Controllability of the 1D fractional heat equation under positivity constraints

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1-d fractional heat equation

$$
\begin{cases}
z_t + (-d_x^2)^s z = u\chi_\omega & (x,t) \in (-1,1) \times (0,T) \\
z \equiv 0 & (x,t) \in (-1,1)^c \times (0,T), \\
z(\cdot,0) = z_0 & x \in (-1,1).
\end{cases}
\qquad (\mathcal{FH})
$$

- $\omega \subset (-1,1)$
- $z_0 \in L^2(-1,1)$.

We are interested in analyzing controllability properties under positivity state/control constraints.

The fractional Laplacian

Fractional Laplacian

$$(-d_x^2)^s z = C_s \, \mathrm{P.V.} \int_{\mathbb{R}} \frac{z(x) - z(y)}{|x-y|^{1+2s}} \, dy.$$

$$C_s = \left(\int_{\mathbb{R}} \frac{1 - \cos(\zeta_1)}{|\zeta|^{1+2s}} \, d\zeta \right)^{-1} = \frac{s 2^{2s} \Gamma\left(\frac{1+2s}{2}\right)}{\pi^{\frac{1}{2}} \Gamma(1-s)}.$$

Fractional Sobolev space

$$H^s(-1,1) := \left\{ z \in L^2(-1,1) : \int_{-1}^{1}\int_{-1}^{1} \frac{|z(x)-z(y)|^2}{|x-y|^{1+2s}} dxdy < \infty \right\}$$

- Intermediate **Hilbert** space between $L^2(-1,1)$ and $H^1(-1,1)$.

Scalar product

$$\langle z, w\rangle_{H^s(-1,1)} := \int_{-1}^{1} zw\, dx + \int_{-1}^{1}\int_{-1}^{1} \frac{(z(x)-z(y))(w(x)-w(y))}{|x-y|^{1+2s}}\, dxdy$$

Fractional Sobolev norm

$$\|z\|_{H^s(-1,1)} := \left(\int_{-1}^{1} |z|^2\, dx + \int_{-1}^{1}\int_{-1}^{1} \frac{|z(x)-z(y)|^2}{|x-y|^{1+2s}}\, dxdy \right)^{\frac{1}{2}}$$

Known controllability results

Theorem (B. and Hernández-Santamaría, IMA J. Math. Control Inf., 2018)

There exists $u \in L^2(\omega \times (0,T))$ such that the fractional heat equation $(\mathcal{FH})$ is

- ***null-controllable*** *at time $T > 0$ if and only if $s > 1/2$.*
- ***approximately controllable*** *at time $T > 0$ for all $s \in (0,1)$.*

Remark: the equation being linear, by translation if $s > 1/2$ we have controllability to trajectories $\widehat{z}$.

Proof

Null controllability: through the moment method, based on the following behavior of the spectrum.

Eigenvalues (Kwaśnicki, J. Funct. Anal., 2012)

$$\lambda_k = \left(\frac{k\pi}{2} - \frac{(1-s)\pi}{4}\right)^{2s} + O\left(\frac{1}{k}\right).$$

Approximate controllability: The result follows from the following property.

Parabolic unique continuation

Given $s \in (0,1)$ and $p^T \in L^2(-1,1)$, let p be the unique solution to the adjoint equation. Let $\omega \subset (-1,1)$ be an arbitrary open set. If $p = 0$ on $\omega \times (0,T)$, then $p = 0$ on $(-1,1) \times (0,T)$.

This, in turn, is a consequence of the **unique continuation** property for the Fractional Laplacian.

Fall and Felli, Comm. Partial Differential Equations, 2014.

- The fractional heat equation **preserves positivity**: if z_0 is a given non-negative initial datum in $L^2(-1,1)$ and u is a non-negative function, then so it is for the solution z of $(\mathcal{FH})$.

  B., Warma and Zuazua, 2019

Question

Can we control the fractional heat dynamics $(\mathcal{FH})$ from any initial datum $z_0 \in L^2(-1,1)$ to any positive trajectory $\widehat{z}$, under positivity constraints on the control and/or the state?

Theorem (B., Warma and Zuazua, 2019)

Let $s > 1/2$, $z_0 \in L^2(-1,1)$ and let $\widehat{z}$ be a positive trajectory, i.e., a solution of $(\mathcal{FH})$ with initial datum $0 < \widehat{z}_0 \in L^2(-1,1)$ and right hand side $\widehat{u} \in L^\infty(\omega \times (0,T))$. Assume that there exists $\nu > 0$ such that $\widehat{u} \geq \nu$ a.e in $\omega \times (0,T)$. Then, the following assertions hold.

1. *There exist $T > 0$ and a non-negative control $u \in L^\infty(\omega \times (0,T))$ such that the corresponding solution z of $(\mathcal{FH})$ satisfies $z(x,T) = \widehat{z}(x,T)$ a.e. in $(-1,1)$. Moreover, if $z_0 \geq 0$, we also have $z(x,t) \geq 0$ for every $(x,t) \in (-1,1) \times (0,T)$.*

2. *Define the minimal controllability time by*
$$T_{min}(z_0,\widehat{z}) := \inf \Big\{ T > 0 : \exists \ 0 \leq u \in L^\infty(\omega \times (0,T)) \text{ s. t. } z(\cdot,0) = z_0 \text{ and } z(\cdot,T) = \widehat{z}(\cdot,T) \Big\}.$$
Then, $T_{min} > 0$.

3. *For $T = T_{min}$, there exists a non-negative control $u \in \mathcal{M}(\omega \times (0,T_{min}))$, the space of Radon measures on $\omega \times (0,T_{min})$, such that the corresponding solution of $(\mathcal{FH})$ satisfies $z(x,T) = \widehat{z}(x,T)$ a.e. in $(-1,1)$.*

Proof - main ingredients

The proof requires two main ingredients:

1. Controllability through L^∞ **controls**, consequence of the following observability inequality

$$\boxed{\|p(\cdot,0)\|^2_{L^2(-1,1)} \leq C\left(\int_0^T \int_\omega |p(x,t)|\, dxdt\right)^2}$$

2. **Dissipativity** of the fractional heat semi-group.

Some preliminary results

Theorem

Let $\{\mu_k\}_{k\geq 1}$ be a sequence of real numbers satisfying the conditions:

1. *There exists $\gamma > 0$ such that $\mu_{k+1} - \mu_k \geq \gamma$ for all $k \geq 1$.*
2. $\sum_{k\geq 1} \mu_k^{-1} < +\infty$.

Then, for any $T > 0$, there is a positive constant $\mathcal{C} = \mathcal{C}(T) > 0$ such that, for any finite sequence $\{c_k\}_{k\geq 1}$ it holds the inequality

$$\sum_{k\geq 1} |c_k|^2 e^{-2\mu_k T} \leq C \left\| \sum_{k\geq 1} c_k e^{-\mu_k t} \right\|^2_{L^1(0,T)} .$$

PROOF:

Under the above hypothesis on $\{\mu_k\}_{k\geq 1}$, the function $F(t) := \sum_{k\geq 1} c_k e^{-\mu_k t}$ satisfies

$$|c_k| \leq C \|F\|_{L^1(0,T)}, \quad \sum_{k\geq 1} |c_k| e^{(\mu_1 - \mu_k)t} \leq C(t) \|F\|_{L^1(0,T)}, \quad C(t) \text{ **uniformly bounded** } \forall t > 0.$$

Then

$$\sum_{k\geq 1} |c_k|^2 e^{-2\mu_k T} = \sum_{k\geq 1} |c_k| e^{(\mu_1 - \mu_k)t} \left(|c_k| e^{(\mu_k - \mu_1)t} e^{-2\mu_k T} \right) \leq C \|F\|^2_{L^1(0,T)} .$$

Some preliminary results

Lemma

Consider the eigenvalue problem for the Dirichlet fractional Laplacian in $(-1,1)$*:*

$$\begin{cases} (-d_x^2)^s \phi_k = \lambda_k \phi_k, & x \in (-1,1) \\ \phi_k = 0, & x \in (-1,1)^c. \end{cases}$$

Then for any open subset $\omega \subset (-1,1)$*, there is a positive constant* $\beta > 0$ *such that* $\|\phi_k\|_{L^1(\omega)} \geq \beta > 0$*.*

PROOF (main idea):

The proof is based on the fact that

$$\int_\omega |\phi_k(x)|\,dx \geq \int_\omega \left| \sin\left(\mu_k(1+x) + \frac{(1-s)\pi}{4}\right) - \frac{c(1-s)}{\sqrt{s}}\mu_k^{-1-2s} \right| dx$$

$$\mu_k := \frac{k\pi}{2} - \frac{(1-s)\pi}{4}$$

L^1 observability

Proposition

For any $T > 0$ and $p_T \in L^2(-1,1)$, let $p \in L^2((0,T); H_0^s(-1,1)) \cap C([0,T]; L^2(-1,1))$ with $p_t \in L^2((0,T); H^{-s}(-1,1))$ be the weak solution of the adjoint system

$$\begin{cases} -p_t + (-d_x^2)^s p = 0, & (x,t) \in (-1,1) \times (0,T) \\ p = 0, & (x,t) \in (-1,1)^c \times (0,T) \\ p(\cdot, T) = p_T(\cdot), & x \in (-1,1). \end{cases}$$

Then, for any $s > 1/2$, there is a constant $C = C(T) > 0$ such that

$$\|p(\cdot,0)\|^2_{L^2(-1,1)} \leq C \left(\int_0^T \int_\omega |p(x,t)|\, dxdt \right)^2.$$

L^∞ controllability

Theorem

For any $z_0 \in L^2(-1,1)$, $s > 1/2$ and $T > 0$, there exists a control function $u \in L^\infty(\omega \times (0,T))$ such that the corresponding unique weak solution z of $(\mathcal{FH})$ with initial datum $z(x,0) = z_0(x)$ satisfies $z(x,T) = 0$ a.e. in $(-1,1)$. Moreover, there is a constant $C > 0$ (depending only on T) such that

$$\|u\|_{L^\infty(\omega\times(0,T))} \leq C\,\|z_0\|_{L^2(-1,1)}\,.$$

PROOF:

Classical duality argument.

Proof of the main result - 1: constrained controllability

- Subtracting $\widehat{z}$ in the equation, it is enough to show that there exist a time $T > 0$ and a control $v \in L^\infty(\omega \times (0,T))$, $v > -\nu$ a.e. in $\omega \times (0,T)$, such that

$$\begin{cases} \xi_t + (-d_x^2)^s \xi = v\chi_\omega, & (x,t) \in (-1,1) \times (0,T) \\ \xi = 0, & (x,t) \in (-1,1)^c \times (0,T) \\ \xi(\cdot,0) = z_0(\cdot) - \widehat{z}_0(\cdot), & x \in (-1,1) \end{cases} \quad \Rightarrow \quad \xi(x,T) = 0.$$

- This is equivalent to the observability inequality

$$\|p(\cdot,\tau)\|^2_{L^2(-1,1)} \leq C(T-\tau)\left(\int_\tau^T \int_\omega |p(x,t)|\, dxdt\right)^2.$$

- Using that the eigenvalues $\{\lambda_k\}_{k\geq 1}$ form a non-decreasing sequence, and the **dissipativity** of the fractional heat semi-group:

$$\begin{aligned} \|p(\cdot,0)\|^2_{L^2(-1,1)} &\leq e^{-2\lambda_1\tau}\, \|p(\cdot,\tau)\|^2_{L^2(-1,1)} \\ &\leq e^{-2\lambda_1\tau} C(T-\tau)\left(\int_0^T \int_\omega |p(x,t)|\, dxdt\right)^2. \end{aligned}$$

Proof of the main result - 1: constrained controllability

- By duality, the control v can be chosen such that

$$\|v\|^2_{L^\infty(\omega\times(0,T))} \leq e^{-2\lambda_1\tau}C(T-\tau)\left\|z_0-\widehat{z}_0\right\|^2_{L^2(-1,1)}.$$

- Taking $\tau = T/2$, we obtain

$$\|v\|^2_{L^\infty(\omega\times(0,T))} \leq e^{-\lambda_1 T}C(T)\left\|z_0-\widehat{z}_0\right\|^2_{L^2(-1,1)}.$$

- The observability constant $\mathcal{C}(T)$ is **uniformly bounded** for any $T>0$. Hence, for T large enough we have

$$\|v\|^2_{L^\infty(\omega\times(0,T))} < \nu.$$

- This implies that $v > -\nu$. Therefore, the control $v > -\nu$ steers ξ from $z_0 - \widehat{z}_0$ to zero in time $T>0$, provided T is large enough. Consequently, z is controllable to the trajectory $\widehat{z}$ in time T.
- If $z_0 \geq 0$, thanks to the **maximum principle**, we also have $z(x,t) \geq 0$ for every $(x,t) \in (-1,1)\times(0,T)$.

Proof of the main result - 2: positivity of T_{min}

- Solution of $(\mathcal{FH})$ in the basis of the eigenfunctions $\{\phi_k\}_{k\geq 1}$:

$$z(x,t) = \sum_{k\geq 1} z_k(t)\phi_k(x).$$

$$z_k(t) = z_k^0 e^{-\lambda_k t} + \int_0^t e^{-\lambda_k(t-\tau)} u_k(\tau)\, d\tau, \quad u_k(t) := \int_\omega u(x,t)\phi_k(x)\, dx.$$

- $z(x,T) = \widehat{z}(x,T)$ a.e. in $(-1,1)$:

$$z_k(T) = \int_{-1}^1 \widehat{z}(x,T)\phi_k(x)\, dx =: \zeta_k \quad \Rightarrow \quad \zeta_k - z_k^0 e^{-\lambda_k T} = \int_0^T e^{-\lambda_k(T-\tau)} u_k(\tau)\, d\tau.$$

- For every $0 \leq \tau \leq T$:

$$\zeta_k - z_k^0 e^{-\lambda_k T} \leq \int_0^T u_k(\tau)\, d\tau \leq \zeta_k e^{\lambda_k T} - z_k^0.$$

- Assume by contradiction that, for every $T > 0$, there exists a non-negative control function u^T steering z_0 to $\widehat{z}(\cdot, T)$ in time T, and that $\widehat{z}(\cdot, T) \neq z_0$. Then:

$$\lim_{T\to 0^+} \int_0^T u_k^T(\tau)\, d\tau = \zeta_k - z_k^0 =: \gamma \quad \Longrightarrow \quad z_k^0 = \zeta_k - \gamma.$$

Proof of the main result - 2: positivity of T_{min}

- $z_0 \in L^2(-1,1)$:

$$\sum_{k\geq 1} |z_k^0|^2 = \sum_{k\geq 1} \left(\zeta_k^2 - 2\gamma\zeta_k + \gamma^2\right) < +\infty \quad \Rightarrow \quad \lim_{k\to+\infty} \left(\zeta_k^2 - 2\gamma\zeta_k + \gamma^2\right) = 0.$$

- Since $\{\phi_k\}_{k\geq 1}$ is an orthonormal complete system in $L^2(-1,1)$, $\phi_k \rightharpoonup 0$ in $L^2(-1,1)$ as $k \to +\infty$. Hence:

$$\begin{aligned} \lim_{k\to+\infty} (\widehat{z}(\cdot,T), \phi_k)_{L^2(-1,1)} &= \lim_{k\to+\infty} \int_{-1}^{1} \widehat{z}(x,T)\phi_k(x)\, dx \\ &= \lim_{k\to+\infty} \zeta_k = 0 \quad \Rightarrow \quad \gamma = 0. \end{aligned}$$

  Consequently

$$0 = z_k^0 - \zeta_k = \int_{-1}^{1} (z_0(x) - \widehat{z}(x,T))\phi_k(x)\, dx, \quad \textit{for all} \;\; k \geq 1.$$

- This is possible if and only if $z_0(x) = \widehat{z}(x,T)$ a.e. in $(-1,1)$, which contradicts our previous assumption.

Proof of the main result - 3: minimal-time control

Constrained controllability of the system $(\mathcal{FH})$ holds in the minimal time T_{min} with controls in the (Banach) space of the Radon measures $\mathcal{M}(\omega \times (0, T_{min}))$ endowed with the norm

$$\|\mu\|_{\mathcal{M}(\omega\times(0,T_{min}))} = \sup \left\{ \int_{\omega\times(0,T_{min})} \varphi(x,t)\, d\mu(x,t) : \varphi \in C(\overline{\omega} \times [0, T_{min}], \mathbb{R}),\ \max_{\overline{\omega}\times[0,T_{min}]} |\varphi| = 1 \right\}.$$

Solutions of $(\mathcal{FH})$ with controls in $\mathcal{M}(\omega \times (0, T_{min}))$ are defined by transposition

Transposition solution

Given $z_0 \in L^2(-1,1)$, $T > 0$, and $u \in \mathcal{M}(\omega \times (0,T))$, the function $z \in L^1((-1,1) \times (0,T))$ is a solution of $(\mathcal{FH})$ defined by transposition if

$$\int_{\omega\times(0,T)} p(x,t)\, du(x,t) = \langle z(\cdot, T), p_T \rangle - \int_{-1}^{1} z_0(x) p(x,0)\, dx,$$

where, for every $p_T \in L^\infty(-1,1)$, the function $p \in L^\infty(Q)$ is the unique solution of

$$\begin{cases} -p_t + (-d_x^2)^s p = 0, & (x,t) \in (-1,1) \times (0,T) \\ p = 0, & (x,t) \in (-1,1)^c \times (0,T) \\ p(\cdot, T) = p_T, & x \in (-1,1). \end{cases}$$

Proof of the main result - 3: minimal-time control

- Denote $T_k := T_{min} + \frac{1}{k}$, $k \geq 1$.

  There exists a sequence of non-negative controls $\{u^{T_k}\}_{k\geq 1} \subset L^\infty(\omega \times (0, T_k))$ such that the corresponding solution z^k of $(\mathcal{FH})$ with $z^k(x,0) = z_0(x)$ a.e. in $(-1,1)$ satisfies $z^k(x, T_k) = \widehat{z}(x, T_k)$ a.e. in $(-1,1)$.
- Extend these controls by $\widehat{u}$ on $(T_k, T_{min} + 1)$ to get a new sequence in $L^\infty(\omega \times (0, T_{min+1}))$.
- $p_T > 0 \;\Rightarrow\; p(x,t) \geq \theta > 0$ for all $(x,t) \in (-1,1) \times (0, T_{min} + 1)$. Then,

$$\begin{aligned}
\theta \left\| u^{T_k} \right\|_{L^1(\omega\times(0,T_{min}+1))} &= \theta \int_0^{T_{min}+1} \int_\omega u^{T_k}(x,t)\,dxdt \\
&\leq \int_0^{T_{min}+1} \int_{-1}^{1} p(x,t) u^{T_k}(x,t)\,dxdt \\
&= \langle z(\cdot, T), p_T \rangle - \int_{-1}^{1} z_0(x) p(x,0)\,dx \leq M.
\end{aligned}$$

- $\{u^{T_k}\}_{k\geq 1}$ is **bounded** in $L^1(\omega \times (0, T_{min+1}))$, hence, it is bounded in the space $\mathcal{M}(\omega \times (0, T_{min+1}))$. Thus, extracting a sub-sequence, we have:

$$u^{T_k} \overset{*}{\rightharpoonup} \widetilde{u} \quad weakly -* \; in\; \mathcal{M}(\omega \times (0, T_{min+1}))\; as\; k \to +\infty.$$

  The limit control $\widetilde{u}$ satisfies the non-negativity constraint.

Proof of the main result - 3: minimal-time control (cont.)

- For any k large enough and $T_{min} < T_0 < T_{min+1}$, we have

$$\int_{\omega\times(0,T_0)} p(x,t)\, du^{T_k}(x,t) = \langle \widehat{z}(\cdot,T_0), p_T\rangle - \int_{-1}^{1} z_0(x)p(x,0)\, dx.$$

- p_T: first **non-negative** eigenfunction of $(-d_x^2)^s$

$$p \in C([0,T]; D((-d_x^2)^s)) \hookrightarrow C([0,T]\times[-1,1]).$$

- By definition of *weak** limit, letting $k \to +\infty$, we obtain

$$\int_{\omega\times(0,T_0)} p(x,t)\, d\,\widetilde{u}(x,t) = \langle \widehat{z}(\cdot,T_0), p_T\rangle - \int_{-1}^{1} z_0(x)p(x,0)\, dx,$$

which implies that $z(x,T_0) = \widehat{z}(x,T_0)$ a.e. in $(-1,1)$.

- Taking the limit as $T_0 \to T_{min}$ and using the fact that

$$|\widetilde{u}|(\omega\times(T_{min},T_0)) = |\hat{u}|(\omega\times(T_{min},T_0)) = 0, \quad as\ T_0 \to T_{min}$$

we deduce that $z(x,T_{min}) = \widehat{z}(x,T_{min})$ a.e. in $(-1,1)$.

Numerical simulations

- We consider the problem of steering the initial datum $z_0(x) = \frac{1}{2}\cos\left(\frac{\pi}{2}x\right)$ to the target trajectory $\widehat{z}$ solution of $\mathcal{FH}$ with initial datum $\widehat{z}_0(x) = 6\cos\left(\frac{\pi}{2}x\right)$ and right-hand side $\widehat{u} \equiv 1$.
- We choose $s = 0.8$ and $\omega = (-0.3, 0.8) \subset (-1, 1)$ as the control region.
- The approximation of the minimal controllability time is obtained by solving the following constrained minimization problem:

minimize T

$$
\begin{cases}
T > 0 \\
z_t + (-d_x^2)^s z = u\chi_\omega, & a.\,e.\ in\ (-1,1) \times (0,T) \\
z(\cdot,0) = z_0 \geq 0, & a.\,e.\ in\ (-1,1) \\
z \geq 0, & a.\,e.\ in\ (-1,1) \times (0,T) \\
u \geq 0, & a.\,e.\ in\ \omega \times (0,T).
\end{cases}
$$

Numerical simulations

To perform the simulations, we apply a FE method for the space discretization of the fractional Laplacian on a uniform space-grid

$$x_i = -1 + \frac{2i}{N_x}, \quad i = 1, \ldots, N_x,$$

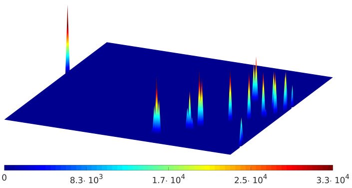
with $N_x = 20$. Moreover, we use an explicit Euler scheme for the time integration on the time-grid

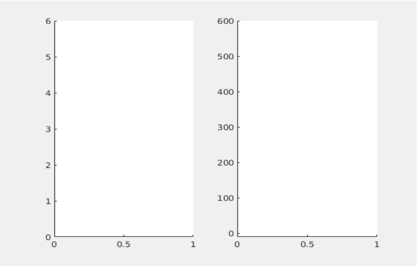
$$t_j = \frac{Tj}{N_t}, \quad j = 0, \ldots, N_t,$$

with N_t satisfying the **Courant-Friedrich-Lewy** condition. In particular, we choose here $N_t = 100$.

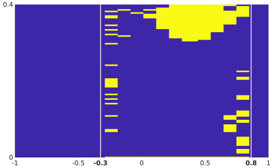
Numerical simulations

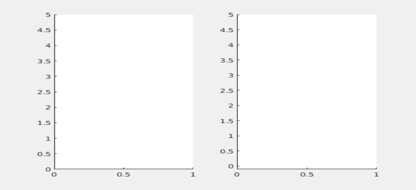
- We obtain the minimal time $T_{min} \simeq 0,2101$.
- In this time horizon, the fractional heat equation $\mathcal{FH}$ is controllable from the initial datum z_0 to the desired trajectory $\widehat{z}(\cdot, T)$ by maintaining the positivity of the solution.





- The impulsional behavior of the control is lost when extending the time horizon beyond T_{min}.



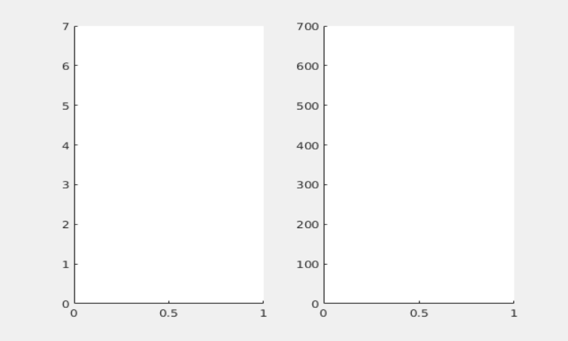


This control has been computed by solving the minimization problem:

$$\min \ \|z(\cdot,T)-\widehat{z}(\cdot,T)\|_{L^2(-1,1)}$$

$$\begin{cases} T>0 \\ z_t+(-d_x^2)^s z = u\chi_\omega, & a.\,e.\ in\ (-1,1)\times(0,T) \\ z(\cdot,0)=z_0\geq 0, & a.\,e.\ in\ (-1,1) \\ z\geq 0, & a.\,e.\ in\ (-1,1)\times(0,T) \\ u\geq 0, & a.\,e.\ in\ \omega\times(0,T). \end{cases}$$

- Finally, when considering a time horizon $T < T_{min}$, constrained controllability fails.



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