Optimal approximation of internal controls for a wave-type problem with fractional Laplacian

Ionel Rovenţa

Department of Mathematics, University of Craiova, Romania

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The control for the continuous wave equation

Given $T \geq 2$ and $(u^0, u^1) \in L^2((0,1),\mathbb{C}) \times H^{-1}((0,1),\mathbb{C})$ there exists a control function $v \in C^0([0,T],\mathbb{C})$ such that the solution of the wave equation

$$\begin{cases} u''(t,x) - u_{xx}(t,x) = 0 & t \in (0,T),\ x \in (0,1), \\ u(t,0) = 0 & t \in (0,T), \\ u(t,1) = v(t) & t \in (0,T), \\ u(0,x) = u^0(x), \quad u'(0,x) = u^1(x) & x \in (0,1), \end{cases} \tag{1}$$

satisfies

$$u(T,x) = u'(T,x) = 0 \qquad (x \in (0,1)).$$

The discrete model of the wave equation

Let $N \in \mathbb{N}^*$ and $h = \frac{1}{N+1}$. We consider the following semi-discrete space approximation of the wave equation by the explicit finite-differences method:

$$\begin{cases} u_j''(t) - \frac{u_{j+1}(t) - 2u_j(t) + u_{j-1}(t)}{h^2} = 0 & 1 \leq j \leq N,\ t > 0, \\ u_0(t) = 0 & t \in (0,T), \\ u_{N+1}(t) = v_h(t) & t \in (0,T), \\ u_j(0) = u_j^0, \quad u_j'(0) = u_j^1 & 1 \leq j \leq N. \end{cases} \tag{2}$$

The discrete controllability problem

Given $T \geq 2$, $h > 0$ *and* $((u_j^0, u_j^1))_{1 \leq j \leq N} \in \mathbb{C}^{2N}$, *there exists a control function* $v_h \in C^0([0,T])$ *such that the solution of the system of equations* (2) *verifies*

$$u_j(T) = u_j'(T) = 0 \qquad (j = 1, 2, ..., N).$$

The convergence property does not holds

The main problem

The sequence of discrete controls $(v_h)_{h>0}$ converges to a control v of the continuous wave equation?

In general, there exist high-frequency spurious solutions generated by the discretization process that make the discrete controls diverge when the mesh-size h goes to zero. (Glowinski - Li - Lions ('90), Infante - Zuazua ('99), Micu ('03), Zuazua ('05), etc...)

Basically, this difficulty can be overcame by using an appropriate filtering technique to eliminate the short wave length components of the solutions/initial data of the discrete system, i.e. the large frequencies (of order $|n| = N$) of the discretized problem.

The aim

- To filtering the initial data in an optimal range in order to restore the uniform controllability property.

- To obtain a relation between the range of filtration and the minimal time of control.

The main result

Theorem (Lissy, Rovenţa, Math. Comp. 2019)

Let $(u^0, u^1) \in L^2((0,1),\mathbb{C}) \times H^{-1}((0,1),\mathbb{C})$ *a filtered initial data such that*

$$(u^0, u^1)_{1\leq j\leq N} = \sum_{1\leq |n|\leq f(N)} a_n \Phi^n(x).$$

Then, for any $T > \frac{4}{1-\sin\left(\frac{\pi\Gamma(f)}{2}\right)}$*, there exists a control* $v_h \in C^0([0,T],\mathbb{C})$ *bringing the solution of (2) to (0,0) such that the sequence* $(v_h)_{h>0}$ *is bounded in* $C^0([0,T],\mathbb{C})$*.*

Micu [Numer. Math. ('02)] proved that there exists regular initial data (with $f(N) = N$) for which there exists no sequence of discrete controls uniformly bounded in $L^2(0,T)$.

By adding a vanishing viscosity term the problem appearing in the high frequencies has been solved (Micu, SICON ('08)).

The continuous model for the hinged beam

The boundary controlled transversal vibrations of a 1–D beam with hinged boundary conditions are modelled by the following equation

$$\begin{cases} u''(t,x) + u_{xxxx}(t,x) = 0 & (t,x) \in (0,T) \times (0,1) \\ u(t,0) = u(t,1) = u_{xx}(t,0) = 0 & t \in (0,T) \\ u_{xx}(t,1) = v(t) & t \in (0,T) \\ u(0,x) = u^0(x) & x \in (0,1) \\ u'(0,x) = u^1(x) & x \in (0,1), \end{cases} \tag{3}$$

The vector $\begin{pmatrix} u \\ u' \end{pmatrix}$ represents the state and v is the control acting on the extremity $x = 1$ of the beam.

The same result for the beam

The convergence of the approximate boundary controls corresponding to initial data in the finite energy space cannot be guaranteed(Leon and Zuazua (ESAIM COCV, 2002)).

By adding a vanishing viscosity term the problem appearing in the high frequencies has been solved (Bugariu, Micu, Roventa, Math. Comp,. (2016)).

What about clamped beam equation?

The controllability problem associated to the beam clamped at both extremities

Let $T > 0$. For every initial data $(y_0, y_1) \in L^2(0,1) \times H^{-2}(0,1)$ there exists a control $v \in L^2(0,T)$ such that the solution of

$$\begin{cases} \ddot{y}(x,t) + \partial_x^4 y(x,t) = 0, & (x,t) \in (0,1) \times (0,T) \\ y(0,t) = y(1,t) = 0, & t \in (0,T) \\ \partial_x y(0,t) = 0, \quad \partial_x y(1,t) = v(t), & t \in (0,T) \\ y(x,0) = y_0(x), \quad \dot{y}(x,0) = y_1(x), & x \in (0,1), \end{cases} \tag{4}$$

verifies

$$y(x,T) = \dot{y}(x,T) = 0 \qquad (x \in (0,1)).$$

The main result

(Cindea, Micu, Roventa, SICON, 2017)

It is proved the convergence of the approximate boundary controls corresponding to initial data in the finite energy space, if the initial data is filtered in the range δN).

- since the discrete finite-difference operator is no more the square of finite-differences discrete Laplacian, the eigenvalues cannot be explicitly computed.
- for the eigenvales: algebraic computations combined with Rouché's theorem
- for the eigenvectors: asymptotic estimates and a discrete multiplier method.

Between the beam and the wave

- the approximation of controls for the wave equation:

$$\lambda_n = \frac{2}{h} \sin \frac{n\pi h}{2}$$

- the approximation of controls for hinged beam equation:

$$\lambda_n = \frac{4}{h^2} \sin^2 \frac{n\pi h}{2}$$

- What is between the BEAM and WAVE?
- We consider the approximation of controls a problem with fractional Laplacian:

$$\lambda_n = \left(\frac{2}{h} \sin \frac{n\pi h}{2} \right)^{\alpha}, \quad \alpha \in (1, 2).$$

Internal controls for a problem with fractional Laplacian using finite-difference method

We consider the 1-D Dirichlet-Laplace operator Δ_D on $H^2(0,1) \cap H_0^1(0,1)$ with state space $L^2(0,1)$.
$-\Delta_D : D(\Delta_D) \to L^2(0,1)$ is a positive definite operator with compact resolvent, the k-th eigenvalue is $\lambda_k = k^2\pi^2$, with associated normalized eigenvector $e_k(x) := \sqrt{2}\sin(k\pi x)$.

Let $s > 0$. Then, we introduce

$$D((-\Delta_D)^s) = \{f \in L^2(\Omega) | \sum_{k=1}^{\infty} \lambda_k^{2s} \left| < f, e_k >_{L^2(0,1)} \right|^2 < \infty\}.$$

For $f = \sum_{k=1}^{\infty} < f, e_k >_{L^2(0,1)} e_k \in D((-\Delta_D)^s)$, we define the spectral fractional Laplace operator

$$(-\Delta_D)^s f(x) = \sum_{k=1}^{\infty} \lambda_k^s < f, e_k >_{L^2(0,1)} e_k(x).$$

The continuous fractional control problem

Let $T > 0$, $\alpha \in (1,2)$ and $0 < a < b < 1$.

$$\begin{cases} u''(t,x) + (-\Delta_D)^\alpha u(t,x) = \chi_{(a,b)}(x) v(t,x), & t \in (0,T),\ x \in (0,1), \\ u(t,0) = 0, & t \in (0,T), \\ u(t,1) = 0, & t \in (0,T), \\ u(0,x) = u^0(x), \quad u'(0,x) = u^1(x), & x \in (0,1), \end{cases} \tag{5}$$

where $v \in L^2((0,T) \times (0,1))$ is a control, assumed to be localized in (a,b).

We study the existence of a control function $v \in L^2((0,T) \times (0,1))$ such that the solution of equation (5) verifies

$$u(T) = u'(T) = 0.$$

The space of initial conditions:

$$\mathcal{H} = D((-\Delta_D)^{\alpha/2})\times L^2(0,1) = \left\{ (u^0, u^1) = \sum_{k\in\mathbb{Z}^*} a_k \Phi^k(x) \,|\, \sum_{k\in\mathbb{Z}^*} |a_k|^2 < \infty \right\},$$

where

$$\Phi^k(x) = \frac{1}{sgn(k)|k|^\alpha} \left(\sin(k\pi x), -i\, sgn(k)|k|^\alpha \sin(k\pi x)\right)^T, \qquad k \in \mathbb{Z}^*.$$

Note that $(i\, sgn(k)|k|^\alpha)_{k\in\mathbb{Z}^*}$ is the family of eigenvalues of the elliptic operator associated to the continuous problem (5).

The discrete control problem

Let $N \in \mathbb{N}^*$ and $h = \frac{1}{N+1}$. We introduce

$$U(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \\ \dots \\ u_N(t) \end{pmatrix}, \quad V_h(t) = \begin{pmatrix} v_1(t) \\ v_2(t) \\ \dots \\ v_N(t) \end{pmatrix}, \quad U^0 = \begin{pmatrix} u_1^0 \\ u_2^0 \\ \dots \\ u_N^0 \end{pmatrix} \quad \text{and} \quad U^1 = \begin{pmatrix} u_1^1 \\ u_2^1 \\ \dots \\ u_N^1 \end{pmatrix}.$$

We consider the following semi-discrete space approximation:

$$\begin{cases} U''(t) + \mathcal{A}_h^\alpha U(t) = \mathcal{B}_h(V_h(t)), \quad t \in (0,T), \\ U(0,\cdot) = U^0, \quad U'(0,\cdot) = U^1. \end{cases} \tag{6}$$

$\mathcal{A}_h \in \mathcal{M}_N(\mathbb{R})$ is the discrete Laplacian matrix given by

$$\mathcal{A}_h := \frac{1}{h^2}\begin{pmatrix} 2 & -1 & 0 & \dots & \dots & \dots & 0 \\ -1 & 2 & -1 & 0 & \dots & \dots & 0 \\ 0 & -1 & 2 & -1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 0 & -1 & 2 & -1 \\ 0 & \dots & \dots & \dots & 0 & -1 & 2 \end{pmatrix}.$$

The discrete control operator $\mathcal{B}_h \in \mathcal{M}_N(\mathbb{R})$ is given by

$$\mathcal{B}_h(V_h(t)) := (\chi_{(a,b)}(jh)v_j(t))^T_{1\leq j\leq N}.$$

Uniformly boundedness of the sequence of discrete controls

Let $f : \mathbb{N}^* \to \mathbb{N}^*$ be a increasing filtration function.

Theorem

Let $T > 0$ and $\alpha \in (1, 2)$. Consider $(a_k)_{k \in \mathbb{Z}^*} \in l^2(\mathbb{Z}^*, \mathbb{C})$. Assume that

$$\lim_{N \to \infty} f(N) = \infty \text{ and } \Gamma := \limsup_{N \to \infty} \frac{f(N)}{N} < 1. \tag{7}$$

For $(u_j^0, u_j^1)_{1 \le j \le N} = \sum_{1 \le |n| \le f(N)} a_n \Phi_h^n$, there exists a uniformly bounded sequence of controls $(V_h)_{h>0}$ for the discrete control problem (6) verifying that for any $h > 0$, the solution $(u_1, \ldots u_N)$ of (6) verifies $(u_1(T), \ldots, u_N(T)) = (0, \ldots, 0)$.

The uniform controllability may fail if we do not filtrate the initial conditions

Theorem

Let $T > 0$ and $\alpha \in (1,2)$. There exists a sequence $(a_k)_{k\in\mathbb{Z}^*} \in l^2(\mathbb{Z}^*, \mathbb{C})$ for which no sequence of controls $(V_h)_{h>0}$ for the discrete control problem (6) with initial condition $(u_j^0, u_j^1)_{1\le j\le N} = \sum_{1\le|n|\le N} a_n \Phi_h^n$ can be uniformly bounded in h.

The convergence result

Theorem

Let $(u^0, u^1) = \sum_{n\in\mathbb{Z}^*} a_n \Phi^n \in \mathcal{H}$ *be the initial datum for the continuous problem* (5) *and let us consider the initial datum of the semi-discrete problem* (6)

$$(u_j^0, u_j^1)_{1\le j\le N} = \sum_{1\le |n|\le f(N)} a_n \tilde{\Phi}^n(h), \tag{8}$$

where $\tilde{\Phi}^n(h)$ *is the discretization of the eigenfunction* Φ^n*.*
Let $v \in L^2((0,T)\times(0,1))$ *be a weak limit of a subsequence of the bounded controls* $(V_h)_{h>0}$*, with initial datum given in* (8)*. Then* v *is a control for the continuous problem* (5)*.*

A variational lemma for the continuous moment problem

Lemma

Problem (5) is controllable if and only if, for any initial datum $(u^0, u^1) \in \mathcal{H}$ decomposed as in (14), there exists $v \in L^2((0,T) \times (0,1))$ such that

$$\langle \overline{w}^1, u^0 \rangle_{D((-\Delta_D)^{\alpha/2})', D((-\Delta_D)^{\alpha/2})} - \int_0^1 u^1(x) \overline{w}^0(x)\, dx = \int_a^b \int_0^T v(t,x) \overline{w}(t,x)\, dt\, dx, \quad (9)$$

for all $(w^0, w^1) \in \mathcal{H}' = L^2(0,1) \times D((-\Delta_D)^{\alpha/2})'$, where w is the the solution of the adjoint problem

$$\begin{cases} w''(t,x) + (-\Delta_D)^\alpha w(t,x) = 0 & t \in (0,T),\ x \in (0,1), \\ w(T,x) = w^0(x), \quad w'(T,x) = w^1(x) & x \in (0,1). \end{cases} \quad (10)$$

The continuous moment problem

Theorem

The initial datum $(u^0, u^1) \in \mathcal{H}$ *is controllable if and only if there exists* $v \in L^2((0,T)\times(0,1))$ *such that*

$$\int_a^b \int_0^T v(t,x) e^{-i\nu_n t} \Phi^n(x)\, dt\, dx = 2a_n \qquad (n \in \mathbb{Z}^*), \qquad (11)$$

where $\nu_n = \pi^\alpha sgn(n)|n|^\alpha$ *and* $(i\nu_n)_{n\in\mathbb{Z}^*}$ *are the eigenvalues of the elliptic operator associated to the continuous problem* (10).

The discrete adjoint system

We consider $\mathcal{L}_h^\alpha \in \mathcal{M}_{2N\times 2N}(\mathbb{R})$ given by

$$\mathcal{L}_h^\alpha := \begin{pmatrix} 0 & -I_n \\ \\ \mathcal{A}_h^\alpha & 0 \end{pmatrix}, \tag{12}$$

where $I_n \in \mathcal{M}_n(\mathbb{R})$ is the identity matrix of size n.

For $Z(t) = (W(t), W'(t))^T$ and $Z^T = (W^0, W^1)^T$, we consider the discrete adjoint system

$$\left\{ \begin{array}{l} Z'(t) + \mathcal{L}_h^\alpha Z(t) = 0, \ t > 0, \\ Z(T) = Z^T. \end{array} \right. \tag{13}$$

The eigenvalues and eigenvectors for the discrete adjoint system

The eigenvalues of the discretization matrix $\mathcal{L}_h^\alpha$ are given by the family $(i\,\lambda_n(h))_{1\leq|n|\leq N}$, where

$$\lambda_n(h) = sgn(n)\left(\frac{2}{h}\sin\left(\frac{|n|\pi h}{2}\right)\right)^\alpha, \qquad 1\leq|n|\leq N,$$

and the corresponding eigenvectors are

$$\Phi_h^n = \frac{1}{\lambda_n(h)}\begin{pmatrix} \varphi_h^n \\ -i\,\lambda_n(h)\varphi_h^n \end{pmatrix}, \qquad 1\leq|n|\leq N,$$

where

$$(\varphi_h^n)_{1\leq|n|\leq N} = (\sin(n\pi h), \sin(2n\pi h), \cdots, \sin(Nn\pi h)) \in \mathbb{C}^N.$$

The discrete moment problem

Theorem

The discrete problem (2) is controllable if and only if, for any initial datum $Z^0 = \sum_{1\le|k|\le N} \alpha_k \Phi_h^k \in \mathbb{C}^{2N}$ *there exists a function* $V_h \in \left(L^2(0,T)\right)^N$ *such that*

$$h \sum_{1\le j\le N} \chi_{(a,b)}(jh) \sin(j\pi nh) \int_0^T e^{-i\lambda_n t} v_j(t)\, dt = 2\alpha_n, \qquad 1 \le |n| \le N.$$

The biorthogonals

A sequence $(\theta_m)_{1\le|m|\le N} \subset L^2\left(-\frac{T}{2},\frac{T}{2}\right)$ is *biorthogonal to the family of exponential functions* $\left(e^{i\lambda_n t}\right)_{1\le|n|\le N}$ *in* $L^2\left(-\frac{T}{2},\frac{T}{2}\right)$ if

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} \theta_m(t) e^{i\overline{\lambda}_n t} dt = \delta_{mn} \qquad (1 \le |m|, |n| \le N) .$$

An explicit formula for the discrete control

Once we have a biorthogonal sequence $(\theta_m)_{1\leq|m|\leq N}$ to the family $(e^{i\lambda_n t})_{1\leq|n|\leq N}$ in $L^2\left(-\frac{T}{2},\frac{T}{2}\right)$ we can construct a control as:

$$V_h(t)=\sum_{1\leq|n|\leq f(N)}\frac{2a_nB_h^n}{||B_h^n||^2}e^{-i\lambda_n\frac{T}{2}}\Theta_n\left(t-\frac{T}{2}\right),$$

where B_h^n is given by

$$B_h^n=h(\chi_{(a,b)}(jh)\sin(j\pi nh))_{1\leq j\leq N}.$$

Construction of the biorthogonals

We construct and evaluate an explicit biorthogonal sequence to the family $(e^{i\lambda_n t})_{1\leq |n|\leq N}$ in $L^2\left(-\frac{T}{2},\frac{T}{2}\right)$ in the following way:

1. We construct an entire function P_m, with the property that $P_m(\lambda_n)=\delta_{mn}$.

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2. We obtain an estimate of the product P_m on the real axis.

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1. We construct an entire function P_m, with the property that $P_m(\lambda_n) = \delta_{mn}$.
2. We obtain an estimate of the product P_m on the real axis.
3. We construct a smart multiplier M_m with rapid decay on the real axis such that $P_m M_m \in L^1(R) \cap L^2(R)$ and $M_m(\lambda_n) = \delta_{mn}$.

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4. The Fourier transform of the entire function $\psi_m(z) := P_m(z)M_m(z)$ gives the element θ_m of a biorthogonal sequence to the family $(e^{i\lambda_n t})_{1\le|n|\le N}$ in $L^2\left(-\frac{T}{2},\frac{T}{2}\right)$.

Open problems

- to consider non-uniform meshes
- to obtain general controllability results depending on gap conditions between the eigenvalues


Thank
you