROBLEM

For given $\mathcal{A}, \mathcal{B}, T$: does it exist, for any $y_0 \in X$, a $u \in L^2(0,T;U)$ such that $y(T) = 0$?

Scalar heat equation

(Fattorini-Russell, '74), (Lebeau-Robiano, '95), (Fursikov-Imanuvilov, '96).

NC holds at any time $T > 0$, for any Distributed/Boundary control domain, in any dimension.

System of coupled heat equations - As many controls as equations

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System of coupled heat equations - Less controls than equations

NC property

- may depend on the geometry/position of the control domain,
- may depend on the control time T,

There is some kind of **hyperbolic behavior** for such parabolic PDEs.

Remarks

- distributed control is no more equivalent to boundary control,
- very few results are available in dimension $d > 1$.

Proposition

A function $u \in L^2(0,T;U)$ *is a null-control at time* $T > 0$ *for the problem*

$$\partial_t y + \mathcal{A}y = \mathcal{B}u, \quad y(0) = y_0,$$

if and only if the function $v(t) = u(T-t)$ *satisfies*

$$\int_0^T \left(v(t), \mathcal{B}^\star e^{-t\mathcal{A}^\star} \phi \right)_U dt = -\left\langle y_0, e^{-T\mathcal{A}^\star} \phi \right\rangle_X, \quad \forall \phi \in X.$$

Assume for the moment

- *Scalar control :* $U = \mathbb{R}, \qquad \mathcal{B}^\star : D(\mathcal{A}^\star) \to \mathbb{R}$
- *Non-negative real point spectrum :* $\Lambda := \sigma_p(\mathcal{A}^\star) \subset (0,+\infty)$

$$\mathcal{A}^\star \phi_\lambda = \lambda \phi_\lambda, \text{ with } \|\phi_\lambda\| = 1, \ \forall \lambda \in \Lambda.$$

Moment equation

For any $y_0 \in X$, a null-control $v \in L^2(0,T;\mathbb{R})$ should satisfy (**at least**)

$$(\mathcal{B}^\star \phi_\lambda) \left(\int_0^T v(t) e^{-\lambda t}\, dt \right) = -\left\langle y_0, e^{-\lambda T} \phi_\lambda \right\rangle_X, \quad \forall \lambda \in \Lambda.$$

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Necessary condition 1 :

Moment equation

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Necessary condition 1 :

$$\boxed{\mathcal{B}^\star \phi_\lambda \neq 0, \ \ \forall \lambda \in \Lambda.}$$

$\rightsquigarrow$ This is nothing but the Fattorini-Hautus condition for Approximate Controllability

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Examples

$\partial_t y - \partial_x^2 y = u(t)\delta_{x_0}$, on $(0,1)$,
App. Cont. $\Leftrightarrow x_0 \notin \mathbb{Q}$, **(Dolecki, '73)**

$$\partial_t y + \begin{pmatrix} -\partial_x^2 & 1 \\ 0 & -d\partial_x^2 \end{pmatrix} y = 0, \quad y(t,0) = \begin{pmatrix} 0 \\ u(t) \end{pmatrix}.$$

App. Cont. $\Leftrightarrow \left(d = 1 \text{ or } \sqrt{d} \notin \mathbb{Q}\right)$, **(Ammar Kodja-Benabdallah-Gonzalez Burgos-de Teresa, '14)**

Moment equation

For any $y_0 \in X$, a null-control $v \in L^2(0,T;\mathbb{R})$ should satisfy (at least)

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App. Cont. $\Leftrightarrow \sqrt{d} \notin \mathbb{Q}$.

Moment equation

For any $y_0 \in X$, we look for $v \in L^2(0,T;\mathbb{R})$ such that

$$(\mathcal{B}^\star \phi_\lambda)\left(\int_0^T v(t)e^{-\lambda t}\,dt\right) = -\left\langle y_0, e^{-\lambda T}\phi_\lambda\right\rangle_X, \quad \forall \lambda \in \Lambda.$$

Necessary condition 2 : Strong enough observation of eigenfunctions

$$\left|\left\langle y_0, \frac{e^{-\lambda T}\phi_\lambda}{\mathcal{B}^\star \phi_\lambda}\right\rangle_X\right| \leq \sqrt{T}\|v\|_{L^2(0,T)}, \quad \forall \lambda \in \Lambda, \quad \forall y_0 \in X,$$

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$$\Longrightarrow \sup_{\lambda \in \Lambda} \frac{e^{-\lambda T}}{|\mathcal{B}^\star \phi_\lambda|} < +\infty.$$

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We obtain a **minimal null-control time** condition

$$\boxed{T \geq \limsup_{\substack{\lambda \to +\infty \\ \lambda \in \Lambda}} \frac{-\ln |\mathcal{B}^\star \phi_\lambda|}{\lambda}.}$$

Moment equation

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$$\boxed{T \geq \limsup_{\substack{\lambda\to+\infty\\ \lambda\in\Lambda}} \frac{-\ln|\mathcal{B}^\star\phi_\lambda|}{\lambda}.}$$

Example

Let $x_0 \notin \mathbb{Q}$ **(D, '73)**

$$\partial_t y - \partial_x^2 y = u(t)\delta_{x_0}, \quad \text{on } (0,1),$$

$$\boxed{T \geq \limsup_{k\to\infty} \frac{-\ln|\sin(k\pi x_0)|}{k^2\pi^2} := T_0(x_0)}$$

Moreover, for any $\tau \in [0,+\infty]$, there exists $x_0 \notin \mathbb{Q}$ s.t. $T_0(x_0) = \tau$.

Moment equation

For any $y_0 \in X$, we look for $v \in L^2(0,T;\mathbb{R})$ such that

$$(\mathcal{B}^\star \phi_\lambda) \left(\int_0^T v(t) e^{-\lambda t}\, dt \right) = -\left\langle y_0, e^{-\lambda T} \phi_\lambda \right\rangle_X, \ \ \forall \lambda \in \Lambda.$$

Necessary condition 3 : Let $\lambda, \mu \in \Lambda$, with $\lambda \neq \mu$.

$$\boxed{\psi_\lambda = \phi_\lambda / \mathcal{B}^\star \phi_\lambda}$$

$$\begin{cases} \displaystyle\int_0^T v(t) e^{-\lambda t}\, dt = -\left\langle y_0, e^{-\lambda T} \psi_\lambda \right\rangle_X, \\ \displaystyle\int_0^T v(t) e^{-\mu t}\, dt = -\left\langle y_0, e^{-\mu T} \psi_\mu \right\rangle_X, \end{cases}$$

MOMENT EQUATION

For any $y_0 \in X$, we look for $v \in L^2(0,T;\mathbb{R})$ such that

$$(\mathcal{B}^\star \phi_\lambda)\left(\int_0^T v(t)e^{-\lambda t}\,dt\right) = -\left\langle y_0, e^{-\lambda T}\phi_\lambda\right\rangle_X, \quad \forall \lambda \in \Lambda.$$

NECESSARY CONDITION 3 : Let $\lambda, \mu \in \Lambda$, with $\lambda \neq \mu$. $\boxed{\psi_\lambda = \phi_\lambda / \mathcal{B}^\star \phi_\lambda}$

$$\int_0^T v(t)\frac{e^{-\lambda t} - e^{-\mu t}}{\lambda - \mu}\,dt = -\left\langle y_0, \frac{e^{-\lambda T}\psi_\lambda - e^{-\mu T}\psi_\mu}{\lambda - \mu}\right\rangle_X.$$

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$$\Rightarrow \sup_{\substack{\lambda,\mu \in \Lambda \\ \lambda \neq \mu}} \left(e^{-\lambda T} \left\| \frac{\psi_\lambda - \psi_\mu}{\lambda - \mu} \right\|_X \right) < +\infty,$$

Moment equation

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$$\Rightarrow \sup_{\substack{\lambda,\mu\in\Lambda\\ \lambda\neq\mu}}\left(e^{-\lambda T}\left\|\frac{\psi_\lambda-\psi_\mu}{\lambda-\mu}\right\|_X\right) < +\infty,$$

We obtain another **minimal null-control time condition**

$$\boxed{T \geq \limsup_{\substack{\lambda\to+\infty\\ \lambda\in\Lambda}}\left(\sup_{\substack{\mu\in\Lambda\\ \mu\neq\lambda}} \frac{\ln\left\|\frac{\psi_\lambda-\psi_\mu}{\lambda-\mu}\right\|_X}{\lambda}\right)}.$$

Moment equation

For any $y_0 \in X$, we look for $v \in L^2(0,T;\mathbb{R})$ such that

$$(\mathcal{B}^\star \phi_\lambda)\left(\int_0^T v(t)e^{-\lambda t}\,dt\right) = -\left\langle y_0, e^{-\lambda T}\phi_\lambda\right\rangle_X, \quad \forall \lambda \in \Lambda.$$

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- **Case 1 :** $(\psi_\lambda)_\lambda$ forms a Riesz basis of X.

$$\boxed{T \geq \limsup_{\substack{\lambda\to+\infty \\ \lambda\in\Lambda}} \left(\sup_{\substack{\mu\in\Lambda \\ \mu\neq\lambda}} \frac{-\ln|\lambda-\mu|}{\lambda}\right).}$$

$\rightsquigarrow$ The minimal control time is related to the **condensation of eigenvalues**.

Moment equation

For any $y_0 \in X$, we look for $v \in L^2(0,T;\mathbb{R})$ such that

$$(\mathcal{B}^\star \phi_\lambda)\left(\int_0^T v(t)e^{-\lambda t}\,dt\right) = -\left\langle y_0, e^{-\lambda T}\phi_\lambda\right\rangle_X, \quad \forall \lambda \in \Lambda.$$

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- Case 1 : $(\psi_\lambda)_\lambda$ forms a Riesz basis of X.

$$\begin{cases} \partial_t y + \begin{pmatrix} -\partial_x^2 & 0 \\ 0 & -d\partial_x^2 \end{pmatrix} y = 0, \\ y(t,0) = \begin{pmatrix} u(t) \\ u(t) \end{pmatrix}. \end{cases}$$

$$\boxed{T \geq \limsup_{k,l\to\infty} \frac{-\ln|k^2-dl^2|}{k^2\pi^2} := T_0(d)}$$

• For any $\tau \in [0,+\infty]$, there exists $\sqrt{d} \notin \mathbb{Q}$ s.t. $T_0(d) = \tau$.

MOMENT EQUATION

For any $y_0 \in X$, we look for $v \in L^2(0,T;\mathbb{R})$ such that

$$(\mathcal{B}^\star \phi_\lambda)\left(\int_0^T v(t)e^{-\lambda t}\,dt\right) = -\left\langle y_0, e^{-\lambda T}\phi_\lambda\right\rangle_X, \quad \forall \lambda \in \Lambda.$$

NECESSARY CONDITION 3 :

$$\boxed{T \geq \limsup_{\substack{\lambda\to+\infty \\ \lambda\in\Lambda}} \left(\sup_{\substack{\mu\in\Lambda \\ \mu\neq\lambda}} \frac{\ln \left\| \frac{\psi_\lambda-\psi_\mu}{\lambda-\mu} \right\|_X}{\lambda}\right)}.$$

- CASE 2 : $(\psi_\lambda)_\lambda$ does not form a Riesz basis of X :

  the quantity $\dfrac{\psi_\lambda - \psi_\mu}{\lambda - \mu}$ may be bounded even if $|\lambda - \mu|$ is small.

MAIN OBSERVATION

Condensation of eigenfunctions may compensate **condensation of eigenvalues**.

Moment equation

For any $y_0 \in X$, we look for $v \in L^2(0,T;\mathbb{R})$ such that

$$(\mathcal{B}^\star \phi_\lambda)\left(\int_0^T v(t) e^{-\lambda t}\, dt\right) = -\left\langle y_0, e^{-\lambda T}\phi_\lambda\right\rangle_X, \ \ \forall \lambda \in \Lambda.$$

Necessary condition 3 :

$$\boxed{T \geq \limsup_{\substack{\lambda\to+\infty\\ \lambda\in\Lambda}} \left(\sup_{\substack{\mu\in\Lambda\\ \mu\neq\lambda}} \frac{\ln \left\| \frac{\psi_\lambda - \psi_\mu}{\lambda-\mu} \right\|_X}{\lambda} \right).}$$

- Case 2 : $(\psi_\lambda)_\lambda$ does not form a Riesz basis of X :

Let $a \in L^2(0,1)$ be small enough.

$$\begin{cases} \partial_t y + \begin{pmatrix} -\partial_x^2 & 1 \\ 0 & -\partial_x^2 + a(x) \end{pmatrix} y = 0, \\ y(t,0) = \begin{pmatrix} 0 \\ u(t) \end{pmatrix}. \end{cases}$$

We can prove that

$$\boxed{\limsup_{\substack{\lambda\to+\infty\\ \lambda\in\Lambda}} \left(\sup_{\substack{\mu\in\Lambda_a\\ \mu\neq\lambda}} \frac{\ln \left\| \frac{\psi_\lambda - \psi_\mu}{\lambda-\mu} \right\|_X}{\lambda} \right) = 0.}$$

Moment equation

For any $y_0 \in X$, we look for $v \in L^2(0,T;\mathbb{R})$ such that

$$(\mathcal{B}^\star \phi_\lambda)\left(\int_0^T v(t) e^{-\lambda t}\,dt\right) = -\left\langle y_0, e^{-\lambda T}\phi_\lambda\right\rangle_X, \quad \forall \lambda \in \Lambda.$$

Necessary condition 4 : Suppose there exists a generalized eigenvector ϕ_λ^1

$$\mathcal{A}^\star \phi_\lambda^1 = \lambda \phi_\lambda^1 + \phi_\lambda, \quad \text{with } \mathcal{B}^\star \phi_\lambda^1 = 0.$$

A simple computation leads to

$$\int_0^T (T-t) e^{-t\lambda} v(t)\,dt = e^{-T\lambda}\left\langle y_0, \frac{\phi_\lambda^1}{\mathcal{B}^\star \phi_\lambda}\right\rangle_X.$$

$$\Longrightarrow \sup_{\lambda \in \Lambda} e^{-T\lambda} \frac{\left\|\phi_\lambda^1\right\|_X}{|\mathcal{B}^\star \phi_\lambda|} < +\infty,$$

We find out another **minimal null-control time**

$$\boxed{T \geq \limsup_{\substack{\lambda \to +\infty \\ \lambda \in \Lambda}} \frac{-\ln \frac{\left\|\phi_\lambda^1\right\|_X}{|\mathcal{B}^\star \phi_\lambda|}}{\lambda}.}$$

Moment equation

For any $y_0 \in X$, we look for $v \in L^2(0,T;\mathbb{R})$ such that

$$(\mathcal{B}^\star \phi_\lambda)\left(\int_0^T v(t)e^{-\lambda t}\,dt\right) = -\left\langle y_0, e^{-\lambda T}\phi_\lambda\right\rangle_X, \quad \forall \lambda \in \Lambda.$$

Necessary condition 4 : Suppose there exists a generalized eigenvector ϕ^1_λ

$$\boxed{T \geq \limsup_{\substack{\lambda\to+\infty\\ \lambda\in\Lambda}} \frac{-\ln \frac{\|\phi^1_\lambda\|_X}{|\mathcal{B}^\star\phi_\lambda|}}{\lambda}.}$$

Let $q \in L^\infty(0,1)$ **(AK-B-GB-dT, '16)**

$$\begin{cases} \partial_t y + \begin{pmatrix} -\partial_x^2 & q(x) \\ 0 & -\partial_x^2 \end{pmatrix} y = 0, \\ y(t,0) = \begin{pmatrix} 0 \\ u(t) \end{pmatrix}. \end{cases}$$

- AC $\Leftrightarrow I_\lambda(q) := \int_0^1 q|\varphi_\lambda|^2\,dx \neq 0, \ \forall \lambda \in \Lambda.$
- It can be shown that $\|\phi^1_\lambda\|_X \sim |I_\lambda(q)|^{-1}$,

$$\boxed{T \geq \limsup_{\substack{\lambda\to+\infty\\ \lambda\in\Lambda}} \frac{-\ln|I_\lambda(q)|}{\lambda} := T_0(q)}$$

- For any $\tau \in [0,+\infty]$ there exists $q \in L^\infty(0,1)$ s.t. $T_0(q) = \tau$.

Summary of this discussion

- A minimal time of null-control may appear for such systems in different kinds of spectral situations:

$$\boxed{T \geq \limsup_{\substack{\lambda\to+\infty\\ \lambda\in\Lambda}} \frac{-\ln|\mathcal{B}^\star\phi_\lambda|}{\lambda}}$$

$$\boxed{T \geq \limsup_{\substack{\lambda\to+\infty\\ \lambda\in\Lambda}} \left(\sup_{\substack{\mu\in\Lambda\\ \mu\neq\lambda}} \frac{\ln\left\|\frac{\psi_\lambda-\psi_\mu}{\lambda-\mu}\right\|_X}{\lambda}\right)} \quad \overset{\checkmark}{\Longleftarrow} \quad \overset{?}{\Longrightarrow} \quad \boxed{T \geq \limsup_{\substack{\lambda\to+\infty\\ \lambda\in\Lambda}} \left(\sup_{\substack{\mu\in\Lambda\\ \mu\neq\lambda}} \frac{-\ln|\lambda-\mu|}{\lambda}\right)}$$

$$\boxed{T \geq \limsup_{\substack{\lambda\to+\infty\\ \lambda\in\Lambda}} \frac{-\ln\frac{\left\|\phi_\lambda^1\right\|_X}{|\mathcal{B}^\star\phi_\lambda|}}{\lambda}}$$

- Are those conditions optimal ?
- Are there more involved spectral situations ?
- What happens when all the phenomena happen simultaneously ?

ASSUMPTIONS : $(\phi_\lambda)_{\lambda\in\Lambda}$ is a complete family of X, $\mathcal{B}^\star\phi_\lambda \neq 0$, $\forall\lambda\in\Lambda$.

MOMENTS EQUATIONS

The function $u \in L^2(0,T)$ is a null control at time $T>0$ **if and only if**

$$\int_0^T u(T-t)e^{-\lambda t}\,dt = -\frac{e^{-\lambda T}}{\mathcal{B}^\star\phi_\lambda}\langle y_0,\phi_\lambda\rangle_X\,,\ \ \forall\lambda\in\Lambda.$$

THEOREM ((Müntz, 1914), (Luxemburg-Korevaar, '71), (Dolecki, '73), (Fattorini-Russel, '74))

- *If* $\sum_{\lambda\in\Lambda}\frac{1}{\lambda}<+\infty$, *there exists a biorthogonal family* $(q_\lambda)_{\lambda\in\Lambda}\subset L^2(0,T)$

$$\int_0^T q_\lambda(t)e^{-\mu t}\,dt = \delta_{\lambda,\mu},\ \ \forall\lambda,\mu\in\Lambda.$$

- *If in addition we assume the* **gap condition** $\inf_{\lambda\neq\mu}|\lambda-\mu|>0$ *then*

$$\|q_\lambda\|_{L^2(0,T)} \leq C_\varepsilon e^{\varepsilon\lambda},\ \ \forall\lambda\in\Lambda.$$

- *Thus, for* $T > \limsup_{\lambda\to\infty}\dfrac{-\ln|\mathcal{B}^\star\phi_\lambda|}{\lambda}$, *the problem is null-controllable at time* T.

ASSUMPTIONS : $(\phi_\lambda)_{\lambda\in\Lambda}$ is a complete family of X, $\mathcal{B}^\star\phi_\lambda \neq 0$, $\forall\lambda\in\Lambda$.

MOMENTS EQUATIONS

The function $u \in L^2(0,T)$ is a null control at time $T > 0$ **if and only if**

$$\int_0^T u(T-t)e^{-\lambda t}\,dt = -\frac{e^{-\lambda T}}{\mathcal{B}^\star\phi_\lambda}\langle y_0, \phi_\lambda\rangle_X, \quad \forall\lambda\in\Lambda.$$

For $T > \limsup_{\lambda\to\infty} \frac{-\ln|\mathcal{B}^\star\phi_\lambda|}{\lambda}$, the construction of the control u is the following

$$u(T-t) = \sum_{\lambda\in\Lambda} \underbrace{-\frac{e^{-\lambda T}}{\mathcal{B}^\star\phi_\lambda}\langle y_0, \phi_\lambda\rangle_X\, q_\lambda(t)}_{:=v_\lambda(t)}.$$

IMPORTANT REMARK : Each term v_λ of the series solves the “partial” moment problem

$$\begin{cases} \displaystyle\int_0^T v_\lambda(t)e^{-\lambda t}\,dt = -\frac{e^{-\lambda T}}{\mathcal{B}^\star\phi_\lambda}\langle y_0, \phi_\lambda\rangle_X, & \\ \displaystyle\int_0^T v_\lambda(t)e^{-\mu t}\,dt = 0, & \forall\mu\in\Lambda\setminus\{\lambda\}. \end{cases}$$

ASSUMPTIONS : $(\phi_\lambda, \phi^1_\lambda)_{\lambda\in\Lambda}$ is complete in X, $\mathcal{B}^\star\phi_\lambda \neq 0, \mathcal{B}^\star\phi^1_\lambda = 0, \forall\lambda\in\Lambda.$

MOMENTS EQUATIONS

The function $u \in L^2(0,T)$ is a null control at time $T>0$ **if and only if**

$$\int_0^T u(T-t)e^{-\lambda t}\,dt = -\frac{e^{-\lambda T}}{\mathcal{B}^\star\phi_\lambda}\langle y_0, \phi_\lambda\rangle_X\,,\quad \forall\lambda\in\Lambda,$$

$$\int_0^T u(T-t)(-t)e^{-\lambda t}\,dt = -\frac{e^{-\lambda T}}{\mathcal{B}^\star\phi_\lambda}\left\langle y_0, \phi^1_\lambda\right\rangle_X - \frac{Te^{-\lambda T}}{\mathcal{B}^\star\phi_\lambda}\langle y_0, \phi_\lambda\rangle_X\,,\quad \forall\lambda\in\Lambda.$$

THEOREM ((Fernandez Cara-Gonzalez Burgos-de Teresa, '10), (AK-B-GB-dT, '16), ...)

- *If* $\sum_{\lambda\in\Lambda}\frac{1}{\lambda} < +\infty$*, there exists a biorthogonal family* $(q_{\lambda,j})_{\substack{\lambda\in\Lambda\\ j\in\{0,1\}}} \subset L^2(0,T)$
$$\int_0^T q_{\lambda,j}(t)(-t)^k e^{-\mu t}\,dt = \delta_{\lambda,\mu}\delta_{j,k},\quad \forall\lambda,\mu\in\Lambda, \forall j,k\in\{0,1\}.$$
- *If in addition we assume the* **gap condition** $\inf_{\lambda\neq\mu}|\lambda-\mu| > 0$ *then*
$$\|q_{\lambda,j}\|_{L^2(0,T)} \leq C_{\varepsilon,T}e^{\varepsilon\lambda},\quad \forall\lambda\in\Lambda,\ \forall j\in\{0,1\}.$$
- *Thus, for* $T > \limsup_{\substack{\lambda\to\infty\\ \lambda\in\Lambda}} \dfrac{\ln \frac{\|\phi^1_\lambda\|_X}{\mathcal{B}^\star\phi_\lambda}}{\lambda}$*, the problem is null-controllable at time* T*.*

ASSUMPTIONS : $(\phi_\lambda)_{\lambda\in\Lambda}$ is a complete family of X, $\inf_{\lambda\in\Lambda} |\mathcal{B}^\star \phi_\lambda| > 0$, and

$$\sum_{\lambda\in\Lambda} \frac{1}{\lambda} < +\infty.$$

NO GAP CONDITION : Let $c(\Lambda) \in [0, +\infty]$ be the **condensation index** of the family Λ
... it only depends on the way the eigenvalues are asymptotically close.

THEOREM ((AK-B-GB-dT, '14))

• *There exists a bi-orthogonal family* $(q_\lambda)_{\lambda\in\Lambda} \subset L^2(0,T)$ *such that*

$$\|q_\lambda\|_{L^2(0,T)} \leq C_{\varepsilon,T} e^{(c(\Lambda)+\varepsilon)\lambda}, \quad \forall \lambda \in \Lambda.$$

This estimate is essentially optimal.

• *For* $\boxed{T > c(\Lambda)}$ *the problem is null-controllable at time* T.

• *If* $(\phi_\lambda)_{\lambda\in\Lambda}$ *is a Riesz basis, the problem is not null-controllable for* $\boxed{T < c(\Lambda).}$

USUAL ASSUMPTIONS : $(\phi_\lambda)_{\lambda\in\Lambda}$ is a complete family of X, $\mathcal{B}^\star\phi_\lambda \neq 0$, $\forall\lambda\in\Lambda$, and

$$\sum_{\lambda\in\Lambda}\frac{1}{\lambda} < +\infty.$$

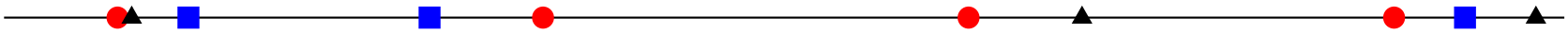
LESS USUAL ASSUMPTION : WEAK GAP CONDITION

$$\exists\rho>0,\ p\geq 1,\quad \mathrm{Card}(\Lambda\cap[\mu,\mu+\rho])\leq p,\ \ \forall\mu>0.$$

TYPICAL EXAMPLE :

$$\Lambda = \bigcup_{i=1}^{p}\Lambda_i,$$

where each family Λ_i satisfies the usual gap condition $\inf\limits_{\substack{\lambda,\mu\in\Lambda_i\\ \lambda\neq\mu}} |\lambda-\mu| > 0.$



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$$\exists\rho>0,\ p\geq 1,\quad \mathrm{Card}(\Lambda\cap[\mu,\mu+\rho])\leq p,\ \ \forall\mu>0.$$

PROPOSITION

We can write

$$\Lambda = \bigcup_{k\geq 1} G_k,$$

with

$$\mathrm{Card}(G_k)\leq p,\ \ \mathrm{diam}(G_k)\leq\rho,$$

$$(\min G_{k+1})-(\max G_k)\geq r,\quad \textit{for some } r>0.$$

Numbering in each group : $\qquad G_k=\{\lambda_{k,1},\dots,\lambda_{k,g_k}\},\qquad \phi_{k,j}:=\phi_{\lambda_{k,j}}.$

DIVIDED DIFFERENCES IN A GIVEN GROUP G_k

We drop the index k

- For any $j \in \{1, ..., g_k\}$, we set $\psi[\lambda_j] := \psi_j = \frac{\phi_j}{\mathcal{B}^\star \phi_j} \in X$.
- For any $i \neq j$ we set

$$\psi[\lambda_i, \lambda_j] := \frac{\psi[\lambda_j] - \psi[\lambda_i]}{\lambda_j - \lambda_i} \in X.$$

- and so on ... following the usual diagram

$$\begin{tikzcd}
\lambda_1 \arrow[r, dash, dashed] & \psi_1 = \psi[\lambda_1] \arrow[dr] & & & \\
& & \psi[\lambda_1, \lambda_2] \arrow[dr] & & \\
\lambda_2 \arrow[r, dash, dashed] & \psi_2 = \psi[\lambda_2] \arrow[ur] \arrow[dr] & & \psi[\lambda_1, \lambda_2, \lambda_3] \arrow[dr] & \\
& & \psi[\lambda_2, \lambda_3] \arrow[ur] \arrow[dr] & & \psi[\lambda_1, \lambda_2, \lambda_3, \lambda_4] \\
\lambda_3 \arrow[r, dash, dashed] & \psi_3 = \psi[\lambda_3] \arrow[ur] \arrow[dr] & & \psi[\lambda_2, \lambda_3, \lambda_4] \arrow[ur] & \\
& & \psi[\lambda_3, \lambda_4] \arrow[ur] & & \\
\lambda_4 \arrow[r, dash, dashed] & \psi_4 = \psi[\lambda_4] \arrow[ur] & & &
\end{tikzcd}$$

THEOREM ((B-B-M, '18))

Assume that $U = \mathbb{R}$ and

$$(\phi_\lambda)_{\lambda\in\Lambda} \text{ is complete in } X,$$

$$\sum_{\lambda\in\Lambda} \frac{1}{\lambda} < +\infty,$$

$$+ \text{ Weak gap condition},$$

and define

$$T_0 = \limsup_{k\to+\infty} \frac{\ln\left(\max_{1\leq l\leq g_k} \|\psi[\lambda_{k,1},\dots,\lambda_{k,l}]\|_X\right)}{\lambda_{k,1}}.$$

Then, we have

- *For $T > T_0$, the system is null-controllable.*
- *For $T < T_0$, the system is not null-controllable.*

REMARK : The negative part ($T < T_0$) was essentially proved in the previous section

Assuming $T > T_0$, we are led to find $v \in L^2(0,T)$ solving a moments problem

$$\int_0^T v(t)e^{-\lambda t}\,dt = \omega_\lambda, \quad \forall \lambda \in \Lambda. \tag{$\star$}$$

THE USUAL WAY FAILS

- Poor (yet optimal !) estimates of the biorthogonal family.
- Impossible to take advantage of any particular structure of the data $(\omega_\lambda)_{\lambda\in\Lambda}$.

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- Poor (yet optimal !) estimates of the biorthogonal family.
- Impossible to take advantage of any particular structure of the data $(\omega_\lambda)_{\lambda\in\Lambda}$.

PROPOSITION (BLOCK BY BLOCK RESOLUTION)

For each G_k, there exists a v_k satisfying

$$\begin{cases} \displaystyle\int_0^T v_k(t)e^{-\lambda_{k,j}t}\,dt = \omega_{k,j}, \quad \forall j \in \{1,\dots,g_k\}, \\ \displaystyle\int_0^T v_k(t)e^{-\lambda t}\,dt = 0, \quad \forall \lambda \in \Lambda \setminus G_k, \end{cases}$$

that satisfies the (essentially optimal) estimate

$$\|v_k\| \leq C_\varepsilon e^{\varepsilon \lambda_{k,1}} \max_{l\in\{1,\dots,g_k\}} |\omega[\lambda_{k,1},\dots,\lambda_{k,l}]|.$$

Finally, $v = \sum_k v_k$ formally solves ($\star$).

TOOLS

- Complex analysis : Laplace transform, Blaschke products, etc ...
- Interpolation theory
- Nice algebraic properties of divided differences

COMMENTS

- Contains (almost) all the known abstract results in the literature for scalar controls of such systems (with weak-gap condition).
- The study of each particular system is reduced to a careful spectral analysis in each group.
- All the constants in the estimates are somehow uniform with respect to Λ. In particular they give interesting information even when $T_0 = 0$:
  - $\rightsquigarrow$ Systems depending on parameters
  - $\rightsquigarrow$ Semi-discrete systems
  - $\rightsquigarrow$ etc ...

MORE OR LESS STRAIGHTFORWARD EXTENSIONS

- For complex eigenvalues as soon as imaginary parts are not too large.
- For eigenvalues with (bounded) algebraic multiplicities.

$$\partial_t y + \begin{pmatrix} -\partial_x^2 & 1 \\ 0 & -\partial_x^2 + f(-\partial_x^2) \end{pmatrix} y = 0, \quad y(t,0) = \begin{pmatrix} 0 \\ u(t) \end{pmatrix},$$

with a bounded $f : \mathbb{R}^+ \to \mathbb{R}$ for instance.

- Weak gap condition with $p = 2$.
- The condensation index $c(\Lambda)$ can be arbitrary large.

$$f(s) = e^{-\alpha s} \Longrightarrow c(\Lambda) = \alpha.$$

$$\partial_t y + \begin{pmatrix} -\partial_x^2 & 1 \\ 0 & -\partial_x^2 + f(-\partial_x^2) \end{pmatrix} y = 0, \quad y(t,0) = \begin{pmatrix} 0 \\ u(t) \end{pmatrix},$$

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- Weak gap condition with $p = 2$.
- The condensation index $c(\Lambda)$ can be arbitrary large.

$$f(s) = e^{-\alpha s} \Longrightarrow c(\Lambda) = \alpha.$$

- Natural choice of groups : $G_k = \{\lambda_{k,1} = k^2\pi^2,\ \lambda_{k,2} = k^2\pi^2 + f(k^2\pi^2)\}$.
  Up to normalization, we have

$$\phi_{k,1}(x) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \sin(k\pi x), \qquad \phi_{k,2}(x) = \begin{pmatrix} 1 \\ f(k^2\pi^2) \end{pmatrix} \sin(k\pi x).$$

$$\phi[\lambda_{k,1}, \lambda_{k,2}] = \frac{\phi_{k,2} - \phi_{k,1}}{\lambda_{k,2} - \lambda_{k,1}} = \frac{\phi_{k,2} - \phi_{k,1}}{f(k^2\pi^2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin(k\pi x), \quad \textbf{bounded !}$$

Theorem

The above system is null-controllable at any time $T > 0$.

$$\partial_t y + \begin{pmatrix} -\partial_x^2 & 1 & 0 \\ 0 & -\partial_x^2 + f(-\partial_x^2) & 0 \\ 0 & 0 & -\partial_x^2 + g(-\partial_x^2) \end{pmatrix} y = 0, \quad y(t,0) = \begin{pmatrix} 0 \\ u(t) \\ u(t) \end{pmatrix}.$$

with $f(s) = e^{-\alpha s}$, and $g(s) = e^{-\beta s}$, $\alpha, \beta > 0$.

- Weak gap condition with $p = 3$.
- The condensation index $c(\Lambda)$ can be computed

$$c(\Lambda) = \alpha + \beta.$$

$$\partial_t y + \begin{pmatrix} -\partial_x^2 & 1 & 0 \\ 0 & -\partial_x^2 + f(-\partial_x^2) & 0 \\ 0 & 0 & -\partial_x^2 + g(-\partial_x^2) \end{pmatrix} y = 0, \quad y(t,0) = \begin{pmatrix} 0 \\ u(t) \\ u(t) \end{pmatrix}.$$

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- Weak gap condition with $p = 3$.
- The condensation index $c(\Lambda)$ can be computed

$$c(\Lambda) = \alpha + \beta.$$

- The results in **(AK-B-GB-dT, '14)** imply the null-controllability for any

$$T > c(\Lambda).$$

However, this minimal control time is over-estimated.

Theorem

Let

$$T_0 = \min(\alpha + \beta, 2\beta).$$

- *If $T > T_0$, the above system is null-controllable at time T.*
- *If $T < T_0$, the above system is not null-controllable at time T.*

$$\begin{cases} \partial_t y + \begin{pmatrix} -\partial_x^2 & 1 \\ 0 & -\partial_x^2 + a(x) \end{pmatrix} y = 0, \\ y(t,0) = \begin{pmatrix} 0 \\ u(t) \end{pmatrix}, \quad y(t,1) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \end{cases}$$

We have already seen that $c(\Lambda_a)$ can be **arbitrary large** but ...

THEOREM

For any $a \in L^2(0,1)$ small enough, the system is null-controllable at any time $T > 0$.

REMARK

If a is not small enough, the result still holds as soon as y_0 belongs to some finite codimension space.

(Bhandari-B, preprint '19)

$$\begin{cases} \partial_t y + \begin{pmatrix} -\partial_x^2 & 0 \\ 1 & -\partial_x^2 \end{pmatrix} y = 0, & \\ -\partial_x y_1 + \beta_1 y_1 = u(t), & \text{at } x = 0 \\ \partial_x y_1 + \beta_1 y_1 = 0, & \text{at } x = 1 \\ -\partial_x y_2 + \beta_2 y_2 = 0, & \text{at } x = 0 \\ \partial_x y_2 + \beta_2 y_2 = 0, & \text{at } x = 1 \end{cases}$$

Motivations : Treatment of Dirichlet boundary data by penalisation.

- Standard approach in numerical analysis.
- For $\beta_i > 0$, we have solutions in the usual energy spaces even for low regularity data $u \in L^2$.
- For β_1 large, we expect that u/β_1 is almost a Dirichlet control

$$-\frac{1}{\beta_1}\partial_x y_1 + y_1 = \frac{u(t)}{\beta_1}, \text{ at } x = 0.$$

(Bhandari-B, preprint '19)

$$\begin{cases} \partial_t y + \begin{pmatrix} -\partial_x^2 & 0 \\ 1 & -\partial_x^2 \end{pmatrix} y = 0, & \\ -\partial_x y_1 + \beta_1 y_1 = u(t), & \text{at } x = 0 \\ \partial_x y_1 + \beta_1 y_1 = 0, & \text{at } x = 1 \\ -\partial_x y_2 + \beta_2 y_2 = 0, & \text{at } x = 0 \\ \partial_x y_2 + \beta_2 y_2 = 0, & \text{at } x = 1 \end{cases}$$

Theorem

For any $\beta_1, \beta_2 > 1$, and any time $T > 0$, there exists a control $u_{\beta_1,\beta_2} \in L^2(0,T)$ s.t.

$$\|u_{\beta_1,\beta_2}\|_{L^2(0,T)} \leq C_T(1+\beta_1)\|y_0\|.$$

Up to a subsequence, $\frac{u_{\beta_1,\beta_2}}{\beta_1}$ converges towards a Dirichlet control as $\beta_1, \beta_2 \to +\infty$.

Remarks :

- Existence of a control for any T, β_i does not need the block moment approach.
- However, the precise estimate **crucially needs** the block moment approach to deal with the condensation of eigenvalues when β_1 and β_2 are close.
- Some extensions to the multi-D case.

SAME ASSUMPTIONS AS BEFORE EXCEPTED THAT U IS NOW ANY HILBERT SPACE

- To simplify : assume algebraically (and geometrically) simple eigenvalues.
- The control problem becomes : find $v \in L^2(0,T;U)$ such that

$$\int_0^T e^{-\lambda t} \left(v(t), \mathcal{B}^\star \phi_\lambda\right)_U \, dt = -e^{-\lambda T} \left\langle y_0, \phi_\lambda \right\rangle_X, \quad \forall \lambda \in \Lambda.$$

- Since $b_\lambda := \mathcal{B}^\star \phi_\lambda$ is now any element in U, we cannot "divide" by b_λ.

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THE USUAL CASE : WHEN THERE IS NO SPECTRAL CONDENSATION (Lagnese, ’83)

- Assume that Λ satisfies the usual gap condition.
  We can use a biorthogonal family $(q_\lambda)_\lambda$ with good estimates and look for v in the following form

$$v(t) = -\sum_{\lambda \in \Lambda} e^{-\lambda T} \left\langle y_0, \phi_\lambda \right\rangle_X q_\lambda(t) \frac{b_\lambda}{\|b_\lambda\|_U^2}.$$

  The minimal control time is

$$T_0 = \limsup_{\substack{\lambda \to +\infty \\ \lambda \in \Lambda}} \frac{-\ln \|b_\lambda\|_U}{\lambda}.$$

SAME ASSUMPTIONS AS BEFORE EXCEPTED THAT U IS NOW ANY HILBERT SPACE

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$$\int_0^T e^{-\lambda t} (v(t), \mathcal{B}^\star \phi_\lambda)_U \, dt = -e^{-\lambda T} \langle y_0, \phi_\lambda \rangle_X , \quad \forall \lambda \in \Lambda.$$

- Since $b_\lambda := \mathcal{B}^\star \phi_\lambda$ is now any element in U, we cannot "divide" by b_λ.

THE GENERAL CASE : TAKING INTO ACCOUNT SPECTRAL CONDENSATION

THEOREM ((B.-Morancey, '19?))

Assume that Λ satisfy the weak gap condition.
There exists an "explicit" matrix M_k of size $g_k \times g_k$ associated to each group G_k such that the minimal NC time of the system is

$$T_0 = \limsup_{k \to \infty} \frac{\ln(\rho(M_k))}{2\lambda_{k,1}}.$$

THE FORMULA NICELY DEGENERATES IN THE EASY CASES:

- When all the observations b_λ (in the group G_k) are colinear : same result as in the scalar case.
- When all the observations b_λ (in the group G_k) are orthogonal : same result as the one by Lagnese's technique $\Rightarrow$ we don't see the effect of spectral condensation.

BOUNDARY CONTROL WITH TWO CONTROLS

$$\partial_t y + \begin{pmatrix} -\partial_x^2 & 1 \\ 0 & -d\partial_x^2 \end{pmatrix} y = 0, \quad y(t,0) = \begin{pmatrix} 0 \\ u_1(t) \end{pmatrix}, \quad y(t,1) = \begin{pmatrix} u_2(t) \\ u_2(t) \end{pmatrix}$$

THEOREM

Let $d \neq 1$.

- **One control :** **(AK-B-GB-dT, '14)**
  Assume that $u_2 = 0$, and $\sqrt{d} \notin \mathbb{Q}$, then the minimal null-control time is
  $$T_0(d) = c(\Lambda_d).$$
- **Two controls :** **(B-M, '19?)**
  For any $d > 0$, there is no minimal null-control time
  $$T_0 = 0.$$

DISTRIBUTED CONTROLS

$$\partial_t y + \begin{pmatrix} -\partial_x^2 & 1 & 0 \\ 0 & -\partial_x^2 + f(-\partial_x^2) & 0 \\ 0 & 0 & -\partial_x^2 + g(-\partial_x^2) \end{pmatrix} y = 1_{\omega_1} u_1(t,x) B_1 + 1_{\omega_2} u_2(t,x) B_2,$$

with $f(s) = e^{-\alpha s}$, and $g(s) = e^{-\beta s}$, $\alpha, \beta > 0$.

THEOREM

Assume that $\omega_1 \cap \omega_2 = \emptyset$ *and* $B_1, B_2 = \begin{pmatrix} 0 \\ * \\ * \end{pmatrix} \in \mathbb{R}^3$ *are linearly independent.*

- **One control :**
  Assume that $u_2 = 0$, *then there is a minimal control time*

$$T_0 = \beta + \min(\alpha, \beta).$$

- **Two controls :**
  There is no minimal control time

$$T_0 = 0.$$

Let us recall the following 1D result with $d \neq 1$

$$\partial_t y + \begin{pmatrix} -\partial_x^2 & 1 \\ 0 & -d\partial_x^2 \end{pmatrix} y = 0, \quad y(t,0) = \begin{pmatrix} 0 \\ u(t) \end{pmatrix}, \quad y(t,1) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Theorem ((AK-B-GB-dT, '14))

- *If $\sqrt{d} \in \mathbb{Q}$, the system is not approximately controllable.*
- *If $\sqrt{d} \notin \mathbb{Q}$, the system has a minimal null-control time $T_0(d)$ that may be any number in $[0, +\infty]$.*

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The same system in 2D : Let $\Omega = (0,L_1) \times (0,L_2)$ be a rectangle and $\Gamma \subset \partial\Omega$.

$$\partial_t y + \begin{pmatrix} -\Delta & 1 \\ 0 & -d\Delta \end{pmatrix} y = 0, \quad y(t,.) = \begin{pmatrix} 0 \\ 1_\Gamma u(t,.) \end{pmatrix}.$$

Theorem ((B-Olive, '19?))

If Γ intersects two **non parallel** *sides of $\partial\Omega$, then the system is null-controllable at any time $T > 0$, for any value of d.*

Let $\Lambda_i = \{k^2 L_i^2/\pi^2, \ \ k \geq 1\}$, $i = 1, 2$ be the eigenvalues of $-\partial_x^2$ on $(0, L_i)$.

IN THE CASE WHERE Γ IS THE UNION OF TWO NON PARALLEL SIDES

The null-control problem is equivalent to:

A STRANGE "MOMENTS" PROBLEM

Find two families $(f_\lambda)_{\lambda\in\Lambda_1} \subset L^2(0,T)$, $(g_\mu)_{\mu\in\Lambda_2} \subset L^2(0,T)$ such that

$$\begin{cases} \displaystyle\int_0^T f_\lambda(t) e^{-(\lambda+\mu)t}\,dt + \int_0^T g_\mu(t) e^{-(\lambda+\mu)t}\,dt = \omega_{\lambda,\mu}, & \forall \lambda \in \Lambda_1, \mu \in \Lambda_2 \\ \displaystyle\int_0^T f_\lambda(t) e^{-d(\lambda+\mu)t}\,dt + \int_0^T g_\mu(t) e^{-d(\lambda+\mu)t}\,dt = \omega_{d\lambda,d\mu}, & \forall \lambda \in \Lambda_1, \mu \in \Lambda_2, \end{cases}$$

with $\displaystyle\sum_{\lambda\in\Lambda_1} \|f_\lambda\|^2_{L^2(0,T)} + \sum_{\mu\in\Lambda_2} \|g_\mu\|^2_{L^2(0,T)} < +\infty.$

Here, $\omega_{\bullet,\bullet}$ are given and satisfy nice exponential estimates.

MAIN DIFFICULTY

It may happen that $\lambda + \mu$ is close from $d(\lambda + \tilde{\mu})$ for some $\lambda \in \Lambda_1$, $\mu, \tilde{\mu} \in \Lambda_2$

$\Longrightarrow$ Spectral condensation troubles $\Longrightarrow$ Block moment approach

Thanks for your attention !

A. Benabdallah, F. Boyer, M. Morancey, A BLOCK MOMENT METHOD TO HANDLE SPECTRAL CONDENSATION PHENOMENON IN PARABOLIC CONTROL PROBLEMS, preprint, https://hal.archives-ouvertes.fr/hal-01949391