Control of quantum states by quasi-adiabatic motions of walls

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The ideal model

In 2016, D. Turaev proposes a smart way to permute eigenstates on the segment.

Consider the operators $H_{\alpha,a}$ defined by

$$H_{\alpha,a}u = -\partial^2_{xx}u \quad \text{on } (0,a)\cup(a,1).$$

on

$$\{u \in H^2((0,a)\cup(a,1)) \text{ such that } u(0)=u(1)=0,$$
$$u(a^-)=u(a^+) \text{ and } \alpha u(t,a)+(1-\alpha)(u'(a^+)-u'(a^-))=0\}$$

Could be seen as

$$H_{\alpha,a}u = -\partial^2_{xx}u + \frac{\alpha}{1-\alpha}\delta_{x=a}$$

An adiabatic theorem

Apply adiabatic theory to $i\partial_t u = -\partial^2_{xx} u + \frac{\alpha(t)}{1-\alpha(t)}\delta_{x=a(t)}$?

Theorem – Bornemann (1998)

Let $H(t)$ be a family of positive self-adjoint operator on X with same domain such that $H(t) : D(H^{1/2}(t)) \to D(H^{-1/2}(t))$ is of class $\mathcal{C}^2$.
Let $t \in [0,1] \mapsto \lambda(t)$ be a continuous curve of simple isolated eigenvalue of $H(t)$ with an associated family of orthogonal projections $P \in \mathcal{C}^1([0,1], \mathcal{L}(X))$
For any initial data $u_0 \in X^{1/2}$ with $\|u_0\|_X = 1$ and for any sequence $\epsilon \to 0$, the solutions $u_\epsilon \in \mathcal{C}^0([0,1], X^{1/2}) \cap \mathcal{C}^1([0,1], X^{-1/2})$ of

$$i\partial_t u_\epsilon(t) = H(\varepsilon t) u_\epsilon(t) \qquad u_\epsilon(0) = u_0$$

satisfy after a time $T = 1/\varepsilon$

$$\langle P(1) u_\epsilon(1/\varepsilon) | u_\epsilon(1/\varepsilon)\rangle_X \xrightarrow[\epsilon \longrightarrow 0]{} \langle P(0) u_\epsilon(0) | u_\epsilon(0)\rangle_X$$

Permutation of quantum states

Consider a potential wall

$$V(x,t) \;=\; I(t)\,\rho^{\eta(t)}(x-a(t)),$$

with $\rho^\eta = \eta\,\rho(\eta\,\cdot\,)$ being an approximation of the Dirac distribution, and the Schrödinger equation

$$\begin{cases} i\partial_t u(t,x) = -\partial^2_{xx} u(t,x) + V(t,x)u(t,x) & x\in(0,1),\;\; t\in(0,T], \\ u(t,0)=u(t,1)=0, & t\in[0,T] \end{cases}$$

generating a unitary propagator Γ_s^t in $L^2((0,1),\mathbb{C})$.

Theorem – A. Duca, R.J. & Dmitry Turaev (2019)

Let $\sigma:\mathbb{N}\to\mathbb{N}$ be any permutation realizable by the ideal model of Dmitry Turaev and let $N\in\mathbb{N}$. Then there exists paths $\eta(t)$, $a(t)$ and $I(t)$ such that for all $k\leq N$, there exists $\alpha_k\in\mathbb{C}$ with $|\alpha_k|=1$ such that

$$\Big\|\;\Gamma_0^T\sin(k\pi x)\;-\;\alpha_k\sin(\sigma(k)\pi x)\;\Big\|_{L^2}\leq\varepsilon\;.$$

Approximate controllability

Consider several potenial walls

$$V(t,x) \;=\; \sum_{j=1}^{J} I_j(t)\rho^{\eta_j(t)}(x-a_j(t))$$

and the Schrödinger equation

$$\begin{cases} i\partial_t u(t,x) = -\partial^2_{xx}u(t,x) + V(t,x)u(t,x) & x\in(0,1),\; t\in(0,T], \\ u(t,0)=u(t,1)=0, & t\in[0,T] \end{cases}$$

generating a unitary propagator Γ_s^t in $L^2((0,1),\mathbb{C})$.

Theorem – A. Duca, R.J. & Dmitry Turaev (2019)

Let $\varepsilon>0$ and let $u_{\rm i}$ and $u_{\rm f}$ in $L^2((0,1),\mathbb{C})$ with $\|u_{\rm i}\|_{L^2}=\|u_{\rm f}\|_{L^2}$.
There exist $J\in\mathbb{N}$, $T>0$ and smooth functions $\{\eta_j\}_{j\le J},\{I_j\}_{j\le J}\subset\mathcal{C}^\infty([0,T],\mathbb{R}^+)$ and $\{a_j\}_{j\le J}\subset\mathcal{C}^\infty([0,T],(0,1))$ such that

$$\|\Gamma_0^T u_{\rm i} \;-\; u_{\rm f}\|_{L^2} \;\le\; \varepsilon\ .$$