Decay of semilinear damped wave equations: cases without geometric control condition

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Benasque 2019

Benasque 2015

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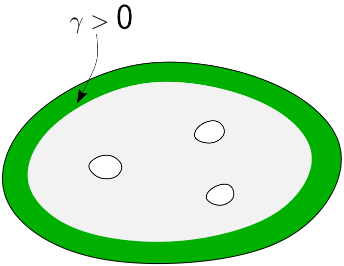
The semilinear damped wave equation

$$\boxed{\partial^2_{tt} u + \gamma(x)\partial_t u = \Delta u - f(x,u)}$$



- Ω is a smooth compact manifold of dimension $d = 2$ with Dirichlet boundary conditions.
- the damping γ is in $\mathbb{L}^\infty(\Omega)$, $\gamma(x) \geq 0$
- f is smooth and of degree p

  $|f(x,u)| + |f'_x(x,u)| \leq C(1+|u|)^p$
  $|f'_u(x,u)| \leq C(1+|u|)^{p-1}$
- f is of the sign of u:

  $f(x,u)u \geq 0$

Notations

$$\boxed{\partial^2_{tt} u + \gamma(x)\partial_t u = \Delta u - f(x,u)}$$

Set $X = H^1_0(\Omega) \times L^2(\Omega)$ and

$$U = \begin{pmatrix} u \\ \partial_t u \end{pmatrix} \qquad A = \begin{pmatrix} 0 & Id \\ \Delta & -\gamma(x) \end{pmatrix} \qquad F(U) = \begin{pmatrix} 0 \\ -f(x,u) \end{pmatrix}$$

- $\Rightarrow$ e^{At} is a dissipative semigroup on X.
- $\Rightarrow$ Since f is of degree $p < \infty$ and Ω is of dimension $d = 2$, $F : X \longrightarrow X$ is defined and Lipschitz on the bounded sets.

We consider in X the equation

$$\boxed{\partial_t U = AU + F(U) \qquad U(t=0) = U_0 \in X}$$

The gradient dynamics

Set $V(x,u) = \int_0^u f(x,\xi)\,\mathrm{d}\xi$. The energy

$$\mathcal{E}(U) = \int_\Omega \frac{1}{2}(|\nabla u|^2 + |\partial_t u|^2) + V(x,u)\,\mathrm{d}x$$

is non-increasing along the trajectories since

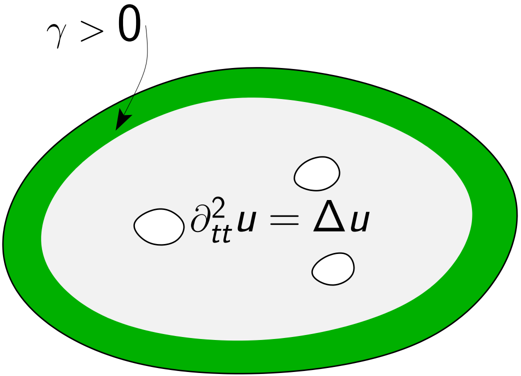
$$\partial_t \mathcal{E}(U(t)) = -\int_\Omega \gamma(x)|\partial_t u|^2\,\mathrm{d}x$$

$\Rightarrow$ **Global existence of solutions**

Motivations

The linear equation is **dissipative** and any solution goes to zero

$$\|e^{At}U\|_X \xrightarrow[t\longrightarrow+\infty]{} 0\ .$$

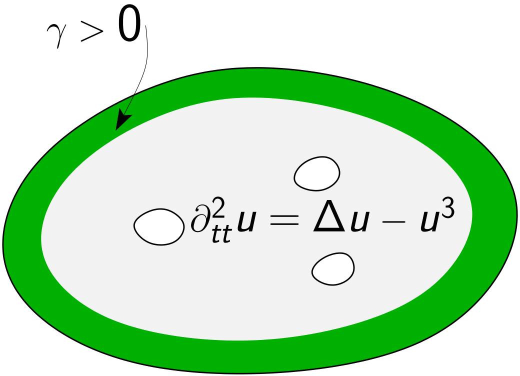


Motivations

The linear equation is **dissipative** and any solution goes to zero

$$\|e^{At}U\|_X \xrightarrow[t\longrightarrow+\infty]{} 0 \ .$$

Do we still have stabilization of the nonlinear problem?
At which rate?

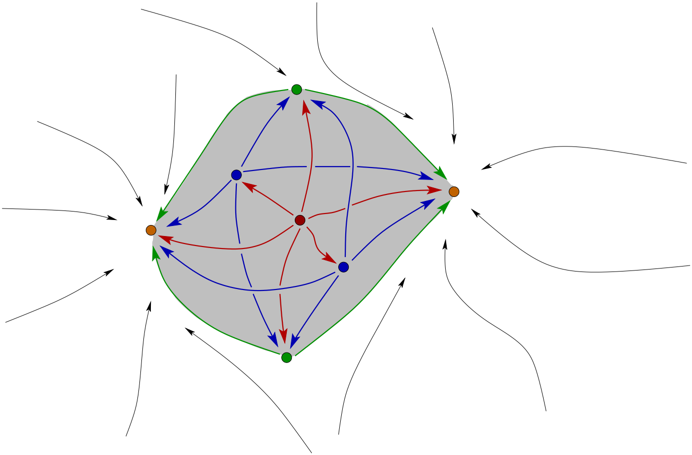


Motivations

Also possible to consider convergence to less trivial dynamics by assuming that f is only asymptotically of the sign of u

$$\forall |u| \geq R \;, \quad f(x,u)u \geq 0 \;.$$

Existence of global attractor with gradient structure?



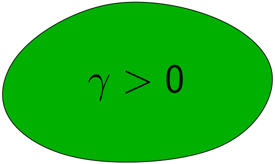
1. A historic result
2. A standard extension
3. The disk with two holes
4. The disk with three holes
5. Conclusion

A historic result

$$\boxed{\partial^2_{tt} u + \gamma(x)\partial_t u = \Delta u - f(x,u)}$$

Assume:

- $\gamma(x) \geq \alpha > 0$ in Ω
- f is of degree $p < \infty$
- $f(x,u)u \geq 0$



Theorem – J.K. Hale (1985) and A. Haraux (1985)

With the above assumptions, any solution $u(t)$ of the damped wave equation converges to 0 in $X = H^1_0(\Omega) \times L^2(\Omega)$. Moreover, the convergence is uniform in bounded sets of X.

Step 1: the trajectories are bounded.

If $f(x,u)u \geq 0$ and f is of degree p, then the energy is well defined, non-negative and bounded on bounded sets.

$$\frac{1}{2}\|U\|_X^2 + \min V \ \leq \ \int_\Omega \frac{1}{2}(|\nabla u|^2 + |\partial_t u|^2) + V(x,u)\,\mathrm{d}x \ \leq \ K(\|U\|_X) \ .$$

Since $\mathcal{E}$ is non-increasing, **the trajectories of bounded sets are bounded**.

A historic result

Step 2: the asymptotic compactness.

The linear semigroup is stabilized:

$$\forall t \geq 0 \ , \quad \|e^{At}\|_{\mathcal{L}(X)} \leq M e^{-\lambda t}$$

Moreover, if f is of degree p, then
$F : \begin{pmatrix} u \\ v \end{pmatrix} \in H^1_0(\Omega) \times L^2(\Omega) \longmapsto \begin{pmatrix} 0 \\ -f(x,u) \end{pmatrix} \in H^1_0(\Omega) \times L^2(\Omega)$ is compact.

$$U(t) = e^{At} U_0 + \int_0^t e^{A(t-s)} F(U(s))\, \mathrm{d}s$$

$\Rightarrow$ **the bounded sets admits compact $\omega-$limit sets.**

A historic result

Step 3: a unique continuation property.

It is sufficient to show that the $\omega-$limit sets consists of equilibrium points. By Lasalle's principle, the trajectories $U(t) = (u, \partial_t u)$ in the $\omega-$limit sets have constant energy. So we have

$$\partial_t \mathcal{E}(U(t)) = -\int_\Omega \gamma(x)|\partial_t u|^2 \,\mathrm{d}x = 0 \ .$$

Since $\gamma(x) \geq \alpha > 0$, we have $\partial_t u \equiv 0$ and thus u is an equilibrium point. Due to the sign assumption, we finally obtain $u \equiv 0$.

A historic result

- Asymptotic compactness $\Leftrightarrow$ high frequencies are not really modified by the nonlinearity
- Unique continuation $\Leftrightarrow$ classify low-frequency solutions

Key arguments where we use γ positive:

1. $\|e^{At}\|_{\mathcal{L}(X)} \leq Me^{-\lambda t}$ has finite integral on $[0,+\infty)$
2. if $\mathcal{E}(U(t))$ is constant, then $\int \gamma(x)|u_t|^2 = 0$ and $u(t)$ is constant.

A standard extension

1. A historic result
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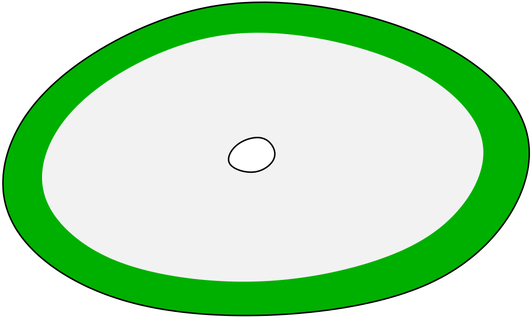
What happens when $\gamma(x)$ may vanish?



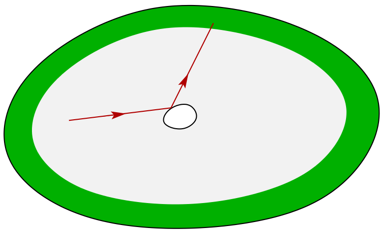
The decay of the linear semigroup

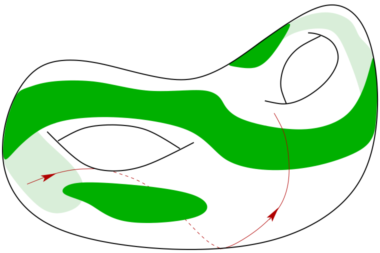
Theorem – J. Rauch and M. Taylor (1974)
C. Bardos, G. Lebeau and J. Rauch (1992)

$$\|e^{At}\|_{\mathcal{L}(X)} \leq Me^{-\lambda t}$$

$$\Longleftrightarrow$$

Any long enough geodesic meets the support of the damping γ





The unique continuation property

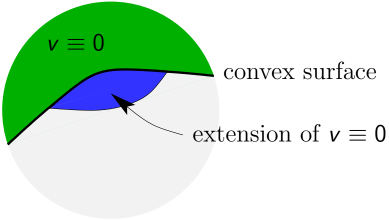
If $U_\infty(t)$ belongs to an $\omega-$limit set,

$$\partial_t \mathcal{E}(U_\infty(t)) = -\int_\Omega \gamma(x)|\partial_t u_\infty|^2 \,\mathrm{d}x = 0 \ .$$

So $v(t) = \partial_t u_\infty(t)$ vanishes in ω the support of γ. Thus, we have

$$v \equiv 0 \text{ in } \omega \times \mathbb{R} \quad \text{ and } \quad \partial_{tt}^2 v = \Delta v - f_u'(x, u_\infty(x,t))v \ .$$

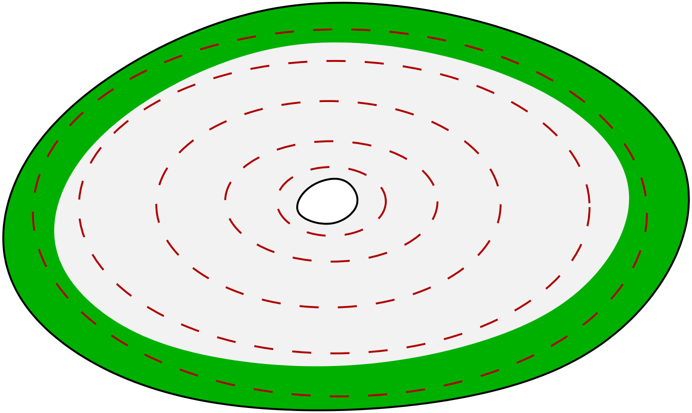
To conclude that $v \equiv 0$ everywhere, we need to use a unique continuation property.
Basically, we may extend the zone where $v \equiv 0$ through **convex surfaces**.



[N. Lerner and L. Robbiano, 1985], [L. Hörmander, 1985], [Tataru, 1996]

The unique continuation property

The stabilization holds for the domain with zero or one hole

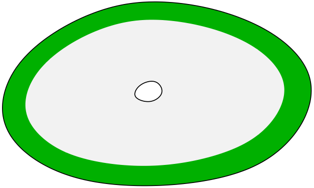


A classic result

$$\boxed{\partial_{tt}^2 u + \gamma(x)\partial_t u = \Delta u - f(x,u)}$$

Assume:

- Ω is a two dimensional convex compact domain with or without a convex hole
- $\gamma(x) \geq \alpha > 0$ in a neighborhood of the exterior boundary of Ω.
- $f(x,u)u \geq 0$ and f of degree p



Theorem

With the above assumptions, the semilinear damped wave equation is stabilized. More precisely, there exists $\lambda > 0$ such that, for any $R > 0$, there exists M_R such that

$$\|(u_0, u_1)\|_{H_0^1 \times L^2} \leq R$$
$$\Longrightarrow \quad \|(u, \partial_t u)(t)\|_{H_0^1 \times L^2} \leq M_R e^{-\lambda t} \xrightarrow[t \longrightarrow +\infty]{} 0 \ .$$

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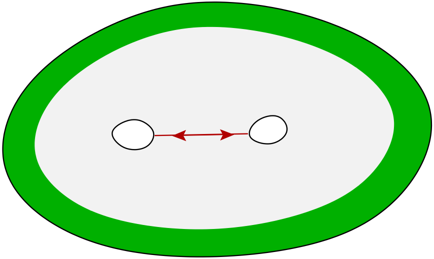
Without the geometric control condition

In some cases, the geometric control condition does not hold, but very few geodesics miss the support of the damping.



$$\|e^{At}U_0\|_{H^1\times L^2} \;\leq\; Me^{-\lambda t^{1/3}}\,\|U_0\|_{H^2\times H^1}$$

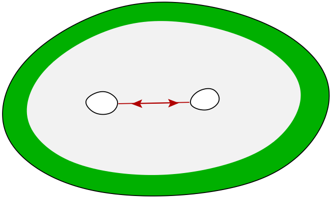
[N. Burq, 1993]
[N. Burq and M. Zworski, 2004]
[R.J. and C. Laurent, 2018]

The disk with two holes

$$\boxed{\partial_{tt}^2 u + \gamma(x)\partial_t u = \Delta u - f(x,u)}$$

Assume:

- Ω is a convex compact domain of dimension 2 with two convex holes
- $\gamma(x) \geq \alpha > 0$ in a neighborhood of the exterior boundary of Ω.
- $f(x,u)u \geq 0$ and f of degree $p < \infty$



Theorem – R.J. and C. Laurent (2018)

With the above assumptions, the semilinear damped wave equation is semi-stabilized. More precisely, $\|(u,\partial_t u)(t)\|_{H_0^1\times L^2} \xrightarrow[t\longrightarrow+\infty]{} 0$.

Moreover, there exists $\tilde\lambda$ *such that, for any* R *and* $\sigma \in (0,1]$, *there exists* $C_{R,\sigma}$ *such that*

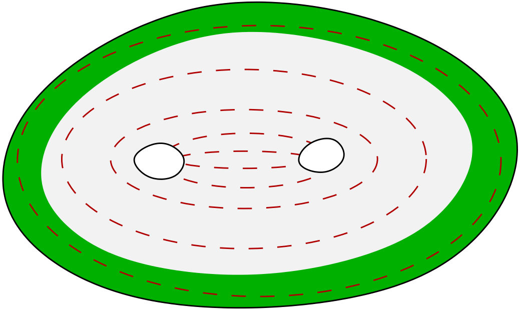
$$\|(u_0,u_1)\|_{H^{1+\sigma}\times H^\sigma} \leq R \implies \|(u,\partial_t u)(t)\|_{H_0^1\times L^2} \leq C_{R,\sigma} e^{-\sigma\tilde\lambda t^{1/3}} .$$

The disk with two holes

Main arguments:

$$\|e^{At}U_0\|_{H^1\times L^2} \le Me^{-\lambda t^{1/3}}\|U_0\|_{H^2\times H^1}$$

$$U(t) = e^{At}U_0 + \int_0^t e^{A(t-s)}F(U(s))\,\mathrm{d}s$$



The disk with two holes

The basic idea to obtain the estimate is the following.

Assume $f(u) = u^3$, for u small in $H^{1+\sigma}(\Omega)$, we have

$$\|f(u)\|_{H^1} \leq \delta \|u\|_{H^{1+\sigma}} \text{ with } \delta \text{ small .}$$

Thus,

$$e^{\sigma\lambda t^{1/3}} U(t) = e^{\sigma\lambda t^{1/3}} e^{At} U_0 + e^{\sigma\lambda t^{1/3}} \int_0^t e^{A(t-s)} F(U(s))\,\mathrm{d}s$$

and

$$\begin{aligned} &\max_{t\in[0,T]} \|e^{\sigma\lambda t^{1/3}} U(t)\|_X \\ &\qquad \leq C + \delta \max_{s\in[0,T]} \|e^{\sigma\lambda s^{1/3}} U(s)\|_X \int_0^T e^{\sigma\lambda(T^{1/3}-s^{1/3})} e^{-\lambda(T-s)^{1/3}}\,\mathrm{d}s\ . \end{aligned}$$

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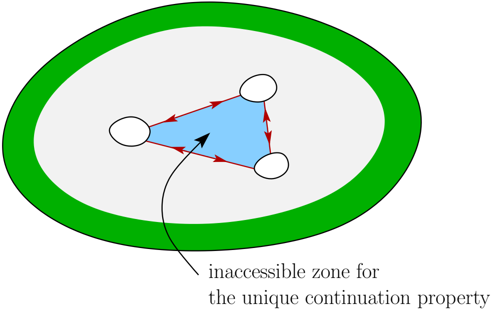
The disk with three holes

If there are three holes or more, with additional technical assumptions, we still have the decay

$$\|e^{At}U_0\|_{H^1\times L^2} \leq Me^{-\lambda t^{1/3}}\|U_0\|_{H^2\times H^1}$$

But the unique continuation property of Lerner-Robbiano-Hörmander-Tataru cannot be used:



An analytic unique continuation property

Theorem – L. Robbiano and C. Zuily (1998) L. Hörmander (1997)

Assume $\omega \neq \emptyset$ *and* $v(t) = \partial_t u(t)$ *solves*

$$v \equiv 0 \text{ in } \omega \times \mathbb{R} \quad \text{ and } \quad \partial_{tt}^2 v = \Delta v - f_u'(x, u(x,t))v \ .$$

Assume moreover that $t \mapsto f_u'(x, u(x,t))$ *is analytic then* $v \equiv 0$ *everywhere.*

[J.K. Hale and G. Raugel, 2003] let us hope that if $f(x,u)$ is analytic in u, then a function u in the attractor should be analytic in time and thus $f_u'(x, u(x,t))$ is also analytic.

An analytic unique continuation property

In the proofs of [J.K. Hale and G. Raugel, 2003], a global solution u is split between the **low-frequencies** $P_n u$ **and the high-frequencies** $Q_n u$. It is used that

$$\|e^{At}U\|_X \leq Me^{-\lambda t}\|U\|_X \implies \|e^{Q_nAQ_nt}Q_nU\|_X \leq Ne^{-\mu t}\|Q_nU\|_X \ .$$

In our case, we would like to obtain

$$\|e^{At}U\|_X \leq Me^{-\lambda t^{1/3}}\|U\|_{D(A)} \implies \|e^{Q_nAQ_nt}Q_nU\|_X \leq h(t)\|Q_nU\|_{D(A)} \ .$$

$\implies$ we adapt the ideas of [J.K. Hale and G. Raugel, 2003] but several technical problems have to be overcome.

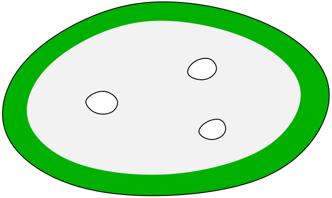
[C.J.K. Batty and Th. Duyckaerts, 2008], [A. Borichev and Y. Tomilov, 2010], [N. Anantharaman and M. Léautaud, 2014]

The disk with three holes

$$\boxed{\partial_{tt}^2 u + \gamma(x)\partial_t u = \Delta u - f(x,u)}$$

Assume:

- Ω is as opposite and the holes are not aligned and small enough
- $f(x,u)$ is analytic in u
- $f(x,u)u \geq 0$ and f of degree $p < \infty$



Theorem – R.J. and C. Laurent (2018)

With the above assumptions, the semilinear damped wave equation is semi-stabilized. More precisely, $\|(u,\partial_t u)(t)\|_{H_0^1\times L^2} \xrightarrow[t\longrightarrow+\infty]{} 0.$
Moreover, there exists $\tilde\lambda$ *such that, for any* R *and* $\sigma\in(0,1]$*, there exists* $C_{R,\sigma}$ *such that*

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Conclusion

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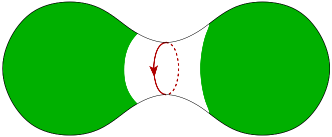
Additional results

- **Other geometries are possible**

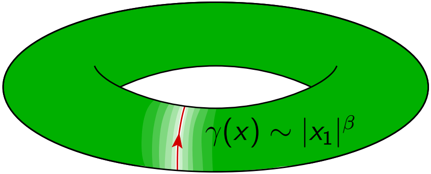
<u>The peanut of rotation</u>



$\|e^{At}U_0\|_{H^1\times L^2} \leq Me^{-\lambda\sqrt{t}}\|U_0\|_{H^2\times H^1}$

[E. Schenck, 2011]
[H. Christianson, E. Schenck, A. Vasy and J. Wunsch, 2014]

<u>The torus with degenerated damping</u>



$\|e^{At}U_0\|_{H^1\times L^2} \leq \frac{C}{(1+t)^{1+2/\beta}}\|U_0\|_{H^2\times H^1}$

[M. Léautaud and N. Lerner, 2015]

- **Higher dimension**

In dimension $d = 3$, assume that f is Sobolev-subcritical, that is of degree p with $p < 3$. It should also be possible to go to f energy-subcritical, that is of degree p with $p < 5$ by using Strichartz estimates, see [B. Dehman, G. Lebeau and E. Zuazua, 2003], [R.J. and C. Laurent, 2013]

Open problem

$$\boxed{\textbf{Global control? At least approximate global controllability?}}$$

The problem is the local controllability close to zero.

Open problem

$$U(t) = e^{At}U_0 + \int_0^t e^{A(t-s)}F(U(s))\,\mathrm{d}s$$

Main open question:

How important is the integrability of the linear decay?

For example, if the linear decay is simply

$$\|e^{At}U_0\|_{H^1\times L^2} \leq \frac{C}{\ln(2+t)}\|U_0\|_{H^2\times H^1}$$

does the asymptotic compactness hold?

Thanks for your attention!

- R.J. and C. Laurent, *Semi-uniform decay for some semilinear damped wave equations*, preprint 2018.
- R.J. and C. Laurent, *A note on the global controllability of the semilinear wave equation*, SIAM Journal on Control and Optimization n°52 (2014), pp. 439–450.
- R.J. and C. Laurent, *Stabilization for the semilinear wave equation with geometric control condition*, Analysis and PDE n°6 (2013), pp. 1089–1119.