Optimality criteria method for problems of optimal design in linearized elasticity

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Linearized elasticity system

Let $\Omega \subseteq \mathbf{R}^d$ be open and bounded, $\mathbf{A} \in \mathrm{L}^\infty(\Omega; \mathrm{Sym}_\mathrm{d}^4)$ satisfying

$$\mathbf{A}\boldsymbol{\xi} : \boldsymbol{\xi} \geq \alpha|\boldsymbol{\xi}|^2, \quad \mathbf{A}^{-1}\boldsymbol{\xi} : \boldsymbol{\xi} \geq \frac{1}{\beta}|\boldsymbol{\xi}|^2, \qquad \boldsymbol{\xi} \in \mathrm{Sym}_\mathrm{d}$$

and f $\in \mathrm{H}^{-1}(\Omega; \mathbf{R}^\mathrm{d})$. Linearized elasticity system with homogeneous Dirichlet boundary condition:

$$\begin{cases} -\mathrm{div}\,(\mathbf{A}e(\mathsf{u})) = \mathsf{f} \\ \mathsf{u} \in \mathrm{H}_0^1(\Omega; \mathbf{R}^d)\,, \end{cases}$$

where $e(\mathsf{u}) = \frac{1}{2}(\nabla \mathsf{u} + \nabla \mathsf{u}^\top)$.

Ω - mixture of two isotropic elastic phases with rigidity tensors $\mathbf{A}_1 = 2\mu_1\mathbf{I}_4 + \left(\kappa_1 - \frac{2\mu_1}{d}\right)\mathbf{I}_2 \otimes \mathbf{I}_2$, $\mathbf{A}_2 = 2\mu_2\mathbf{I}_4 + \left(\kappa_2 - \frac{2\mu_2}{d}\right)\mathbf{I}_2 \otimes \mathbf{I}_2$, where $0 < \mu_1 < \mu_2$ and $0 < \kappa_1 < \kappa_2$.

$$\mathbf{A}(\mathbf{x}) = \chi(\mathbf{x})\mathbf{A}_1 + (1 - \chi(\mathbf{x}))\mathbf{A}_2, \quad \chi \in \mathrm{L}^\infty(\Omega; \{0, 1\}).$$

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Optimal design problem

$$\begin{cases} J(\chi) = \int_\Omega [\chi(\mathbf{x}) g_1(\mathbf{x}, \mathsf{u}(\mathbf{x})) + (1-\chi(\mathbf{x})) g_2(\mathbf{x}, \mathsf{u}(\mathbf{x}))] \, d\mathbf{x} \longrightarrow \min , \\ \chi \in \mathrm{L}^\infty(\Omega; \{0,1\}) , \ \int_\Omega \chi \, d\mathbf{x} = q . \end{cases}$$

$0 < q < |\Omega|$, and g_1, g_2 Caratheodory functions which satisfy growth condition

$$g_{1,2}(\mathbf{x}, \mathsf{u}) \le a|\mathsf{u}|^s + b(\mathbf{x}),$$

for some $a > 0$, $b \in \mathrm{L}^1(\Omega)$ and $1 \le s < \frac{2d}{d-2}$, and u is the solution of

$$\begin{cases} -\mathsf{div}\,(\mathbf{A} e(\mathsf{u})) = \mathsf{f} \\ \mathsf{u} \in \mathrm{H}^1_0(\Omega; \mathbf{R}^d) , \end{cases}$$

where $\mathsf{f} \in \mathrm{H}^{-1}(\Omega; \mathbf{R}^{\mathrm{d}})$, $\mathbf{A} = \mathbf{A}(\mathbf{x}) = \chi(\mathbf{x}) \mathbf{A}_1 + (1-\chi(\mathbf{x})) \mathbf{A}_2$.

Definition (Composite material)

If a sequence of characteristic functions $\chi_n \in \mathrm{L}^\infty(\Omega; \{0,1\})$ and tensors $\mathbf{A}^n(x) = \chi_n(x)\mathbf{A}_1 + (1-\chi_n(x))\mathbf{A}_2$ satisfy

$$\begin{array}{ccc} \chi_n & \overset{*}{\rightharpoonup} & \theta \\ \mathbf{A}^n & \overset{H}{\rightharpoonup} & \mathbf{A}, \end{array}$$

then it is said that $\mathbf{A}$ is homogenised tensor of two-phase composite material with proportions θ of first material and microstructure defined by the sequence (χ_n).

Definition (H-convergence)

A sequence of tensor functions $\mathbf{A}^n$ is said to H-converge to $\mathbf{A}$ if for every f the sequence of solutions of

$$\left\{ \begin{array}{l} -\mathrm{div}\,(\mathbf{A}e(\mathsf{u}_n)) = \mathsf{f} \\ \mathsf{u}_n \in \mathrm{H}^1_0(\Omega; \mathbf{R}^d)\,, \end{array} \right.$$

satisfies $\mathsf{u}_n \rightharpoonup \mathsf{u}$ in $\mathrm{H}^1_0(\Omega;\mathbf{R}^d)$, $\mathbf{A}^n e(\mathsf{u}_n) \rightharpoonup \mathbf{A}e(\mathsf{u})$ in $\mathrm{L}^2(\Omega; \mathrm{Sym_d})$, where u is the solution of the homogenised equation

$$\left\{ \begin{array}{l} -\mathrm{div}\,(\mathbf{A}e(\mathsf{u})) = \mathsf{f} \\ \mathsf{u} \in \mathrm{H}^1_0(\Omega; \mathbf{R}^d)\,, \end{array} \right.$$

Example – simple laminates: if χ_ε depend only on x_1, then

$$\theta(\mathbf{A}-\mathbf{A}_2)^{-1} = (\mathbf{A}_1-\mathbf{A}_2)^{-1} + (1-\theta) f_2(\mathsf{e}_1),$$

where $f_2(\mathsf{e}_1)\boldsymbol{\xi}:\boldsymbol{\xi} = \frac{1}{\mu_2}\left(|\boldsymbol{\xi}\mathsf{e}_1|^2 - (\boldsymbol{\xi}\mathsf{e}_1\cdot\mathsf{e}_1)^2\right) + \frac{1}{2\mu_2+\lambda_2}(\boldsymbol{\xi}\mathsf{e}_1\cdot\mathsf{e}_1)^2$, for any $\boldsymbol{\xi} \in \mathrm{Sym_d}$, with $\lambda_2 = \kappa_2 - 2\mu_2/d$.

Set of all composites:

$$\mathcal{A} := \{(\theta, \mathbf{A}) \in \mathrm{L}^\infty(\Omega; [0,1] \times \mathrm{Sym}_\mathrm{d}^4) : \ \mathbf{A} \in G(\theta) \text{ a.e. on } \Omega\}$$

G–closure problem: for given θ find all possible homogenised (effective) tensors $\mathbf{A}$ - open problem

Theorem (Hashin-Shtrikman bounds on complementary energy)

Let $\boldsymbol{\sigma} \in \mathrm{Sym}_\mathrm{d}$. Any homogenized tensor $\mathbf{A} \in G(\theta)$ satisfies

$$\mathbf{A}^{-1}\boldsymbol{\sigma} : \boldsymbol{\sigma} \geq \mathbf{A}_2^{-1}\boldsymbol{\sigma} : \boldsymbol{\sigma} + \theta \max_{\boldsymbol{\eta} \in \mathrm{Sym}_\mathrm{d}} \left[2\boldsymbol{\sigma} : \boldsymbol{\eta} - (\mathbf{A}_1^{-1} - \mathbf{A}_2^{-1})^{-1}\boldsymbol{\eta} : \boldsymbol{\eta} - (1-\theta) g^c(\boldsymbol{\eta})\right],$$

where $g^c(\boldsymbol{\eta})$ is a nonlocal term given by $g^c(\boldsymbol{\eta}) = \max_{\mathrm{e} \in S^{d-1}} \left(f_2^c(\mathrm{e})\boldsymbol{\eta} : \boldsymbol{\eta}\right)$, and

$$\mathbf{A}^{-1}\boldsymbol{\sigma} : \boldsymbol{\sigma} \leq \mathbf{A}^{-1}\boldsymbol{\sigma} : \boldsymbol{\sigma} + (1-\theta) \min_{\boldsymbol{\eta} \in \mathrm{Sym}_\mathrm{d}} \left[2\boldsymbol{\sigma} : \boldsymbol{\eta} + (\mathbf{A}_1^{-1} - \mathbf{A}_2^{-1})^{-1}\boldsymbol{\eta} : \boldsymbol{\eta} - \theta h^c(\boldsymbol{\eta})\right],$$

where $h^c(\boldsymbol{\eta})$ is a nonlocal term given by $h^c(\boldsymbol{\eta}) = \min_{\mathrm{e} \in S^{d-1}} \left(f_1^c(\mathrm{e})\boldsymbol{\eta} : \boldsymbol{\eta}\right)$, with

$f_i^c(\mathrm{e})\boldsymbol{\xi} : \boldsymbol{\xi} = \mathbf{A}_i\boldsymbol{\xi} : \boldsymbol{\xi} - \frac{1}{\mu_i}|\mathbf{A}_i\boldsymbol{\xi}\mathrm{e}|^2 + \frac{\mu_i + \lambda_i}{\mu_i(2\mu_i + \lambda_i)}\left((\mathbf{A}_i\boldsymbol{\xi})\mathrm{e} \cdot \mathrm{e}\right)^2, i = 1, 2, \boldsymbol{\xi} \in \mathrm{Sym}_d.$
Furthermore, these upper and lower bounds are optimal and optimality is achieved by a rank - d sequential laminate with the lamination directions given by the extremal vectors in the definition of the nonlocal terms $g^c(\boldsymbol{\eta})$ and $h^c(\boldsymbol{\eta})$.

Set of all composites:

$$\mathcal{A} := \{(\theta, \mathbf{A}) \in \mathrm{L}^\infty(\Omega; [0,1] \times \mathrm{Sym}_\mathrm{d}^4) : \ \mathbf{A} \in G(\theta) \text{ a.e. on } \Omega\}$$

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$f_i^c(\mathsf{e})\boldsymbol{\xi} : \boldsymbol{\xi} = \mathbf{A}_i\boldsymbol{\xi} : \boldsymbol{\xi} - \frac{1}{\mu_i}|\mathbf{A}_i\boldsymbol{\xi}\mathsf{e}|^2 + \frac{\mu_i + \lambda_i}{\mu_i(2\mu_i + \lambda_i)}\left((\mathbf{A}_i\boldsymbol{\xi})\mathsf{e} \cdot \mathsf{e}\right)^2$, $i = 1, 2$, $\boldsymbol{\xi} \in \mathrm{Sym}_d$.
Furthermore, these upper and lower bounds are optimal and optimality is achieved by a rank - d sequential laminate with the lamination directions given by the extremal vectors in the definition of the nonlocal terms $g^c(\boldsymbol{\eta})$ *and* $h^c(\boldsymbol{\eta})$.

Original problem:

$$\begin{cases} J(\chi) = \int_\Omega [\chi(\mathbf{x}) g_1(\mathbf{x}, \mathsf{u}(\mathbf{x})) + (1-\chi(\mathbf{x})) g_2(\mathbf{x}, \mathsf{u}(\mathbf{x}))] \, d\mathbf{x} \longrightarrow \min , \\ \chi \in \mathrm{L}^\infty(\Omega; \{0,1\}) , \; \int_\Omega \chi \, d\mathbf{x} = q . \end{cases}$$

Generalized objective function is

$$J(\theta, \mathbf{A}) = \int_\Omega [\theta(\mathbf{x}) g_1(\mathbf{x}, \mathsf{u}(\mathbf{x})) + (1-\theta)(\mathbf{x})) g_2(\mathbf{x}, \mathsf{u}(\mathbf{x}))] \, d\mathbf{x} + l \int_\Omega \theta(\mathbf{x}) \, d\mathbf{x}$$

where u is the solution of the state equation

$$\begin{cases} -\mathrm{div}\,(\mathbf{A} e(\mathsf{u})) = \mathsf{f} \\ \mathsf{u} \in \mathrm{H}_0^1(\Omega; \mathbf{R}^d) . \end{cases}$$

Relaxed problem:

$$\begin{cases} J(\theta, \mathbf{A}) \longrightarrow \min , \\ (\theta, \mathbf{A}) \in \{(\theta, \mathbf{A}) \in \mathrm{L}^\infty(\Omega; [0,1] \times \mathrm{Sym}_\mathrm{d}^4) : \; \mathbf{A} \in G(\theta) \text{ a.e. on } \Omega\} . \end{cases} \tag{1}$$

Compliance minimization

For $g_1 = g_2 = \mathsf{f} \cdot \mathsf{u}$, it follows

$$J(\theta, \mathbf{A}) = \int_\Omega \mathsf{f}(\mathbf{x}) \cdot \mathsf{u}(\mathbf{x})\, d\mathbf{x} + l \int_\Omega \theta(\mathbf{x})\, d\mathbf{x}. \quad (2)$$

By the principle of minimal complementary energy we have

$$\int_\Omega \mathsf{f}(\mathbf{x}) \cdot \mathsf{u}(\mathbf{x}) = \min_{\substack{\tau \in \mathrm{L}^2(\Omega;\mathrm{Sym_d}) \\ -\mathrm{div}\tau = \mathsf{f} \text{ in } \Omega}} \int_\Omega \mathbf{A}^{-1}\tau : \tau\, d\mathbf{x},$$

and the minimum on the right hand side is achieved by $\boldsymbol{\sigma} = \mathbf{A}e(\mathsf{u})$.

Therefore, (2) can be rewritten as

$$J(\theta, \mathbf{A}) = \min_{\substack{\tau \in \mathrm{L}^2(\Omega;\mathrm{Sym_d}) \\ -\mathrm{div}\tau = \mathsf{f} \text{ in } \Omega}} \int_\Omega \mathbf{A}^{-1}\tau : \tau\, d\mathbf{x} + l \int_\Omega \theta(\mathbf{x})\, d\mathbf{x} \quad (3)$$

and

$$\min_{(\theta,\mathbf{A}) \in \mathcal{A}} J(\theta, \mathbf{A}) = \min_{(\theta,\mathbf{A}) \in \mathcal{A}} \min_{\substack{\tau \in \mathrm{L}^2(\Omega;\mathrm{Sym_d}) \\ -\mathrm{div}\tau = \mathsf{f} \text{ in } \Omega}} \int_\Omega \mathbf{A}^{-1}\tau : \tau\, d\mathbf{x} + l \int_\Omega \theta(\mathbf{x})\, d\mathbf{x}.$$

Necessary conditions of optimality

Theorem

If $(\theta^*, \mathbf{A}^*)$ is a minimizer of the objective function (2), and if $\boldsymbol{\sigma}^*$ is the unique corresponding minimizer of (3), then $\boldsymbol{\sigma}^*$ satisfies

$$\begin{cases} \boldsymbol{\sigma}^* = \mathbf{A}^* e(\mathsf{u}) & \textit{in } \Omega \\ -\mathrm{div}\boldsymbol{\sigma}^* = \mathsf{f} & \textit{in } \Omega \\ \mathsf{u} = \mathbf{0} & \textit{on } \partial\Omega, \end{cases} \tag{4}$$

with $\mathsf{u} \in \mathrm{H}_0^1(\Omega, \mathbf{R}^\mathrm{d})$, and $(\theta^*, \mathbf{A}^*)$ satisfies, almost everywhere in Ω ,

$$\mathbf{A}^{*-1}\boldsymbol{\sigma}^* : \boldsymbol{\sigma}^* = g(\theta^*, \boldsymbol{\sigma}^*),$$

where $g(\theta^*, \sigma^*)$ is the lower Hashin-Shtrikman bound on complementary energy, and θ^* is the unique minimizer of the convex minimization problem

$$\min_{0\le\theta\le1} (g(\theta, \sigma^*) + l\theta).$$

Theorem

Let $\tilde{\mathcal{A}} := \{(\theta, \mathbf{A}) \in \mathrm{L}^\infty(\Omega; [0,1] \times \mathrm{Sym}^4_\mathrm{d}) : \ \mathbf{A} \in \mathrm{L}(\theta)$ *a.e. on* $\Omega\}$, *where the set* $\mathrm{L}(\theta)$ *is a set of all sequential laminates* $\mathbf{A}$ *with core* $\mathbf{A}_1$ *and matrix* $\mathbf{A}_2$*, in proportions* θ *and* $(1-\theta)$ *respectively, defined by*

$$\theta(\mathbf{A}^{-1} - \mathbf{A}_2^{-1})^{-1} = (\mathbf{A}_1^{-1} - \mathbf{A}_2^{-1})^{-1} + (1-\theta)\sum_{i=1}^{p} m_i f_2^c(\mathsf{e}_i),$$

where

$$f_2(\mathsf{e}_i)\boldsymbol{\xi} : \boldsymbol{\xi} = \frac{1}{\mu_2}\left(|\boldsymbol{\xi}\mathsf{e}_i|^2 - (\boldsymbol{\xi}\mathsf{e}_i \cdot \mathsf{e}_i)^2\right) + \frac{1}{2\mu_2 + \lambda_2}(\boldsymbol{\xi}\mathsf{e}_i \cdot \mathsf{e}_i)^2,$$

$m_i \geq 0$, $i = 1, \ldots, p$, *and* $\sum_{i=1}^{p} m_i = 1$. *Then*

$$\min_{(\theta,\mathbf{A})\in\mathcal{A}} J(\theta, \mathbf{A}) = \min_{(\theta,\mathbf{A})\in\tilde{\mathcal{A}}} J(\theta, \mathbf{A}).$$

Algorithm

Take some initial θ^0 and $\mathbf{A}^0$. For k from 0 to N:

1. Calculate u^k, the solution of

$$\begin{cases} -\mathsf{div}\,(\mathbf{A}^k e(\mathsf{u}^k)) = \mathsf{f} \\ \mathsf{u}^k \in \mathrm{H}^1_0(\Omega;\mathbf{R}^d)\,, \end{cases}$$

   and define $\sigma^k := \mathbf{A}^k e(\mathsf{u}^k)$.

2. For $\mathbf{x} \in \Omega$, let $\theta^{k+1}(\mathbf{x})$ be the zero of the function

$$\theta \mapsto \frac{\partial g}{\partial \theta}(\theta, \boldsymbol{\sigma}^k(\mathbf{x})) + l\,,$$

   and if a zero doesn't exist, take 0 (or 1) in case when this function is positive (or negative) on $\langle 0, 1\rangle$.

3. Let $(\mathbf{A}^{k+1})(\mathbf{x})$ be a sequential laminate such that $(\mathbf{A}^{k+1})^{-1}\sigma^k : \sigma^k = g(\theta^{k+1}, \sigma^k)$.

Explicit calculations of Hashin Shtrikman lower bound in 2D

$$\mathbf{A}^{-1}\boldsymbol{\sigma} : \boldsymbol{\sigma} \geq \mathbf{A}_2^{-1}\boldsymbol{\sigma} : \boldsymbol{\sigma} + \max_{\boldsymbol{\eta}\in \mathrm{Sym}_2} \left[2\boldsymbol{\sigma} : \boldsymbol{\eta} - (\mathbf{A}_1^{-1} - \mathbf{A}_2^{-1})^{-1}\boldsymbol{\eta} : \boldsymbol{\eta} - (1-\theta)g^c(\boldsymbol{\eta})\right],$$

where

$$g^c(\boldsymbol{\eta}) = \max_{\mathsf{e}\in S^1} \left(f_2^c(\mathsf{e})\boldsymbol{\eta} : \boldsymbol{\eta}\right),$$

and

$$f_2^c(\mathsf{e})\boldsymbol{\eta} : \boldsymbol{\eta} = \mathbf{A}_2\boldsymbol{\eta} : \boldsymbol{\eta} - \frac{1}{\mu_2}|\mathbf{A}_2\boldsymbol{\eta}\mathsf{e}|^2 + \frac{\mu_2+\lambda_2}{\mu_2(2\mu_2+\lambda_2)}(\mathbf{A}_2\boldsymbol{\eta}\mathsf{e}\cdot\mathsf{e})^2.$$

Lemma

Let $\eta_1 \leq \cdots \leq \eta_d$ be the eigenvalues of the symmetric matrix $\boldsymbol{\eta}$. Then

$$g^c(\boldsymbol{\eta}) = \mathbf{A}_2\boldsymbol{\eta} : \boldsymbol{\eta} - \frac{1}{2\mu_2+\lambda_2}\min\left\{(2\mu_2\eta_1 + \lambda_2\mathrm{tr}(\boldsymbol{\eta}))^2, (2\mu_2\eta_d + \lambda_2\mathrm{tr}(\boldsymbol{\eta}))^2\right\}.$$

$$\max_{\boldsymbol{\eta}\in \mathrm{Sym}_2} \left[2\boldsymbol{\sigma} : \boldsymbol{\eta} - (\mathbf{A}_1^{-1} - \mathbf{A}_2^{-1})^{-1}\boldsymbol{\eta} : \boldsymbol{\eta} - (1-\theta)g^c(\boldsymbol{\eta})\right], \tag{5}$$

- $\delta\mu = \mu_2 - \mu_1, \delta\kappa = \kappa_2 - \kappa_1$
- $(\mathbf{A}_1^{-1} - \mathbf{A}_2^{-1})^{-1}\boldsymbol{\eta} : \boldsymbol{\eta} = \frac{2\mu_1\mu_2}{\delta\mu}(\eta_1^2 + \eta_2^2) + \left(\frac{\kappa_1\kappa_2}{\delta\kappa} - \frac{2\mu_1\mu_2}{3\delta\mu}\right)(\eta_1 + \eta_2)^2.$
- von Neumann: $\boldsymbol{\sigma} : \boldsymbol{\eta} \leq \sigma_1\eta_1 + \sigma_2\eta_2$, and equality holds when $\boldsymbol{\sigma}$ and $\boldsymbol{\eta}$ are simultaneously diagonalizable

$$\max_{(\eta_1,\eta_2)\in \mathbf{R}^2} \left[2(\sigma_1\eta_1 + \sigma_2\eta_2) - \frac{2\mu_1\mu_2}{\delta\mu}(\eta_1^2 + \eta_2^2) - \left(\frac{\kappa_1\kappa_2}{\delta\kappa} - \frac{2\mu_1\mu_2}{3\delta\mu}\right)(\eta_1 + \eta_2)^2 - \right. \tag{6}$$
$$\left. -(1-\theta)g^c(\eta_1,\eta_2)\right],$$

$$g^c(\eta_1,\eta_2) = 2\mu_2(\eta_1^2+\eta_2^2)+\lambda_2(\eta_1+\eta_2)^2 - \frac{1}{2\mu_2+\lambda_2}\min\left\{(2\mu_2\eta_1 + \lambda_2 \mathrm{tr}\boldsymbol{\eta})^2, (2\mu_2\eta_2 + \lambda_2\mathrm{tr}\boldsymbol{\eta})^2\right\}.$$

$$\max_{\boldsymbol{\eta}\in\mathrm{Sym}_2} \left[2\boldsymbol{\sigma}:\boldsymbol{\eta} - (\mathbf{A}_1^{-1} - \mathbf{A}_2^{-1})^{-1}\boldsymbol{\eta}:\boldsymbol{\eta} - (1-\theta)g^c(\boldsymbol{\eta})\right], \tag{5}$$

- $\delta\mu = \mu_2 - \mu_1, \delta\kappa = \kappa_2 - \kappa_1$
- $(\mathbf{A}_1^{-1} - \mathbf{A}_2^{-1})^{-1}\boldsymbol{\eta}:\boldsymbol{\eta} = \frac{2\mu_1\mu_2}{\delta\mu}(\eta_1^2+\eta_2^2) + \left(\frac{\kappa_1\kappa_2}{\delta\kappa} - \frac{2\mu_1\mu_2}{3\delta\mu}\right)(\eta_1+\eta_2)^2.$
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$$\max_{(\eta_1,\eta_2)\in\mathbf{R}^2} \left[2(\sigma_1\eta_1+\sigma_2\eta_2) - \frac{2\mu_1\mu_2}{\delta\mu}(\eta_1^2+\eta_2^2) - \left(\frac{\kappa_1\kappa_2}{\delta\kappa} - \frac{2\mu_1\mu_2}{3\delta\mu}\right)(\eta_1+\eta_2)^2 - \right. \\ \left. -(1-\theta)g^c(\eta_1,\eta_2)\right], \tag{6}$$

$$g^c(\eta_1,\eta_2) = 2\mu_2(\eta_1^2+\eta_2^2)+\lambda_2(\eta_1+\eta_2)^2 - \frac{1}{2\mu_2+\lambda_2}\min\left\{(2\mu_2\eta_1+\lambda_2\mathrm{tr}\boldsymbol{\eta})^2, (2\mu_2\eta_2+\lambda_2\mathrm{tr}\boldsymbol{\eta})^2\right\}.$$

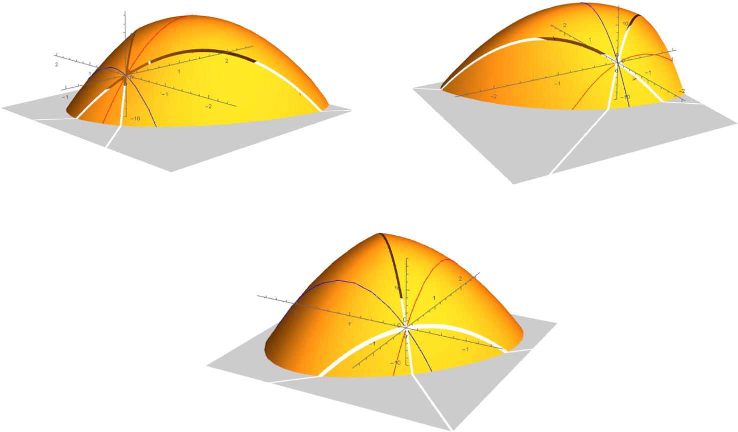


Figure: Graphs of the function $F(\eta_1, \eta_2) =$

$$\begin{cases} 2(\sigma_1\eta_1 + \sigma_2\eta_2) - \frac{2\mu_1\mu_2}{\delta\mu}(\eta_1^2 + \eta_2^2) - \left(\frac{\kappa_1\kappa_2}{\delta\kappa} - \frac{2\mu_1\mu_2}{3\delta\mu}\right)(\eta_1 + \eta_2)^2 - \frac{4(1-\theta)\kappa_2\mu_2}{\kappa_2+\mu_2}\eta_1^2, & |\eta_1| \leq |\eta_2| \\ 2(\sigma_1\eta_1 + \sigma_2\eta_2) - \frac{2\mu_1\mu_2}{\delta\mu}(\eta_1^2 + \eta_2^2) - \left(\frac{\kappa_1\kappa_2}{\delta\kappa} - \frac{2\mu_1\mu_2}{3\delta\mu}\right)(\eta_1 + \eta_2)^2 - \frac{4(1-\theta)\kappa_2\mu_2}{\kappa_2+\mu_2}\eta_2^2, & |\eta_1| \geq |\eta_2| \end{cases}$$

Theorem

If dimension $d = 2$, then for given $\theta \in [0,1]$, $0 < \mu_1 < \mu_2$, $0 < \kappa_1 < \kappa_2$ and matrix $\boldsymbol{\sigma}$ with eigenvalues σ_1 and σ_2, the lower Hashin-Shtrikman bound on complementary energy is explicitly given as follows:

A. *If*

$$(1-\theta)\mu_2(\kappa_2-\kappa_1)|\sigma_1+\sigma_2| < \left(\kappa_1(\kappa_2+\mu_2)+(1-\theta)\mu_2(\kappa_2-\kappa_1)\right)|\sigma_1-\sigma_2|,$$
$$(1-\theta)\kappa_2(\mu_2-\mu_1)|\sigma_1-\sigma_2| < \left(\mu_1(\kappa_2+\mu_2)+(1-\theta)\kappa_2(\mu_2-\mu_1)\right)|\sigma_1+\sigma_2|,$$

then

$$\mathbf{A}^{-1}\boldsymbol{\sigma}:\boldsymbol{\sigma} \geq \theta\mathbf{A}_1^{-1}\boldsymbol{\sigma}:\boldsymbol{\sigma} + (1-\theta)\mathbf{A}_2^{-1}\boldsymbol{\sigma}:\boldsymbol{\sigma} - \\ -(1-\theta)\theta\frac{(\kappa_2+\mu_2)(\kappa_1\kappa_2(\mu_2-\mu_1)|\sigma_1-\sigma_2|+\mu_1\mu_2(\kappa_2-\kappa_1)|\sigma_1+\sigma_2|)^2}{4\kappa_2\mu_2(\kappa_1\mu_1(\mu_2+\kappa_2)+(1-\theta)(\mu_1\mu_2(\kappa_2-\kappa_1)+\kappa_1\kappa_2(\mu_2-\mu_1)))}.$$

B. *If*

$$(1-\theta)\mu_2(\kappa_2-\kappa_1)|\sigma_1+\sigma_2| \geq \left(\kappa_1(\kappa_2+\mu_2)+(1-\theta)\mu_2(\kappa_2-\kappa_1)\right)|\sigma_1-\sigma_2|,$$

then

$$\mathbf{A}^{-1}\boldsymbol{\sigma}:\boldsymbol{\sigma} \geq \mathbf{A}_2^{-1}\boldsymbol{\sigma}:\boldsymbol{\sigma} + \frac{\theta(\kappa_2-\kappa_1)(\kappa_2+\mu_2)(\sigma_1+\sigma_2)^2}{4\kappa_2\left((1-\theta)\kappa_2\mu_2+\kappa_1(\kappa_2+\theta\mu_2)\right)}.$$

C. *If*

$$(1-\theta)\kappa_2(\mu_2-\mu_1)|\sigma_1-\sigma_2| \geq \left(\mu_1(\kappa_2+\mu_2)+(1-\theta)\kappa_2(\mu_2-\mu_1)\right)|\sigma_1+\sigma_2|,$$

then

$$\mathbf{A}^{-1}\boldsymbol{\sigma}:\boldsymbol{\sigma} \geq \mathbf{A}_2^{-1}\boldsymbol{\sigma}:\boldsymbol{\sigma} + \frac{\theta(\mu_2-\mu_1)(\kappa_2+\mu_2)(\sigma_1-\sigma_2)^2}{4\mu_2\left((1-\theta)\kappa_2\mu_2+\mu_1(\mu_2+\theta\kappa_2)\right)}.$$

Theorem (Optimal microstructure)

Let $d = 2$, $\theta \in [0,1]$, $0 < \mu_1 < \mu_2$, $0 < \kappa_1 < \kappa_2$ *and* $\sigma_1,\ \sigma_2$ *eigenvalues of matrix* $\boldsymbol{\sigma}$.

A. *If*

$$
\begin{aligned}
(1-\theta)\mu_2(\kappa_2-\kappa_1)|\sigma_1+\sigma_2| &< \left(\kappa_1(\kappa_2+\mu_2)+(1-\theta)\mu_2(\kappa_2-\kappa_1)\right)|\sigma_1-\sigma_2|,\\
(1-\theta)\kappa_2(\mu_2-\mu_1)|\sigma_1-\sigma_2| &< \left(\mu_1(\kappa_2+\mu_2)+(1-\theta)\kappa_2(\mu_2-\mu_1)\right)|\sigma_1+\sigma_2|,
\end{aligned}
$$

then the optimal microstructure is a simple laminate with layers orthogonal to the eigenvector associated to the smaller eigenvalue by the absolute value of the extremal $\boldsymbol{\eta}$ *in (5).*

B. *If*

$$
(1-\theta)\mu_2(\kappa_2-\kappa_1)|\sigma_1+\sigma_2| \geq \left(\kappa_1(\kappa_2+\mu_2)+(1-\theta)\mu_2(\kappa_2-\kappa_1)\right)(\sigma_2-\sigma_1),
$$

then the optimal microstructure is rank-2 laminate with lamination direction orthogonal to eigenvectors e_1 *and* e_2 *of the extremal* $\boldsymbol{\eta}$ *in (5) and lamination parameters*

$$
\begin{aligned}
m_1 &= \frac{1}{2}+\frac{(\kappa_1(\mu_2+\kappa_2)+(1-\theta)\mu_2(\kappa_2-\kappa_1))(\sigma_2-\sigma_1)}{2(1-\theta)(\kappa_2-\kappa_1)\mu_2(\sigma_1+\sigma_2)}\\
m_2 &= \frac{1}{2}+\frac{(\kappa_1(\mu_2+\kappa_2)+(1-\theta)\mu_2(\kappa_2-\kappa_1))(\sigma_1-\sigma_2)}{2(1-\theta)(\kappa_2-\kappa_1)\mu_2(\sigma_1+\sigma_2)}.
\end{aligned}
$$

C. *If*

$$
(1-\theta)\kappa_2(\mu_2-\mu_1)(\sigma_2-\sigma_1) \geq \left(\mu_1(\kappa_2+\mu_2)+(1-\theta)\kappa_2(\mu_2-\mu_1)\right)|\sigma_1+\sigma_2|,
$$

then the optimal microstructure is rank-2 laminate with lamination direction orhogonal to eigenvectors e_1 *and* e_2 *of the extremal* $\boldsymbol{\eta}$ *in (5) and lamination parameters*

$$
\begin{aligned}
m_1 &= \frac{1}{2}+\frac{(\mu_1(\mu_2+\kappa_2)+(1-\theta)\kappa_2(\mu_2-\mu_1))(\sigma_1+\sigma_2)}{2(1-\theta)(\mu_2-\mu_1)\kappa_2(\sigma_2-\sigma_1)}\\
m_2 &= \frac{1}{2}+\frac{(\mu_1(\mu_2+\kappa_2)+(1-\theta)\kappa_2(\mu_2-\mu_1))(\sigma_1+\sigma_2)}{2(1-\theta)(\mu_2-\mu_1)\kappa_2(\sigma_1-\sigma_2)}.
\end{aligned}
$$

Example 1.

We consider two dimensional problem of compliance minimization

$$J(\theta, \mathbf{A}) = \int_\Omega \mathsf{f} \cdot \mathsf{u} \, d\mathbf{x} \longrightarrow \min,$$

where $\Omega \subseteq \mathbf{R}^2$ is a ball $B(\mathbf{0}, 2)$, $\mu_1 = 63$, $\kappa_1 = 58$, $\mu_2 = 75$, $\kappa_2 = 139$, while u is the state function for

$$\begin{cases} -\mathsf{div}\,(\mathbf{A}e(\mathsf{u})) = \mathsf{f} \\ \mathsf{u} \in \mathrm{H}^1_0(\Omega; \mathbf{R}^d) \,. \end{cases}$$

where we take $\mathsf{f} = 10\mathsf{e}_r$ for the right-hand side.

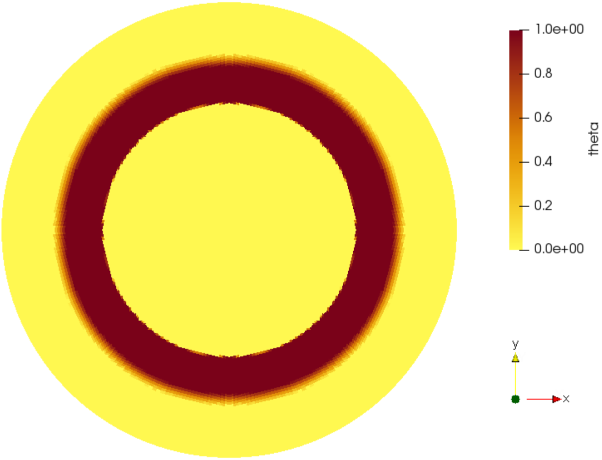
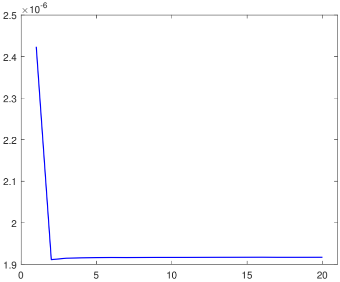
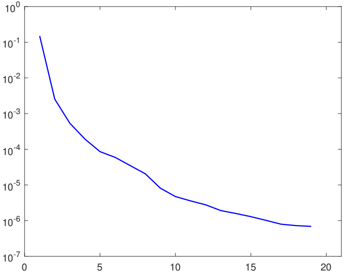


Figure: Optimal distribution of materials in a ball with volume constraint 25% of the first material.



(a) Cost functional J.



(b) $\|\theta^k - \theta^{k+1}\|_{L^2}$ in terms of the iteration number k.

Figure: Convergence history for compliance minimization in a ball.

Example 2.

We consider two dimensional problem of compliance minimization

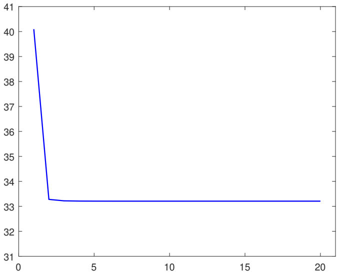
$$J(\theta, \mathbf{A}) = \int_\Omega \mathsf{f} \cdot \mathsf{u} \, d\mathbf{x} + \int_{\Gamma_N} \mathsf{g} \cdot \mathsf{u} \, dS \longrightarrow \min,$$

where $\Omega \subseteq \mathbf{R}^2$ is a rectangle, for u being the state function

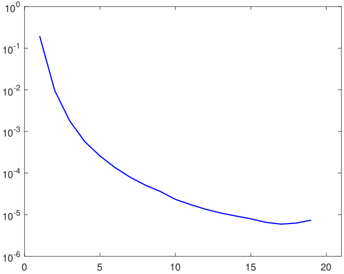
$$\begin{cases} -\mathrm{div}\,(\mathbf{A}e(\mathsf{u})) = \mathsf{f} & \text{in } \Omega \\ \mathsf{u} = \mathbf{0} & \text{on } \Gamma_D \\ \mathbf{A}e(\mathsf{u})\mathsf{n} = \mathsf{g} & \text{on } \Gamma_N, \end{cases}$$

where we take f $= (0, -10)^\top$ for the right-hand side and g $= \mathbf{0}$. The boundary part Γ_D corresponds to $\partial\Omega \cap (B((-2,0), 0.1) \cup B((2,0), 0.1))$, while Γ_N corresponds to the rest of the boundary.

Figure: Optimal distribution of materials in a rectangle with volume constraint 50% of the first material.



(a) Cost functional J.



(b) $\|\theta^k - \theta^{k+1}\|_{L^2}$ in terms of the iteration number k.

Figure: Convergence history for compliance minimization in a rectangle.

Example 3.

We consider two dimensional problem of compliance minimization

$$J(\theta, \mathbf{A}) = \int_\Omega \mathsf{f} \cdot \mathsf{u}\, d\mathbf{x} + \int_{\partial\Omega} \mathsf{g} \cdot \mathsf{u}\, dS \longrightarrow \min,$$

where $\Omega \subseteq \mathbf{R}^2$ is a rectangle, for u being the state function

$$\begin{cases} -\text{div}\,(\mathbf{A}e(\mathsf{u})) = \mathbf{0} & \text{in } \Omega \\ \mathsf{u} = \mathbf{0} & \text{on } \Gamma_D \\ \mathbf{A}e(\mathsf{u})\mathsf{n} = \mathsf{g} & \text{on } \Gamma_N, \end{cases}$$

where we take $\mathsf{g} = (0, -10)^\top$ on $\partial\Omega \cap B((0,1), 0.2)$.

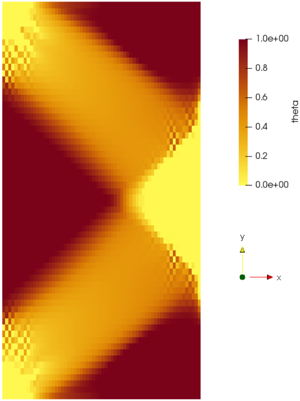
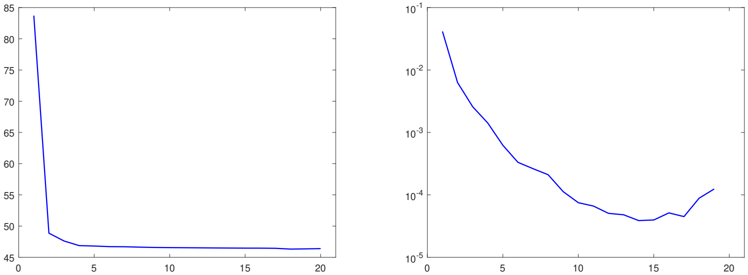


Figure: Optimal distribution of materials in a rectangle with volume constraint 60% of the first material.



(a) Cost functional J.

(b) $\|\theta^k - \theta^{k+1}\|_{L^2}$ in terms of the iteration number k.

Figure: Convergence history for compliance minimization in a rectangle.

Thank you for your attention!