Controllability and stabilization of linear water waves model

Gravity and gravity-capillary cases

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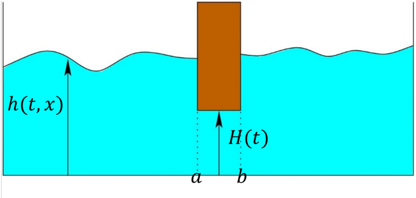
7. Stabilizations issues
   - Polynomially decay for gravity-capillary case
   - Polynomially decay for gravity case

- **A model of wave energy converter (WEC):**
  - Fluid: Navier-Stokes and Euler seem out of reach so we use viscous Saint-Venant;
  - Domain: 2-dimensional; bounded by free surface, fixed bottom (flat);
  - Waves: Shallow water regime.



Unknown functions : h, H and q, Q with $q = vh$.

▶ **The governing equations (Maity, San Martin, Takahashi and MT, 2018):**

$$
\begin{aligned}
\frac{\partial h}{\partial t} + \frac{\partial q}{\partial x} &= 0 && (t>0,\ x \in \mathcal{I} \cup \mathcal{E}),\\
\frac{\partial}{\partial t}\left(\frac{q}{H}\right) + \frac{\partial}{\partial x}\left(\frac{1}{2}\frac{q^2}{H^2} + gH + \frac{p}{\rho}\right) &= 0 && (t>0,\ x \in \mathcal{I}),\\
\frac{\partial}{\partial t}\left(\frac{q}{h}\right) + \frac{\partial}{\partial x}\left(\frac{1}{2}\frac{q^2}{h^2} + gh\right) &= \frac{\mu}{\rho}\frac{\partial}{\partial x}\left(\frac{1}{h}\frac{\partial q}{\partial x}\right) && (t>0,\ x \in \mathcal{E}),\\
\left[\left(\frac{1}{2}\frac{q^2}{H^2} + gH + \frac{p}{\rho}\right) - \frac{\mu}{H\rho}\frac{\partial q}{\partial x}\right](t,a^+) &= \left[\left(\frac{1}{2}\frac{q^2}{h^2} + gh\right) - \frac{\mu}{h\rho}\frac{\partial q}{\partial x}\right](t,a^-), \\
\left[\left(\frac{1}{2}\frac{q^2}{H^2} + gH + \frac{p}{\rho}\right) - \frac{\mu}{H\rho}\frac{\partial q}{\partial x}\right](t,b^-) &= \left[\left(\frac{1}{2}\frac{q^2}{h^2} + gh\right) - \frac{\mu}{h\rho}\frac{\partial q}{\partial x}\right](t,b^+), \\
M\ddot{H}(t) &= -Mg + \int_a^b p(t,x)\,\mathrm{d}x && (t>0).
\end{aligned}
$$

▶ Global existence (Maity, San Martin, Takahashi and MT, 2018) :

Assume that the initial data are close to an equilibrium with $H = \overline{H} > 0$, Then we have solutions with

$$H \in \overline{H} + H^2(0,\infty), \quad h \in \overline{h} + H^1(0,\infty; H^1(\mathcal{E})) \cap C_b^1([0,\infty); L^2(\mathcal{E})),$$

$$q \in C_b([0,\infty); H^1(0,\ell))$$

$$q_{|\mathcal{E}} \in H^1(0,\infty; L^2(\mathcal{E})) \cap C_b([0,\infty); H^1(\mathcal{E})) \cap L^2(0,\infty; H^2(\mathcal{E})),$$

$$q_{|\mathcal{I}} \in H^1(0,\infty; \mathcal{P}_1(\mathcal{I}))$$

$$p_{|\mathcal{I}} \in \frac{Mg}{|\mathcal{I}|} + L^2(0,\infty; \mathcal{P}_2(\mathcal{I}))$$

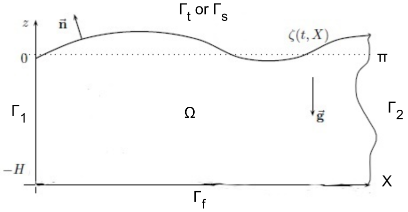
$$\int_{\mathcal{E}} h(t,x)\,\mathrm{d}x + H(t)(b-a) = M\frac{|\mathcal{E}|}{|\mathcal{I}|} + \overline{H}\ell \quad (t \geq 0).$$

▶ Open questions for floating bodies :

- Non vertical walls for the floating body
- 2D Saint-Venant (even for vertical walls)
- Solid moving also horizontally
- Control issues (see Ringwood's book)
- Other fluid models like Euler (see Lannes), Navier-Stokes,. . .

- **Assumptions:**
  - Fluid: inviscid, incompressible, irrotational, constant density;
  - Domain: 2-dimensional; bounded by free surface, fixed bottom (flat) and two sides end; nozero vertical walls (no beaches);
  - Waves: Small-amplitude waves.

    Our domain is as follows:



We always assume that a rigid generator locates on the left boundary, which can produce velocity $u(z,t) = \bar{u}(t)h(z)$.

- **Notations:**

x: horizontal variable, z: vertical variable

$\zeta(x,t)$: The elevation of the free surface;

$V(x,z,t)$: The velocity of the fluid particle (x,z) at time t;

$\phi(x,z,t)$: The velocity potential in gravity waves;

$\psi(x,z,t)$: The velocity potential in gravity-capillary waves;

Γ_1: The left end of the domain $\Gamma_1 = \{(0,z)| -1 < z < 0\}$;

Γ_2: The right end $\Gamma_2 = \{(\pi,z)| -1 < z < 0\}$;

Γ_f: The fixed bottom $\Gamma_f = \{(x,-1)|0 < x < \pi\}$;

Γ_s: The undistured free surface $\Gamma_s = \{(x,0)|0 < x < \pi\}$;

Γ_t: No-flat free surface $\Gamma_t = \{(x,\zeta(x,t))|0 < x < \pi\}$.

I. Gravity waves

- 1.1 Governing equations

- Incompressible: $$\nabla \cdot V = 0 \quad \text{in } \Omega \tag{1}$$

- Irrotational:
$$V = \nabla\phi \qquad \text{in } \Omega \tag{2}$$
- Bernoulli equation:
$$\phi_t + \frac{1}{2}|\nabla\phi|^2 + gy = c(t) - \frac{p}{\rho} \qquad \text{in } \Omega$$
where we used the continuity of pressure on the free surface (by shifting ϕ to make the constant zero). And then:
$$\phi_t + \frac{1}{2}|\nabla\phi|^2 + g\zeta = 0 \qquad \text{on } \Gamma_t \tag{3}$$
- Kinematic condition:
This means that the fluid particle always stays on the free surface.
$$V \cdot \vec{n}\sqrt{1 + |\nabla\zeta|^2} = \zeta_t \qquad \text{on } \Gamma_t \tag{4}$$
where $\vec{n}$ is unit outer normal.

- Impermeable boundary Γ_2 and Γ_f:

$$V \cdot \vec{n} = \partial_{\vec{n}}\phi = 0 \quad \text{on } \Gamma_2 \cup \Gamma_f \tag{5}$$

- Boundary condition on Γ_1:

$$\partial_{\vec{n}}\phi = u \quad \text{on } \Gamma_1 \tag{6}$$

So one obtains a Laplace equation with mixed boundary condition:

$$\begin{cases} \Delta\phi = 0 & \text{in } \Omega \\ \phi_t + \frac{1}{2}|\nabla\phi|^2 + g\zeta = 0 & \text{on } \Gamma_t \\ \sqrt{1+|\nabla\zeta|^2}\partial_{\vec{n}}\phi = \zeta_t & \text{on } \Gamma_t \\ \partial_{\vec{n}}\phi = u & \text{on } \Gamma_1 \\ \partial_{\vec{n}}\phi = 0 & \text{on } \Gamma_2 \cup \Gamma_f \end{cases} \tag{7}$$

1.2 Formulate the equation

To do this, we introduce several important operators:

- Dirichlet and Neumann map

  Consider following equations:

$$\begin{cases} \Delta\Phi = 0 & \text{in } \Omega \\ \Phi = \varphi & \text{on } \Gamma_t \\ \partial_{\vec{n}}\Phi = 0 & \text{on } \Gamma_1 \cup \Gamma_2 \cup \Gamma_f \end{cases} \qquad \begin{cases} \Delta\Psi = 0 & \text{in } \Omega \\ \Psi = 0 & \text{on } \Gamma_t \\ \partial_{\vec{n}}\Psi = u & \text{on } \Gamma_1 \\ \partial_{\vec{n}}\Psi = 0 & \text{on } \Gamma_2 \cup \Gamma_f \end{cases}$$

  Let D denote "Dirichlet map" defined by

$$D : \varphi \longmapsto D\varphi = \Phi$$

  And let N denote "Neumann map" defined by

$$N : u \longmapsto Nu = \Psi$$

- Dirichlet-Neumann, Neumann-Neumann map:
  Let A denote "Dirichlet-Neumann" map defined by

$$A : \varphi \longmapsto \sqrt{1 + |\nabla\zeta|^2}\partial_{\vec{n}} D\varphi|_{\Gamma_t}$$

  Let B denote "Neumann-Neumann" map defined by

$$B : u \longmapsto -\partial_{\vec{n}} N u|_{\Gamma_t}$$

1.2.1 The Zakharov/Craig-Sulem formulation

The general idea is that the free surface and the trace of the velocity potential at the surface fully define the flow.
Now we focus on the boundary conditions on Γ_t in (7) and do some trivial calculations. Let $\phi|_{z=\zeta} = \underline{\phi}$ and then the original water waves problem can be written into following equatons:

$$\begin{cases} \partial_t \zeta - A\underline{\phi} = 0 \\ \partial_t \underline{\phi} + g\zeta + \frac{1}{2}|\nabla \underline{\phi}|^2 - \frac{(A\underline{\phi} + \nabla\zeta \cdot \nabla\underline{\phi})^2}{2(1+|\nabla\zeta|^2)} = 0 \end{cases} \tag{8}$$

1.3 Linearization

Based on the "small amplitude" assumption, we linearize the boundary condition on free surface. It means that we can neglect some high order terms.

- Bernoulli equation:

$$\phi_t + g\zeta = 0 \qquad \text{on } \Gamma_s \tag{9}$$

- Kinematic condition:

$$\partial_{\vec{n}}\phi = \zeta_t \qquad \text{on } \Gamma_s \tag{10}$$

Combining these two equations we have

$$\phi_{tt} + \partial_{\vec{n}}\phi = 0 \quad \text{on } \Gamma_s$$

By using Dirichlet-Neumann map A_g and Neumann-Neumann map B, the boundary condition of original equation on Γ_s can be written into second order ODE.

$$\begin{cases} \phi_{tt} + A_g\phi = Bu \quad \text{on } \Gamma_s \\ \phi(x,0) = \phi_0(x), \quad \phi_t(x,0) = \phi_1(x) \end{cases} \tag{11}$$

Notice that Bernoulli equation gives the relation between ϕ and ζ, then above equation can be transformed into:

$$\begin{cases} \zeta_{tt} + A_g\zeta = B(-u_t) = B\tilde{u} \quad \text{on } \Gamma_s \\ \zeta(x,0) = \zeta_0(x), \quad \zeta_t(x,0) = \zeta_1(x) \end{cases} \tag{12}$$

For variable $\zeta(x,t)$, conservation of volume implies:

$$\int_0^\pi \zeta(x)dx = 0$$

It means that A_g can be defined on zero mean space:

$$L_0^2(\Gamma_s) = \{f(x) \in L^2(\Gamma_s) | \int_0^\pi f(x)dx = 0\}$$

Remark: i. Zero mean condition holds iff $\int_{\Gamma_1} h(z)dz = 0$.
ii. It is worth noting that A_g and B keep the zero mean property.

$$\int_0^\pi A\zeta = \int_0^\pi \partial_{\vec{n}}\Phi = \int_{\partial\Omega} \partial_{\vec{n}}\Phi = \int_\Omega \Delta\Phi = 0$$

$$\int_0^\pi B\tilde{u} = \int_{\Gamma_1} (-\bar{u}_t)h(z) - \int_{\partial\Omega} \partial_{\vec{n}}\Psi = -\int_\Omega \Delta\Psi = 0$$

Theorem 1 (S.Mottelet, 2000, SIAM)

i.The operator $A_g : D(\mathcal{A})(= H^1(\Gamma_s) \cap L_0^2(\Gamma_s)) \to L_0^2(\Gamma_s)$ is self-adjoint, strictly positive and has compact resolvents;
ii. The operator $B : L_0^2(\Gamma_1) \to L_0^2(\Gamma_s)$ is linear bounded.

II. Gravity-capillary waves

Now we consider a fluid in which both surface tension and gravity are significant.

2.1 Governing equations

The only difference is free surface Bernoulli equation.

- Bernoulli equation: The pressure is discontinuous on the free surface because of the existence of surface tension. And the pressure is given by

$$p = p_a - \sigma\kappa \quad \text{on } \Gamma_t$$

where σ is surface tension coefficient, κ is mean curvature of the surface. Then Bernoulli equation has the following form:

$$\psi_t + \frac{1}{2}|\nabla\psi|^2 + g\zeta = \frac{\sigma\kappa}{\rho} = T\kappa \; (T = \frac{\sigma}{\rho}) \quad \text{on } \Gamma_t$$

In our case, $\kappa(\zeta) = -\nabla \cdot \frac{(-\nabla\zeta,1)}{\sqrt{1+|\nabla\zeta|^2}} = \frac{\zeta xx}{(1+\zeta_x^2)^{\frac{3}{2}}}$. Therefore, we obtain the governing equation for gravity-capillary waves:

$$\begin{cases} \Delta\psi = 0 & \text{in } \Omega \\ \psi_t + \frac{1}{2}|\nabla\psi|^2 + g\zeta = T\kappa & \text{on } \Gamma_t \\ \sqrt{1+|\nabla\zeta|^2}\,\partial_{\vec{n}}\psi = \zeta_t & \text{on } \Gamma_t \\ \partial_{\vec{n}}\psi = u & \text{on } \Gamma_1 \\ \partial_{\vec{n}}\psi = 0 & \text{on } \Gamma_2 \cup \Gamma_f \end{cases} \tag{13}$$

2.2 Formulate the equation

2.2.1 The Zakharov/Craig-Sulem formulation

Similarly, we consider the boundary condition of equation (13) on Γ_t. Let $\psi|_{z=\zeta} = \underline{\psi}$ and then

$$\begin{cases} \partial_t \zeta - A\underline{\psi} = 0 \\ \partial_t \underline{\psi} + g\zeta + \frac{1}{2}|\nabla \underline{\psi}|^2 - \frac{(A\underline{\psi} + \nabla\zeta \cdot \nabla\underline{\psi})^2}{2(1+|\nabla\zeta|^2)} = T\kappa \end{cases} \tag{14}$$

We can see that above equations are strongly nonlinear. And if they are solvable and the whole problem will be done.

There are a few results about controllability and stabilization of nonlinear water waves, which, of course, is based on different formulation of control models.

2.2.2 Alazard's results for gravity-capillary waves

Consider Cauchy problem as follows:

$$\begin{cases} \partial_t \zeta - A\underline{\psi} = 0 \\ \partial_t \underline{\psi} + g\zeta + \frac{1}{2}|\nabla \underline{\psi}|^2 - \frac{(A\underline{\psi} + \nabla\zeta \cdot \nabla\underline{\psi})^2}{2(1+|\nabla\zeta|^2)} = T\kappa - P_{ext} \end{cases} \tag{15}$$

with initial data $\zeta|_{t=0} = \zeta_0, \quad \underline{\psi}|_{t=0} = \psi_0$. Where P_{ext}, external pressure, is the control in this system and Alazard's results are:

- Exactly controllable in arbitrary time for small data when ζ and $\underline{\psi}$ are periodic w.r.t horizontal variable x;
- The energy is exponentially decay in time for some pressure law relating P_{ext} to $(\zeta, \underline{\psi})$.

The problem about control of nonlinear water waves is almost open so far.

2.3 Linearization

In this case, Bernoulli equation and kinematic condition can be linearized as

$$\begin{cases} \partial_{\vec{n}}\psi = \zeta_t & \text{on} \quad \Gamma_s \\ \psi_t = \zeta - T\zeta_{xx} & \text{on } \Gamma_s \end{cases}$$

Based on the structure of the boundary condition, we consider the equation of ψ_t:

$$\begin{cases} \Delta\psi_t = 0 & \text{in} \quad \Omega \\ \partial_{\vec{n}}\psi_t = \zeta_{tt} & \text{on} \quad \Gamma_s \\ \psi_t = \zeta - T\zeta_{xx} & \text{on } \Gamma_s \\ \partial_{\vec{n}}\psi_t = u_t = \tilde{u} & \text{on} \quad \Gamma_1 \\ \partial_{\vec{n}}\psi_t = 0 & \text{on} \quad \Gamma_2 \cup \Gamma_f \end{cases} \tag{16}$$

Similarly, we decompose it into two parts:

$$\begin{cases} \Delta\gamma = 0 & \text{in } \Omega \\ \gamma = T\zeta_{xx} - \zeta & \text{on } \Gamma_s \\ \partial_{\vec{n}}\gamma = 0 & \text{on } \Gamma_1 \cup \Gamma_2 \cup \Gamma_f \end{cases} \qquad \begin{cases} \Delta\theta = 0 & \text{in } \Omega \\ \theta = 0 & \text{on } \Gamma_s \\ \partial_{\vec{n}}\theta = \tilde{u} & \text{on } \Gamma_1 \\ \partial_{\vec{n}}\theta = 0 & \text{on } \Gamma_2 \cup \Gamma_f \end{cases}$$

Define $F = AL$ and $Lu = -Tu''$, where L is Sturm-Liouville operator. Above abstract formulation holds if and only if $\psi_t = \gamma + \theta$ and

$$\zeta_{tt} = \partial_{\vec{n}}\psi_t|_{\Gamma_s} = \partial_{\vec{n}}\gamma|_{\Gamma_s} + \partial_{\vec{n}}\theta|_{\Gamma_s} = -(A+F)\zeta + B\tilde{u} \qquad (17)$$

For the sake of energy conservation, it needs the boundary condition (BC):

$$(BC) : \zeta'(0) = 0 = \zeta'(\pi)$$

Theroem 2 (Russell M.Reid, 1995, SIAM)

$A_c = \mathcal{A} + F : D(A) = H^3(\Gamma_s) \cap L_0^2(\Gamma_s)(BC) \longrightarrow L_0^2(\Gamma_s)$ is self-adjoint, strictly positive and has compact resolvent.

Then one gets another second order ODE as follows:

$$\begin{cases} \zeta_{tt} + A_c\zeta = B(-u_t) = B\tilde{u} \quad \text{on } \Gamma_s \\ \zeta(x,0) = \zeta_0(x), \quad \zeta_t(x,0) = \zeta_1(x) \end{cases} \tag{18}$$

III. Spectral results $(k \geq 1)$

two cases	eigenvalues	eigenvectors
gravity A_g	$\lambda_k = k \tanh k$	$\varphi_k = \sqrt{\frac{2}{\pi}} \cos kx$
gravity-capillary A_c	$\mu_k = (k^3 + k) \tanh k$	$\varphi_k = \sqrt{\frac{2}{\pi}} \cos kx$

IV. Known results

- Gravity waves:

  1. L.Russell and M.Reid (1985):
  Null controllability in infinite time;
  No null controllability in finite time.

  2. Mottelet (2000)(Rigid case):
  No approximate controllability in finite time;
  Approximate controllability holds in infinite time;
  Strong stabilizability holds (nonuniformly decay).

- Gravity-capillary waves:

  3. M.Reid (1995):
  Conditional reachability of $\mathcal{D}(A)$ in finite time.

V. Stabilization issues

For the stability system of gravity-capillary waves (18) with the feedback $\tilde{u} = -B^*\partial_t\zeta$, we get the polynomially decay based on Ammari and Tucsnak's work in 2001.

Theorem 3 (Tucsnak & Su, 2019)

For $(\zeta_0, \zeta_1) \in \mathcal{H}_1 \times \mathcal{H}_{\frac{1}{2}}$ and a class of "strategic" shape function $h(z)$, there exists $C_1 > 0$ such that for all $t > 0$, we have

$$\|(\zeta(x), \zeta_t(x)\|^2_{\mathcal{H}_{\frac{1}{2}} \times \mathcal{H}} \leq \frac{C_1}{(1+t)^{\frac{3}{2}}} \|(\zeta_0, \zeta_1)\|^2_{\mathcal{H}_1 \times \mathcal{H}_{\frac{1}{2}}} \tag{19}$$

Remark: The space $\mathcal{H}_\alpha$ is defined by:

$$\mathcal{H}_\alpha = \{\phi \in \mathcal{H}_0 = L^2_0(\Gamma_s) | \sum_{k\geq 1} |\mu_k|^{2\alpha} |\langle \phi, \psi_k\rangle|^2 < \infty\} = D(A_c^\alpha)$$

• Sketch of the proof:

Step 1: Ingham's inequality and strategic condition (SC).

The eigenvalues and corresponding eigenvectors of

$\mathcal{A}_c = \begin{bmatrix} 0 & I \\ -A_c & 0 \end{bmatrix}$ are :

$$\nu_k = i\sqrt{\mu_k} \sim k^{\frac{3}{2}}, \quad \phi_k = \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{\nu_k}\varphi_k \\ \varphi_k \end{bmatrix}$$

Notice that $\sqrt{\mu_{k+1}} - \sqrt{\mu_k} \longrightarrow \infty$, this implies that Ingham's inequality is available.

And we call $h(z)$ is a "strategic" shape, if it satisfies

$$(SC) \begin{cases} \|h'(z)\|_{L^\infty(-1,0)} < \dfrac{(1-\varepsilon)\tanh 1}{1-\frac{2}{e}} |h(0)|, \ \varepsilon \in (0,1) \\ \displaystyle\int_{-1}^{0} h(z)dz = 0 \end{cases} \tag{20}$$

For a strategic shape $h(z)$, we have $|B^*\varphi_k| \geqslant \frac{C}{k}$;

Step 2: Observability inequality.

Based on $h(z)$ in Step 1, the solution of undamped problem

$$\begin{cases} w_{tt} + A_c w = 0 \\ w(0) = \zeta_0, w_t(0) = \zeta_1 \end{cases}$$

satisfies $\int_0^\tau \|B^* w_t\|^2 \geq C\|(\zeta_0, \zeta_1)\|^2_{\mathcal{H}_{\frac{1}{6}} \times \mathcal{H}_{-\frac{1}{3}}}$.

Step 3: Apply the result of Ammari and Tucsnak in 2001, we can conclude that

$$\|(\zeta(x), \zeta_t(x)\|^2_{\mathcal{H}_{\frac{1}{2}} \times \mathcal{H}} \leq \frac{C_1}{(1+t)^{\frac{3}{2}}} \|(\zeta_0, \zeta_1)\|^2_{\mathcal{H}_1 \times \mathcal{H}_{\frac{1}{2}}}$$

for all $(\zeta_0, \zeta_1) \in \mathcal{H}_1 \times \mathcal{H}_{\frac{1}{2}}$.

Remark: We give two shape functions satisfying (SC):

i. Linear example: $h(z) = z + \frac{1}{2}$

$$\int_{-1}^{0} (z + \frac{1}{2}) dz = 0, \quad |h'(z)| = 1 < \frac{(1-\varepsilon_0)\tanh 1}{1 - \frac{2}{e}} \cdot \frac{1}{2}$$

where $\varepsilon_0 < 1 - \frac{2(1-\frac{2}{e})}{\tanh 1} \in (0,1)$.

ii. Nonlinear example: $h(z) = \cos\frac{1}{2}\pi(z + \frac{3}{2})$

$$h(0) = \cos\frac{3}{4}\pi \neq 0, \quad \int_{-1}^{0} \cos\frac{1}{2}\pi(z + \frac{3}{2}) dz = 0$$

$$|h'(z)| < \frac{(1-\varepsilon_1)\tanh 1}{1 - \frac{2}{e}} \cdot \frac{\sqrt{2}}{2}$$

where $\varepsilon_1 < 1 - \frac{\pi(1-\frac{2}{e})}{\sqrt{2}\tanh 1} \in (0,1)$.

For gravity waves, Mottelet has shown that the system is strongly stable, but not exponentially stable. It is known that the resolvent operator of such system must be unbounded on the imaginary axis. We shall get the decay rate by estimating the upper bound of resolvent operator.

Theorem 4 (Tucsnak & Su, 2019)

For $(\zeta_0, \zeta_1) \in \mathbb{H}_1 \times \mathbb{H}_{\frac{1}{2}}$ and strategic shape $h(z)$, there exists $C_2 > 0$, such that for all $t > 0$, we have

$$\|(\zeta(x), \zeta_t(x)\|^2_{\mathbb{H}_{\frac{1}{2}} \times \mathbb{H}} \leq \frac{C_2}{(1+t)^{\frac{1}{6}}} \|(\zeta_0, \zeta_1)\|^2_{\mathbb{H}_1 \times \mathbb{H}_{\frac{1}{2}}} \qquad (21)$$

Remark:
$\mathbb{H}_\beta = \{\phi \in \mathbb{H}_0 = L^2_0(\Gamma_s) | \sum_{k\geq 1} |\lambda_k|^{2\beta} |\langle \phi, \psi_k \rangle|^2 < \infty\} = D(A_g^\beta)$

• Sketch of the proof:

Denote $X = \mathbb{H}_{\frac{1}{2}} \times \mathbb{H}$ and $\mathcal{A}_g = \begin{bmatrix} 0 & I \\ -A_g & -BB^* \end{bmatrix}$ with domain $D(\mathcal{A}_g) = \mathbb{H}_1 \times \mathbb{H}_{\frac{1}{2}}$. It is easy to verify that $i\mathbb{R} \in \rho(\mathcal{A}_g)$ and $\mathcal{A}_g$ generates a contraction semigroup. We shall prove

$$\sup_{|\omega|\geq 1} \frac{1}{\omega^6} \|(i\omega I - \mathcal{A}_g)^{-1}\| < \infty, \qquad \forall\, \omega \in \mathbb{R} \tag{22}$$

Step1: By contradiction argument we assume that the equation (22) is false, and then
$\forall\, n > 0,\ \exists\, \{\omega_n\} \in \mathbb{R}^+, \omega_n \longrightarrow \infty\ (n \longrightarrow \infty)$ and

$$z_n = (u_n, v_n) \in D(\mathcal{A}_g), \quad \|z_n\|_X^2 = 1$$

such that

$$\omega_n^6 (i\omega_n I - \mathcal{A}_g) z_n \longrightarrow 0 \quad \text{in } X \tag{23}$$

which further leads to

$$\begin{aligned} \omega_n^3 B^* v_n &\longrightarrow 0 \quad \text{in } \mathbb{C} \\ \omega_n^3 B B^* v_n &\longrightarrow 0 \quad \text{in } \mathbb{H} \end{aligned} \tag{24}$$

Step 2: Denote $\mathcal{A}_0 = \begin{bmatrix} 0 & I \\ -A_g & 0 \end{bmatrix}$. Notice that

$$\omega_n^3 (i\omega_n I - \mathcal{A}_g) z_n = \omega_n^3 (i\omega_n I - \mathcal{A}_0) z_n + \begin{bmatrix} 0 \\ \omega_n^3 B B^* v_n \end{bmatrix}$$

$$(23) + (24) \Longrightarrow \omega_n^3 (i\omega_n I - \mathcal{A}_0) z_n \longrightarrow 0 \quad \text{in } X \tag{25}$$

It is known that the eigenvalues and eigenvectors of $\mathcal{A}_0$ are

$$\alpha_k = i\beta_k = i\sqrt{\lambda_k} \sim k^{\frac{1}{2}}, \ \phi_k = \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{\alpha_k}\varphi_k \\ \varphi_k \end{bmatrix}$$

Then $z_n = \sum_{k\in\mathbb{Z}^*}\langle z_n, \phi_k\rangle \phi_k$ and (25) implies
$\forall\ \varepsilon > 0, \exists N_\varepsilon > 0\ s.t.$

$$\sum_{k\in\mathbb{Z}^*} |\omega_n^3(\omega_n - \beta_k)|^2 |\langle z_n, \phi_k\rangle|^2 < \varepsilon \tag{26}$$

Lemma (Liu & Rao, 2007)

Given $\varepsilon > 0$ and N_ε from Step 2, for each $n \geq N_\varepsilon$, there exists $k(n) \in \mathbb{Z}^*$ and constant $\gamma > 0,\ s.t.$

$$\begin{aligned} &|\omega_n(\omega_n - \beta_{k(n)})|^2 < \varepsilon \\ &|\omega_n(\omega_n - \beta_k)|^2 > \gamma, \quad \forall\ k \neq k(n) \end{aligned} \tag{27}$$

Step 3: Using above Lemma, we obtain from (26) that

$$\gamma \sum_{k\neq k(n)} \omega_n^4 |\langle z_n, \phi_k\rangle|^2 + |\omega_n^3(\omega_n - \beta_{k(n)})|^2 |\langle z_n, \phi_{k(n)}\rangle^2 < \varepsilon$$

Denote $\psi_n = \langle z_n, \phi_{k(n)}\rangle \phi_{k(n)}$, then from above inequality we have

$$\|z_n - \psi_n\|_X^2 = \sum_{k\neq k(n)} |\langle z_n, \phi_k\rangle|^2 < \frac{\varepsilon}{\gamma\omega_n^4}$$

Combining with $\|z_n\|_X^2 = 1$, then

$$\|\psi_n\|_X^2 = |\langle z_n, \phi_{k(n)}\rangle|^2 > 1 - \frac{\varepsilon}{\gamma\omega_n^4} > 1 - \frac{\varepsilon}{\gamma}$$

Let $\tilde{B} = \begin{bmatrix} 0 \\ B \end{bmatrix}$ and $\tilde{B} \in \mathcal{L}(\mathbb{C}, X)$, Estimate the norm of $\omega_n^2 \tilde{B}^* z_n$ in $\mathcal{C}$,

$$|\omega_n^2 \tilde{B}^*(z_n - \psi_n)| \leq \omega_n^2 \|B^*\| \|z_n - \psi_n\|_X \leq C_2 \sqrt{\frac{\varepsilon}{\gamma}}$$

$$|\omega_n^2 \tilde{B}^* \psi_n| \geq \omega_n^2 |\langle z_n, \phi_{k(n)} \rangle| |B^* \varphi_{k(n)}| \geq C_1 \sqrt{1 - \frac{\varepsilon}{\gamma}}$$

$$\begin{aligned} |\omega_n^2 \tilde{B}^* z_n| &\geq |\omega_n^2 \tilde{B}^* \psi_n| - |\omega_n^2 \tilde{B}^*(z_n - \psi_n)| \\ &\leq C_1 \sqrt{1 - \frac{\varepsilon}{\gamma}} - C_2 \sqrt{\frac{\varepsilon}{\gamma}} > \delta > 0 \end{aligned} \tag{28}$$

for some contant $\delta > 0$. This contradicts with (24).
We get equation (21) by using Borichev and Tomilov's results.

Thanks for your attention !