SOME CONTROLLABILITY RESULTS FOR TIME-DISCRETE PARABOLIC SYSTEMS

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Preliminaries

The controlled heat equation

$\Omega \subset \mathbb{R}^N, \quad T > 0, \quad \omega \subset \Omega.$

$$\begin{cases} y_t - \Delta y = v\chi_\omega & \text{in } Q = \Omega \times (0,T) \\ y = 0 & \text{on } \partial\Omega \times (0,T) \\ y(x,0) = y^0(x) & \text{in } \Omega \end{cases} \tag{1}$$

$v = v(x,t)$ is a control function

Null controllability

System (1) is said to be null-controllable at time T if, for any $y_0 \in L^2(\Omega)$, there exists a control $v \in L^2(\omega \times (0,T))$ such that the corresponding solution satisfies

$$y(\cdot, T) = 0 \quad \text{in } \Omega.$$

The observability inequality

Consider the adjoint equation

$$\begin{cases} -q_t - \Delta q = 0 & \text{in } Q = \Omega \times (0,T) \\ q = 0 & \text{on } \partial\Omega \times (0,T) \\ q(x,T) = q^T(x) & \text{in } \Omega \end{cases} \tag{2}$$

Then, (1) is null-controllable if and only if there exists $C_{obs} > 0$ such that the following observability inequality holds

$$|q(0)|_{L^2(\Omega)} \leq C_{obs} \left(\iint_{\omega \times (0,T)} |q|^2 dxdt \right)^{\frac{1}{2}}, \quad \forall q_T \in L^2(\Omega).$$

In 1996, Fursikov & Imanuvilov used global Carleman estimates which readily yield the observability inequality.

Time-discrete setting

Discrete framework

For any given $M \in \mathbb{N}^*$, we set $\triangle t = T/M$ and introduce the following discretization for the time variable

$$0 = t_0 < t_1 < \ldots < t_M = T,$$

with $t_n = n\triangle t$ and $n \in [\![0, M]\!]$. We also introduce $t_{n+\frac{1}{2}} = (t_{n+1} + t_n)/2$, for $n \in [\![0, M]\!]$

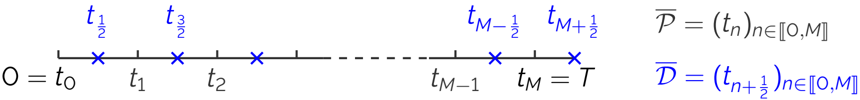

Figure: Discretization of the time variable and its notation.

Time-discrete heat equation

For any time discrete control sequence $v = \{v^{n+\frac{1}{2}}\}_{n\in[\![0,M-1]\!]} \subset L^2(\Omega)$, consider the sequence $y = \{y^n\}_{n\in[\![0,M]\!]} \subset L^2(\Omega)$ verifying

$$\begin{cases} \dfrac{y^{n+1} - y^n}{\triangle t} - \Delta y^{n+1} = \chi_\omega v^{n+\frac{1}{2}}, & n \in [\![0, M-1]\!], \\ y^{n+1}_{|\partial\Omega} = 0, & n \in [\![0, M-1]\!], \\ y^0 = y_0. \end{cases} \tag{3}$$

where y^n (resp. $v^{n+\frac{1}{2}}$) denotes an approximation of y (resp. v) at time t_n (resp. $t_{n+\frac{1}{2}}$).

- System (3) is precisely an Implicit Euler discretization of the heat equation.
- For fixed $\triangle t$ and each n, this system can be regarded as a system of controlled elliptic equations.

What about null controllability...

As in the continuous case, we can formulate the notion of null-controllability: is it true that for any $y_0 \in L^2(\Omega)$ there exists a control sequence $\{v^{n+\frac{1}{2}}\}_{n\in[\![0,M-1]\!]}$ such that the corresponding solution of the time-discrete heat equation satisfies

$$y^M = 0 \quad ?$$

What about null controllability...

As in the continuous case, we can formulate the notion of null-controllability: is it true that for any $y_0 \in L^2(\Omega)$ there exists a control sequence $\{v^{n+\frac{1}{2}}\}_{n \in [\![0, M-1]\!]}$ such that the corresponding solution of the time-discrete heat equation satisfies

$$y^M = 0 \quad ?$$

Theorem. (C. Zheng, 08)

Assume that $\omega \subset\subset \Omega$. For any given $\triangle t > 0$, the time-discrete heat equation is neither null or approximately controllable.

The observability inequality fails

- The (time-discrete) observability inequality does not hold, except for the trivial case $\omega = \Omega$, this is, having an observability inequality like

$$|q^{\frac{1}{2}}|_{L^2(\Omega)} \leq C \left(\sum_{n=0}^{M-1} \triangle t \int_{\omega} |q^{n+\frac{1}{2}}|^2 \right)^{1/2}$$

where $q = \{q^{n+\frac{1}{2}}\}_{n \in [\![0,M]\!]}$ solves the adjoint system

$$\begin{cases} \dfrac{q^{n-\frac{1}{2}} - q^{n+\frac{1}{2}}}{\triangle t} - \Delta q^{n-\frac{1}{2}} = 0, & n \in [\![1, M]\!], \\ q^{n-\frac{1}{2}}_{|\partial\Omega} = 0, & n \in [\![1, M]\!], \\ q^{M+\frac{1}{2}} = q_T. \end{cases}$$

is, in general, FALSE!

Relaxation of the problem and previous results

- C. Zheng ('08) relaxed the null-controllability by considering the projection of the solution over a class of low frequency Fourier components. More precisely, consider

$$\mathcal{C}_K = \text{span}\,\{\phi_k \text{ associated to } \lambda_k \text{ s.t. } \lambda_k \leq K(\triangle t)^{-r}\}$$

  for any fixed $r \in (0, 2)$, some positive constant $K = K(r, T, \Omega, \omega)$ and where (ϕ_k, λ_k) are the eigenfunctions and eigenvalues of the Dirichlet Laplacian.
  Then, there exists a control $v = \{v^{n+\frac{1}{2}}\}_{n \in [\![0,M]\!]}$ (uniformly bounded w.r.t. $\triangle t$) such that

$$\boxed{\Pi_{\mathcal{C}_K} y^M = 0.}$$

Relaxation of the problem and previous results (cont.)

- Later, Ervedoza & Valein (2010) proved that any controllable parabolic equation is also controllable after discretization in time by an appropriate filtering of high frequencies. In fact, they proved an observability inequality of the form

$$|q^{\frac{1}{2}}|^2_{L^2(\Omega)} \leq C_1 \sum_{n=0}^{M-1} \triangle t \int_\omega |q^{n+\frac{1}{2}}|^2 + C_2(\triangle t)^\beta |q^T|_{L^2(\Omega)},$$

where $C_1, C_2 > 0$ are uniform w.r.t to $\triangle t$ and $\beta > 0$ is a fixed constant. This inequality in turn implies that

$$|y^M|_{L^2(\Omega)} \leq \sqrt{C_2}(\triangle t)^{\beta/2} |y_0|_{L^2(\Omega)}.$$

- Some other (similar) results for wave-like, KdV, and Schrödinger equations in Ervedoza, Cheng & Zuazua '08, D. Xu '19 and Zhang, Zheng and Zuazua '09.

Old vs. New

The results from Cheng and Ervedoza & Valein rely (heavily) on spectral analysis techniques. This fact allow us only to consider linear systems and time-independent coefficients.

In Ervedoza & Valein, they assume that the system under study is controllable at the continuous level and are devoted to estimate the "difference" between continuous and discrete case.

Our approach

To derive a Carleman estimate for the time-discrete parabolic operator. This will enable us to study the controllability of more general kind of systems and problems.

Carleman estimates

A Carleman inequality in the continuous framework

A Carleman estimate is a weighted energy estimate of the form

$$\iint_Q e^{2\tau\theta\varphi}\tau\theta|\nabla q|^2 + \iint_Q e^{2\tau\theta\varphi}(\tau\theta)^3|q|^2 \leq C\left(\iint_Q e^{2\tau\theta\varphi}|F|^2 + \iint_{\omega\times(0,T)} e^{2\tau\theta\varphi}(\tau\theta)^3|q|^2\right)$$

for the solutions to

$$\begin{cases} -q_t - \Delta q = F(x,t), & \text{in } Q \\ q = 0 & \text{on } \partial\Omega\times(0,T) \\ q(x,T) = q_T(x) \quad \text{in } \Omega. & \end{cases} \tag{4}$$

The weight $e^{2\tau\theta\varphi}$ is a function composed by: a parameter $\tau > 0$ and

- an x-dependent function

$$\varphi(x) = e^{\lambda\psi(x)} - e^{\lambda K} < 0, \quad K > \|\psi\|_{C(\bar{\Omega})}, \quad \lambda > 0$$

- a time-dependent function

$$\theta(t) = \frac{1}{t(T-t)}$$

A Carleman inequality in the continuous framework

Theorem. (Fursikov & Imanuvilov '96)

For $\lambda \geq 1$ sufficiently large, there exists $C > 0$ and $\tau_0 \geq 1$ depending on Ω, ω and λ such that

$$\iint_Q e^{2\tau\theta\varphi}\tau\theta|\nabla q|^2 + \iint_Q e^{2\tau\theta\varphi}(\tau\theta)^3|q|^2 \leq C\left(\iint_Q e^{2\tau\theta\varphi}|F|^2 + \iint_{\omega\times(0,T)} e^{2\tau\theta\varphi}(\tau\theta)^3|q|^2\right)$$

for all $\tau \geq \tau_0(T+T^2)$ and all solutions q to the equation (4).

Proof.

The proof relies on:

- A suitable change of variables.
- Identifying some dominant terms.
- A LOT of integration by parts.

The change of variables

Continuous case

The change of variable is

$$z(x,t) = e^{\tau\theta(t)\varphi(x)} q(x,t), \quad \text{where} \quad \theta(t) = \frac{1}{t(T-t)}$$

The starting point is to obtain the equation satisfied by

$$\begin{cases} e^{\tau\theta\varphi}\left(\partial_t(e^{-\tau\theta\varphi}z) + \Delta(e^{-\tau\theta\varphi}z)\right) = -e^{\tau\theta\varphi}F, \\ z(\cdot,0) = z(\cdot,T) = 0. \end{cases}$$

Then, after a long procedure that involves integration by parts several times in time and space, we can obtain the desired inequality.

Our main goal is to obtain a time-discrete Carleman *a la* Fursikov & Imanuvilov.

We will try to follow their strategy as close as possible. Obviously, time-discretization introduces additional difficulties that need to be taken into account.

Here, we will only focus on the selection of the Carleman weight and the change of variable.

Some useful definitions and tools

For functions $u = \{u^n\}_{n\in[\![0,M]\!]}$ and $v = \{v^{n+\frac{1}{2}}\}_{n[\![0,M]\!]}$ we may define

- Discrete integrals

$$\int_0^T u := \sum_{n=1}^{M} \triangle t\, u^n \qquad \text{and} \qquad \int\!\!\!\!-_0^T v := \sum_{n=0}^{M-1} \triangle t\, v^{n+\frac{1}{2}}.$$

- Time-discrete derivatives

$$(D_t u)^{n+\frac{1}{2}} := \frac{u^{n+1} - u^n}{\triangle t} \qquad \text{and} \qquad (\overline{D}_t v)^n := \frac{v^{n+\frac{1}{2}} - v^{n-\frac{1}{2}}}{\triangle t}$$

- Time-discrete integration by parts

$$\int\!\!\!\!-_0^T \int_\Omega (D_t u) v = -(u^0, v^{\frac{1}{2}})_{L^2(\Omega)} + (u^M, v^{M+\frac{1}{2}})_{L^2(\Omega)} - \int_0^T \int_\Omega (\overline{D}_t v) u$$

The change of variables in the discrete case

Discrete case

In the discrete case, we cannot exploit the fact that the function θ blows up as $t \to 0$ and $t \to T$. We need to change to

$$\theta(t) = \frac{1}{(t + \delta T)(T + \delta T - t)} \quad \text{with} \quad 0 < \delta < 1/2$$

With this new function, we can propose the change of variable

$$z^{n+\frac{1}{2}} = e^{\tau \theta(t_{n+\frac{1}{2}})\varphi(x)} q^{n+\frac{1}{2}}, \quad n \in [\![0, M]\!]$$

where we recall that $q = \{q^{n+\frac{1}{2}}\}$ solves the equation

$$\begin{cases} \dfrac{q^{n-\frac{1}{2}} - q^{n+\frac{1}{2}}}{\triangle t} - \Delta q^{n-\frac{1}{2}} = F^{n-\frac{1}{2}}, & n \in [\![1, M]\!], \\ q^{n-\frac{1}{2}}_{|\partial\Omega} = 0, & n \in [\![1, M]\!], \\ q^{M+\frac{1}{2}} = q_T. & \end{cases}$$

The change of variables in the discrete case

Following the methodology of the continuous case, we shall look for the equation verified by

$$e^{\tau\theta^{n+\frac{1}{2}}\varphi}\left(\overline{D}_t(e^{-\tau\theta^{n+\frac{1}{2}}\varphi}z^{n+\frac{1}{2}}) + \Delta(e^{-\tau\theta^{n+\frac{1}{2}}\varphi}z^{n+\frac{1}{2}})\right) = -e^{\tau\theta\varphi}F^{n+\frac{1}{2}}$$

Lemma. (Time-discrete derivative of the weight)

Provided $\frac{\triangle t\tau}{(T^3\delta^2)} \leq \kappa$, we have

$$\overline{D}_t e^{\tau\theta\varphi} = \tau\varphi\theta^{n-\frac{1}{2}}e^{\tau\theta^{n-\frac{1}{2}}\varphi} + \triangle t\, e^{\tau\theta^{n-\frac{1}{2}}\varphi}\left(\frac{\tau}{\delta^3T^4} + \frac{\tau^2}{\delta^4T^6}\right)\mathcal{O}_{\lambda,\kappa}(1)$$

A time-discrete Carleman estimate

Theorem. (F. Boyer & V. H.-S. '19)

For $\lambda \geq 1$ sufficiently large, there exists $C > 0$, $\tau_0 \geq 1$, and $\epsilon_0 > 0$ depending on Ω, ω and λ such that

$$\begin{aligned}
&\fint_0^T \int_\Omega e^{2\tau\theta\varphi} \tau\theta |\nabla q|^2 + \fint_0^T \int_\Omega e^{2\tau\theta\varphi} (\tau\theta)^3 |q|^2 \\
&\leq C \left(\fint_0^T \int_\Omega e^{2\tau\theta\varphi} |F|^2 + \fint_0^T \int_\omega e^{2\tau\theta\varphi} (\tau\theta)^3 |q|^2 \right) \\
&\quad + C(\triangle t)^{-1} \left(\int_\Omega \left| (e^{\tau\theta\varphi} q)^{\frac{1}{2}} \right|^2 + \int_\Omega \left| (e^{\tau\theta\varphi} q)^{M+\frac{1}{2}} \right|^2 + \int_\Omega \left| (e^{\tau\theta\varphi} \nabla q)^{M+\frac{1}{2}} \right|^2 \right)
\end{aligned}$$

for all $\tau \geq \tau_0(T + T^2)$, and for all $\triangle t > 0$ and $0 < \delta \leq 1/2$ satisfying

$$\frac{\tau^4 \triangle t}{\delta^4 \min\{T^3, T^6\}} \leq \epsilon_0$$

Controllability results

Heat equation with potential

Consider the system

$$\begin{cases} \dfrac{y^{n+1}-y^n}{\triangle t} - \Delta y^{n+1} + a^{n+1} y^{n+1} = v^{n+\frac{1}{2}}, & n \in [\![0, M-1]\!], \\ y^{n+1}_{|\partial\Omega} = 0, & n \in [\![0, M-1]\!], \\ y^0 = y_0. \end{cases} \tag{5}$$

Theorem. $\phi(\triangle t)$-controllability (F. Boyer & V. H.-S. '19)

Let us consider $T > 0$ and $\triangle t$ sufficiently small. Then, for any $y_0 \in L^2(\Omega)$ and any function ϕ verifying

$$\liminf_{\triangle t \to 0} \frac{\phi(\triangle t)}{e^{-C_2/(\triangle t)^{1/4}}} > 0,$$

there exists a time-discrete control v such that

$$⨍_0^T \int_\omega |v|^2 \leq C |y_0|^2_{L^2(\Omega)},$$

and the associated solution y to (5) verifies

$$|y^M|_{L^2(\Omega)} \leq C\sqrt{\phi(\triangle t)}\,|y_0|_{L^2(\Omega)},$$

where the positive constant C depends only on ϕ, T and $\|a\|_\infty$.

Sketch of the proof

The controllability result is achieved in two steps:

- **First step: controllability in** H^{-1}.
  We choose some $T_0 < T$ and set $M_0 = \left\lfloor \frac{T_0}{\triangle t} \right\rfloor$. Consider the adjoint system

$$\begin{cases} \dfrac{q^{n-\frac{1}{2}} - q^{n+\frac{1}{2}}}{\triangle t} - \Delta q^{n-\frac{1}{2}} + a^n q^{n-\frac{1}{2}} = 0, & n \in [\![1, M_0]\!], \\ q^{n-\frac{1}{2}}_{|\partial\Omega} = 0, & n \in [\![1, M_0]\!], \\ q^{M_0+\frac{1}{2}} = q_T. \end{cases}$$

  Applying our Carleman estimate with $F = -a^n q^{n-\frac{1}{2}}$, we can prove the relaxed observability inequality of the form

$$|q^{\frac{1}{2}}|_{L^2(\Omega)} \leq C_{obs} \left(\fint_0^{T_0} \int_\omega |q|^2 + e^{-\frac{C_2}{(\triangle t)^{1/4}}} |\nabla q_T|^2_{L^2(\Omega)} \right)^{\frac{1}{2}},$$

  With this, we can readily prove a controllability result in H^{-1}, i.e.,

$$|y^{M_0}|^2_{H^{-1}(\Omega)} \leq C_{T_0} e^{-\frac{C_2}{(\triangle t)^{1/4}}} |y_0|^2_{L^2(\Omega)} \quad \text{with} \quad \fint_0^{T_0} \int_\omega |v|^2 \leq C |y_0|^2_{L^2(\Omega)}$$

Sketch of the proof (Cont.)

- **Second step: do nothing!**

  We set $v^{n+\frac{1}{2}} = 0$ for $n \in [\![M_0, M-1]\!]$ and consider the uncontrolled system

$$\begin{cases} \dfrac{y^{n+1} - y^n}{\triangle t} - \Delta y^{n+1} + a^{n+1} y^{n+1} = 0, & n \in [\![M_0, M-1]\!], \\ y^{n+1}_{|\partial\Omega} = 0, & n \in [\![M_0, M-1]\!], \\ y^{M_0+1} = y^{M_0}. \end{cases} \tag{6}$$

from which we can obtain the energy estimate

$$-\sqrt{\triangle t}\, |y^{M_0+1}|_{H_0^1(\Omega)} \leq C |y^{M_0}|_{H^{-1}(\Omega)}.$$

We can iterate for indices $n \in [\![M_0+1, M]\!]$ to deduce that

$$|y^M|^2_{L^2(\Omega)} \leq C \underbrace{e^{-\tilde{C}_2/(\triangle t)^{1/4}}}_{\phi(\triangle t)} |y_0|^2_{L^2(\Omega)},$$

Controllability of a semilinear heat equation

Using the previous result, we can also prove the controllability of the system

$$\begin{cases} \dfrac{y^{n+1} - y^n}{\triangle t} - \Delta y^{n+1} + f(y^{n+1}) = v^{n+\frac{1}{2}} \chi_\omega, & n \in [\![0, M-1]\!], \\ y^{n+1}_{|\partial\Omega} = 0, & n \in [\![0, M-1]\!], \\ y^0 = y_0. \end{cases}$$

where $f \in C^1(\mathbb{R})$ is a globally Lipschitz function with $f(0) = 0$.

The proof is classical and follows well-known results (for instance, C. Fabre, J. P. Puel & E. Zuazua, '95).

Controllability of coupled systems

$$\begin{cases} \dfrac{y_1^{n+1}-y_1^n}{\triangle t} - \Delta y_1^{n+1} + a_{11}^{n+1} y_1^{n+1} + a_{12}^{n+1} y_2^{n+1} = v^{n+\frac{1}{2}} \chi_\omega, & n \in [\![0, M-1]\!] \\ \dfrac{y_2^{n+1}-y_2^n}{\triangle t} - \Delta y_2^{n+1} + a_{21}^{n+1} y_1^{n+1} + a_{22}^{n+1} y_2^{n+1} = 0, & n \in [\![0, M-1]\!] \\ y_1^0 = y_{1,0}, \quad y_2^0 = y_{2,0}, \end{cases}$$

Theorem.

Assume that for $\omega_0 \subset \omega$, the coefficient a_{21} verifies

$$a_{21}^n \geq a_0 > 0 \quad \text{or} \quad -a_{21}^n \geq a_0 > 0 \quad \forall x \in \omega_0,\ n \in [\![1, M]\!].$$

Then, there exists constants C and C_2 such that

$$|q_1^{\frac{1}{2}}|_{L^2(\Omega)}^2 + |q_2^{\frac{1}{2}}|_{L^2(\Omega)}^2 \leq C \left(\fint_0^T \int_\omega |q_1|^2 + e^{-\frac{C_2}{(\triangle t)^{1/5}}} \left[|\nabla q_1^{M+\frac{1}{2}}|_{L^2(\Omega)}^2 + |\nabla q_2^{M+\frac{1}{2}}|_{L^2(\Omega)}^2 \right] \right)$$

for the solutions to the adjoint system

$$\begin{cases} \dfrac{q_1^{n-\frac{1}{2}} - q_1^{n+\frac{1}{2}}}{\triangle t} - \Delta q_1^{n-\frac{1}{2}} + a_{11}^n q_1^{n-\frac{1}{2}} + a_{21}^n q_2^{n-\frac{1}{2}} = 0, & n \in [\![1, M]\!], \\ \dfrac{q_2^{n-\frac{1}{2}} - q_2^{n+\frac{1}{2}}}{\triangle t} - \Delta q_2^{n-\frac{1}{2}} + a_{12}^n q_1^{n-\frac{1}{2}} + a_{22}^n q_2^{n-\frac{1}{2}} = 0, & n \in [\![1, M]\!], \end{cases}$$

Some perspectives

Work in progress and open problems

- Other control problems as **insensitizing controls** can be addressed.
- Fully discrete Carleman estimates ? **Doable**
- Controllability of slightly super linear case ?
- Our approach works for internal control. For boundary controllability of a single equation: **OK**.
- For boundary controllability of coupled systems, we need to change the approach: time-discrete moment method (work in progress).

Thank you for your attention!