On the regularity and the singular support of the minimum time function

for affine-control problems with Hörmander vector fields and smooth manifold targets

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VIII Partial differential equations, optimal design and numerics

Based on joint works with Paolo Albano (University of Bologna) and Piermarco Cannarsa (University of Rome Tor Vergata)

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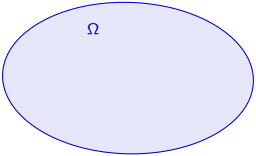
Eikonal equation

Assumption (H)

- $\Omega \subset \mathbb{R}^n$ is a bounded open set with boundary, Γ, given by a C^∞-manifold of dimension $n-1$.
- $\Omega' \supset \Omega$ is an open set of $\mathbb{R}^n$ and $\{X_1, \ldots, X_N\}$ is a system of Hörmander vector fields on Ω', i.e., $Lie(\{X_i\}_{i=1}^N)[x] = \mathbb{R}^n$ for all $x \in \Omega'$ ($N \leq n$).

We consider the Dirichlet problem

$$\begin{cases} \sum_{j=1}^N (X_j T)^2(x) = 1 & \text{in } \Omega, \\ T = 0 & \text{on } \Gamma. \end{cases}$$



We study the regularity and the structure of the singular support of its viscosity solution T.

Hörmander bracket generating condition

- Taking two C^∞ vector fields on Ω',

$$X(x) = \sum_{i=1}^{n} f_i(x)\partial_{x_i}, \quad Y(x) = \sum_{i=1}^{n} g_i(x)\partial_{x_i}, \quad x \in \Omega',$$

  where $f_i, g_i \in C^\infty(\Omega')$, the introduce the Lie bracket

$$[X, Y](x) = \sum_{i=1}^{n} h_i(x)\partial_{x_i} \quad \text{where} \quad h_i = \sum_{j=1}^{n} \left(f_j \partial_{x_j} g_i - g_j \partial_{x_j} f_i \right).$$

- Lie algebra generated by $\{X_i\}_{i=1}^N$: $\mathrm{Lie}(\{X_i\}_{i=1}^N) = \bigcup_{k=1}^\infty \mathrm{Lie}^k(\{X_i\}_{i=1}^N)$, where $\mathrm{Lie}^k(\{X_i\}_{i=1}^N)$ is defined recursively by taking

$$\mathrm{Lie}^1(\{X_i\}_{i=1}^N) = \mathrm{span}\,\{X_i\}_{i=1}^N$$

  and for $k \geq 1$

$$\mathrm{Lie}^{k+1}(\{X_i\}_{i=1}^N) = \mathrm{span}\left(\mathrm{Lie}^k(\{X_i\}_{i=1}^N) \cup \left\{[X, X_j] : X \in \mathrm{Lie}^k(\{X_i\}_{i=1}^N),\, j = 1, \dots, N\right\}\right).$$

- Hörmander bracket condition: $Lie(\{X_i\}_{i=1}^N)[x] = \mathbb{R}^n$ for all $x \in \Omega'$.

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Hörmander bracket generating condition. Cont.

Definition

Degree of nonholonomy at x:

the smallest integer $r = r(x) \geq 1$ such that $Lie^r(\{X_1, \ldots, X_N\}) = \mathbb{R}^n$.

Example (Nonholomonic integrator)

In $\mathbb{R}^3$,

$$f_1(x) = \begin{pmatrix} 1 \\ 0 \\ x_2 \end{pmatrix}, \quad f_2(x) = \begin{pmatrix} 0 \\ 1 \\ -x_1 \end{pmatrix}, \quad [f_1, f_2](x) = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}.$$

Thus, $Lie^2(\{f_1, f_2\})[x] = \text{span}\{f_1, f_2, [f_1, f_2]\}[x] = \mathbb{R}^3$.

MINIMUM TIME PROBLEM

The solution T of

$$\begin{cases} \sum_{j=1}^{N}(X_j T)^2(x) = 1 \quad \text{in} \quad \Omega, \\ T = 0 \quad \text{on} \quad \Gamma \end{cases}$$

can be caracterized as the minimum time function: for $x \in \overline{\Omega}$,

$$T(x) = \min \tau_\Gamma(x,u) \text{ over all controls } u : [0, +\infty[\to \overline{B}_1(0) \subset \mathbb{R}^N$$

where τ_Γ is the *transfer time* to Γ

$$\tau_\Gamma(x,u) = \inf \left\{ t \geq 0 \ : \ y^{x,u}(t) \in \Gamma \right\}$$

and $y^{x,u}(\cdot)$ is the unique solution of the Cauchy problem: for $t \geq 0$,

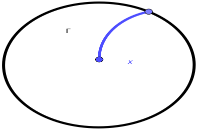
$$y'(t) = \sum_{j=1}^{N} u_j(t) X_j(y(t)), \quad y(0) = x.$$

Sub-Riemannian distance

- $X_1, \ldots, X_N$ are smooth vector fields linearly independent for all $x \in \mathbb{R}^n$.
- $\Delta(x) := \operatorname{span}\{X_1(x), \ldots, X_N(x)\}$ for all $x \in \mathbb{R}^n$.
- Let $g(\cdot,\cdot)$ be a Riemannian metric on $\mathbb{R}^n$, associated with a smooth positive definite symmetric matrix $Q(x)$, that is $g_x(v, w) = \langle Q(x)v, w\rangle$ for all $x \in \mathbb{R}^n$, v and w in $\mathbb{R}^n$.
- (Δ, g): sub-Riemannian distribution of rank $N \leq n$ on $\mathbb{R}^n$.
- Sub-Riemmanian distance

$$d_{SR}(x,y) := \inf\left\{ \int_0^1 g_{\gamma(t)}(\dot\gamma(t), \dot\gamma(t))dt : \ \dot\gamma(t) \in \Delta(\gamma(t)) \text{ a.e. on } [0,1], \right.$$
$$\left. \gamma(0) = x, \gamma(1) = y \right\}.$$

- Let (Δ, g) be a sub-Riemannian structure on $\mathbb{R}^n$, then the topology defined by d_{SR} coincides with the original topology of $\mathbb{R}^n$. In particular, the sub-Riemannian distance d_{SR} is continuous on $\mathbb{R}^n \times \mathbb{R}^n$.



$T(\cdot) = d_{SR}(\Gamma, \cdot)$, Sub-Riemannian distance from Γ.

Viscosity solutions

When T is continuous, it is the viscosity solution of

$$\begin{cases} \sum_{j=1}^{N} (X_j T)^2(x) = 1 & \text{in } \Omega, \\ T = 0 & \text{on } \Gamma. \end{cases}$$

Crandall, Ishii, Lions (1992), User's guide to viscosity solutions of second order partial differential equations.

Known facts

Under the Hörmander condition

- the control problem is locally controllable (Chow-Rashevsky Theorem).
- T is finite and continuous.
- The Dirichlet problem has a unique viscosity solution.

Theorem

T is $1/r_\Omega$- Hölder continuous, where

$$r_\Omega = \max_{x\in\overline{\Omega}} \min\{k \geq 1 : Lie^k(\{X_i\}_{i=1}^N)[x] = \mathbb{R}^n\}.$$

(Nagel-Stein-Wainger 1985, Evans-James 1989).

For instance, $[X_1, [X_1, X_2]]$ has length 3.

NON-DEGENERATE EQUATIONS

- Our eikonal equation can be recast as

$$\langle A(x)DT(x), DT(x)\rangle = 1 \quad \text{in} \quad \Omega,$$

where $A(\cdot)$ is a suitable positive semidefinite $n \times n$ matrix with smooth entries and $DT = (\partial_{x_1} T, \ldots, \partial_{x_n} T)$.
- If for any $x \in \Omega$, $\mathrm{span}\{X_1, \ldots, X_N\}(x) = \mathbb{R}^n$, then the equation is nondegenerate. Any viscosity solution T is locally Lipschitz on $\overline{\Omega}$ and, under mild assumptions ($x \mapsto \langle A(x)p, p\rangle$ semiconvex), T is locally semiconcave on $\overline{\Omega}$.
- Petrov's condition (optimal control): here, if for any $x \in \Gamma$,

$$\sup_{u\in U}\langle \sum_{i=1}^{N} u_i X_i(x), \nabla d^{\Gamma}(x)\rangle > 0,$$

then T is locally Lipschitz on Ω (Veliov 1997), and moreover locally semiconcave (Cannarsa, Sinestrari 1995).

DEFINITION

A function $f : U \to \mathbb{R}$ is locally semiconcave in U if for every $V \Subset U$ there exists a constant C such that $D^2 f \leq CI$ in $\mathcal{D}'(V)$ (in the sense of quadratic forms). f is said to be semiconvex if $-f$ is semiconcave.

DEGENERATE EQUATIONS: DEFINITIONS

Hamiltonian: for any $(x,p) \in \Omega' \times \mathbb{R}^n$,

$$h(x,p) = \sum_{j=1}^{N} \langle X_j(x), p\rangle^2 = \left(\sup_{u\in \overline{B}_1(0)} \sum_{j=1}^{N} u_j \langle X_j(x), p\rangle\right)^2.$$

DEFINITION

- Characteristic set:
  $\mathrm{Char}(\{X_i\}_{i=1}^N) = \{(x,p) \in \Omega' \times \mathbb{R}^n \setminus \{0\} : h(x,p) = 0\}$.
- Characteristic (boundary) points
  $E = \{x \in \Gamma : \mathrm{span}(\{X_i(x)\}_{i=1}^N) \subset T_\Gamma(x)\}$.

EXAMPLE (HEISENBERG VECTOR FIELDS)

In $\mathbb{R}^3$, $X_1 = \partial_{x_1}$, $X_2 = \partial_{x_2} + x_1\partial_{x_3}$.

$$\mathrm{Char}(X_1, X_2) = \{(x_1, x_2, x_3, 0, -x_1p_3, p_3) : (x_1,x_2,x_3) \in \Omega,\ p_3 \neq 0\}$$

is a smooth submanifold of $\mathbb{R}^6$ of codimension 2.

Singular time-optimal trajectories may occur, and these may destroy the regularity of T (well known idea, see Sussmann (1992), Agrechev (1998), and then Trélat, Cannarsa and Rifford...)

Degenerate equations: definitions

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Example (Heisenberg vector fields)

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PONTRYAGIN MAXIMUM PRINCIPLE

Pontryagin Hamiltonian: $H(x, p, u) = \sum_{j=1}^{n} u_j \langle X_j(x), p \rangle$.

THEOREM

For every time-optimal trajectory starting from x, $y = y^{x,u}$, there exist $p(\cdot) \in AC([0, T(x)]; \mathbb{R}^n)$ and $\nu \in \{0, 1\}$ such that, for a.e. $t \in [0, T(x)]$,

1. $(p(\cdot), \nu) \neq 0$.
2. $p_k'(t) = -\sum_{j=1}^{N} u_j(t) \langle \partial_{x_k} X_j(y(t)), p(t) \rangle$ *for every* $k = 1, \ldots, n$.
3. $p(T(x)) \in N_\Gamma(T(x))$.
4. $H(y(t), p(t), u(t)) = \max_{v \in \overline{B}_1(0)} H(y(t), p(t), v)$.
5. $H(y, p, u) \equiv \nu$.

Note that $p : [0, T(x)] \to \mathbb{R}^n \setminus \{0\}$.

DEFINITION

An extremal lift is a 4-tuple (y, p, ν, u) solving (1)-(4). The extremal lift is normal if $\nu \neq 0$ and abnormal if $\nu = 0$. An optimal trajectory is said singular if it admits an abnormal extremal lift.

Pontryagin maximum principle

Pontryagin Hamiltonian: $H(x,p,u) = \sum_{j=1}^{n} u_j \langle X_j(x), p \rangle$.

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EXAMPLE

Strict abnormal minimizers may destroy the regularity of T.

EXAMPLE (LIU-SUSSMANN 1994)

In $\mathbb{R}^3$ consider vector fields

$$X_1 = \partial_{x_1}, \qquad X_2 = (1 - x_1)\partial_{x_2} + x_1^2 \partial_{x_3}.$$

Then, there exists a bounded open set $\Omega \subset \mathbb{R}^3$, with C^∞ boundary, such that the viscosity solution of the Dirichlet problem

$$\begin{cases} (X_1 T)^2 + (X_2 T)^2 = 1 \qquad \text{in} \quad \Omega, \\ T|_\Gamma = 0, \end{cases}$$

fails to be locally Lipschitz in Ω.

This is a minimum time function T that is not better than Hölder continuous somewhere!

Singular time-optimal trajectories

Definition

y is time-optimal singular if there exists an extremal lift such that $\langle p(t), X_j(y(t))\rangle = 0$ for all $j = 1, \dots, n$, i.e. $(y(t), p(t)) \in \mathrm{Char}(\{X_j\}_{j=1}^N)$.

Theorem

Let $x \in \Omega$ and let $y = y^{x,u}$ be a time-optimal trajectory. Then y is singular $\iff y^{x,u}(T(x)) \in E = \{x \in \Gamma : \mathrm{span}(\{X_i(x)\}_{i=1}^N) \subset T_\Gamma(x)\}$.

Proof. Let (y, p) be such that $(y(t), p(t)) \in \mathrm{Char}(\{X_j\}_{j=1}^N)$. The function $t \mapsto h(y(t), p(t))$ is constant. This implies that

$$y(T(x)) \in E \Leftrightarrow h(y(T(x)), p(T(x))) = 0 \Leftrightarrow (y(t), p(t)) \in \mathrm{Char}(\{X_j\}_{j=1}^N).$$

Theorem (Derridj, 1972)

$E \subset \Gamma$ is a closed set and $\mathcal{H}^{n-1}(E) = 0$.

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Singular-time optimal trajectories

Definition

We say that a function $f : \overline{\Omega} \to \mathbb{R}$ is Lipschitz at a point $x_0 \in \overline{\Omega}$ if there exists a neighbourhood U of x_0 and a constant $L \geq 0$ such that

$$|f(x) - f(x_0)| \leq L|x - x_0| \qquad \forall x \in U \cap \overline{\Omega}.$$

Theorem

Assume (H) and let $x_0 \in \Omega$. Then T fails to be Lipschitz at x_0 if and only if there exists a singular time-optimal $y^{x_0,u}$.

Interior regularity

Theorem (Interior regularity)

Under assumption (H), the following properties are equivalent:

1. *the minimum time problem admits no singular time-optimal trajectory;*
2. *T is locally semiconcave in Ω;*
3. *T is locally Lipschitz continuous in Ω.*

Proof. $2 \Rightarrow 3$ is well-known. The theorem in the previous slide shows that $3 \Rightarrow 1$. The implication $1 \Rightarrow 2$ follows by a combination of

Theorem (Cannarsa-Sinestrari, 1995)

For a smooth controlled system, the minimum time is locally semiconcave whenever the target is a noncharacteristic smooth compact manifold.

and

Theorem (Derridj, 1972)

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Boundary regularity

Definition

We say that a function $f : \overline{\Omega} \to \mathbb{R}$ is Hölder continuous of exponent $\alpha \in]0,1]$ at a point $x_0 \in$ if there exist a neighborhood $U \subset$ of x_0 and a constant $C \geq 0$ such that

$$|f(x) - f(x_0)| \leq C|x - x_0|^{\alpha} \qquad \forall x \in U \cap \overline{\Omega}.$$

Theorem

Assume (H). Then:

1. *for any $x \in \Gamma \setminus E$, T is C^{∞} in a neighborhood of x;*
2. *for any $x \in E$, T is Hölder continuous at x of exponent $1/k(x)$, with $k(x)$ given by*

$$k(x) = \min\left\{k \geq 1 \ : \ Lie^{k}(\{X_i\}_{i=1}^{N})[x] = \mathbb{R}^n\right\} \qquad (x \in \Omega).$$

EXCLUDING SINGULAR-TIME OPTIMAL TRAJECTORIES

- $\{X_i\}_{i=1}^N$ is *strongly bracket generating* on Ω if for every $v = (v_1, \dots, v_N) \in \mathbb{R}^N \setminus \{0\}$,

$$\mathrm{span}\{X_i\}_{i=1}^N[x] + \mathrm{span}\left\{\sum_{j=1}^N v_j[X_j, X_i]\right\}_{i=1}^N [x] = \mathbb{R}^n \qquad \forall x \in \Omega.$$

  Example: Heisenberg vector field.
- Γ is noncharacteristic and $\mathrm{Char}(X_1, \dots, X_N)$ is a sympletic manifold.
- Systems of vector fields admitting, in general, singular time-optimal trajectories may have a better behaviour when Ω enjoys specific properties.

EXAMPLE (LIU–SUSSMAN'S EXAMPLE)

In $\mathbb{R}^3$ consider vector fields

$$X_1 = \partial_{x_1}, \quad X_2 = (1 - x_1)\partial_{x_2} + x_1^2\partial_{x_3}$$

and let Ω be a bounded convex open set with C^∞ boundary. Then there are no singular time-optimal trajectory.

Excluding singular-time optimal trajectories

- $\{X_i\}_{i=1}^N$ is *strongly bracket generating* on Ω if for every $v = (v_1, \ldots, v_N) \in \mathbb{R}^N \setminus \{0\}$,

$$\mathrm{span}\{X_i\}_{i=1}^N[x] + \mathrm{span}\left\{\sum_{j=1}^N v_j[X_j, X_i]\right\}_{i=1}^N [x] = \mathbb{R}^n \qquad \forall x \in \Omega.$$

  Example: Heisenberg vector field.
- Γ is noncharacteristic and $\mathtt{Char}(X_1, \ldots, X_N)$ is a sympletic manifold.
- Systems of vector fields admitting, in general, singular time-optimal trajectories may have a better behaviour when Ω enjoys specific properties.

Example (Liu-Sussman's example)

In $\mathbb{R}^3$ consider vector fields

$$X_1 = \partial_{x_1}, \quad X_2 = (1 - x_1)\partial_{x_2} + x_1^2 \partial_{x_3}$$

and let Ω be a bounded convex open set with C^∞ boundary. Then, there are no singular time-optimal trajectory.

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Singular support

What happens if singular-time optimal trajectories may appear?

Definition

For any $x \in \Omega$ and any $k \in \mathbb{N} \cup \{\infty\}$,

$$x \notin \operatorname{sing\,supp}_{Lip} T \iff \exists \text{ an open set } U \ni x : T \in Lip(U).$$

$$x \notin \operatorname{sing\,supp}_{C^k} T \iff \exists \text{ an open set } U \ni x : T \in C^k(U).$$

Finally, $\operatorname{sing\,supp} T := \operatorname{sing\,supp}_{C^\infty} T$.

Properties under Assumption (H)

- $\operatorname{sing\,supp}_{Lip} T$ is a closed set
- $\operatorname{sing\,supp}_{Lip} T$ has null Lebesgue measure (Nguyen, 2010).
- T is locally semiconcave on $\Omega \setminus \operatorname{sing\,supp}_{Lip} T$.

SINGULAR SUPPORT

THEOREM

Under Assumption (H), $\operatorname{sing\,supp}_{C^\infty} T = \operatorname{sing\,supp}_{C^{1,1}} T$ *.*

Proof. We show that $\Omega \setminus \operatorname{sing\,supp}_{C^{1,1}} T \subseteq \Omega \setminus \operatorname{sing\,supp} T$.

- For any $x \in \Omega \setminus \operatorname{sing\,supp}_{C^{1,1}} T$, there exists a unique (nonsingular) optimal trajectory starting from x, say y^{x,u_x}.

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- $y^{x,u_x}(T(x)) := \xi_x$ is noncharacteristic point.
- We introduce the system of characteristic: for $\xi \in V \subset \Gamma$ nghd of ξ_x,

$$\begin{cases} -\dot{X}(t) = \nabla_p H(X(t), P(t)), & X(0) = \xi, \\ \dot{P}(t) = \nabla_x H(X(t), P(t)), & P(0) = g(\xi) := H(\xi, \nu(\xi))^{-1} \nu(\xi). \end{cases}$$

- Since $T \in C^{1,1}$ in a nghb of x, there are no conjugate times for x, and T is of class $C^{1,1}$ on a neighborhood of $y^{x,u}([0, T(x)))$ (Cannarsa-S. 2015).
- By the method of characteristics, $T \in C^\infty$ on a nghb of x.

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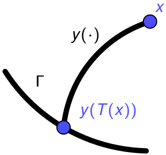

Singular support

In the complement of a closed set of measure zero T has the same regularity of the data of the Dirichlet problem:

Theorem

$\operatorname{sing\,supp} T$ *is a closed set of Lebesgue measure zero.*

Proof. We show that $\operatorname{sing\,supp}_{C^{1,1}} T$ has null Lebesgue measure.

$$\operatorname{sing\,supp}_{C^{1,1}} T = \operatorname{sing\,supp}_{Lip} T \cup \left(\operatorname{sing\,supp}_{C^{1,1}} T \setminus \operatorname{sing\,supp}_{Lip} T\right).$$

Now note that

- $\operatorname{sing\,supp}_{Lip} T$ has null Lebesgue measure by Nguyen, 2010.
- T is locally semiconcave in $\Omega \setminus \operatorname{sing\,supp}_{Lip} T \Rightarrow T$ has a second order expansion a.e. on $\Omega \setminus \operatorname{sing\,supp}_{Lip} T$ by Alexandroff $\Rightarrow$ There exists a set of full measure in $\Omega \setminus \operatorname{sing\,supp}_{Lip} T$ which lies in the complement of $\operatorname{sing\,supp}_{C^{1,1}} T \setminus \operatorname{sing\,supp}_{Lip} T$ (follows from Cannarsa- S. 2015).

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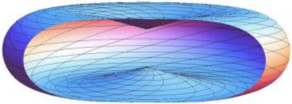
Some Known facts and Open Problems when the target is a singleton

- Known fact: semiconcavity for the sub-Riemannian distance for vector fields admitting no singular minimizing controls in a bracket generating subRiemannian manifold. [1]
- At a point x along a *strictly* abnormal minimizer leaving from x_0, the distance from x_0 can not be expected to be Lipschitz at x.
- Lack of semiconcavity for some classes of problems with *normal-abnormal* minimizers.[2]
- Open Problem: "Sard conjectures" in sub-Riemannian geometry: the distance from a point is not smooth on a set that is the complement of an open and dense set, but it is not known whether it has measure zero. [3]



A sub-Riemannian sphere

[1] P. Cannarsa and L. Rifford, Semiconcavity results for optimal control problems admitting no singular minimizing controls, 2008.

[2] A. Montanari, D. Morbidelli. On the lack of semiconcavity of the sub- Riemannian distance in a class of Carnot groups, 2016.

[3] See A. Figalli and L. Rifford. Mass transportation on sub-Riemannian manifolds, 2010, and L. Rifford and E. Trélat, Morse-Sard type results in sub-Riemannian geometry, 2005.

Thank you for the attention!

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