An optimal control problem governed by heat equation with non convex constraints applied to selective laser melting process

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Joint work with Serge Nicaise and Luc Paquet

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VIII Partial differenial equations, optimal design and numerics

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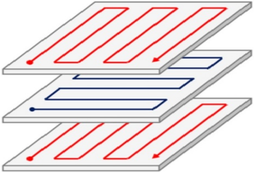
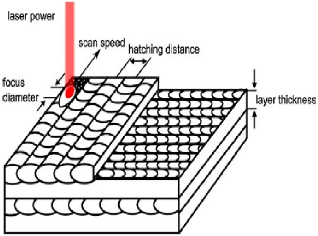
What is selective laser melting ?





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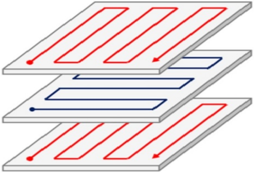

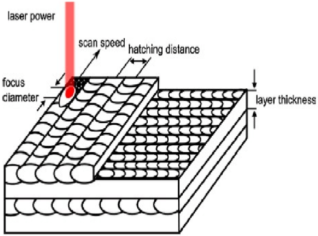

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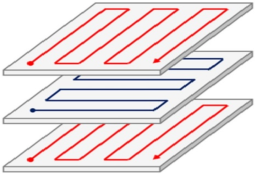

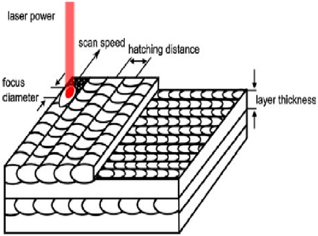

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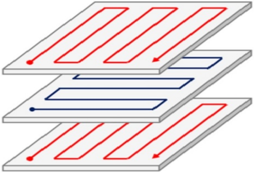
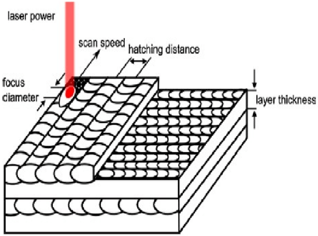




- A 3d object is sliced into thin layers,
- one layer of powder is spread on the build plateform,
- a high energy produced by a Gaussian laser beam scans the surface of the powder bed following a given trajectory,
- 3d object created by melting metallic powder by a layer-by-layer process.

Main motivation : Control temperature gradient in laser melting process by acting on laser trajectory

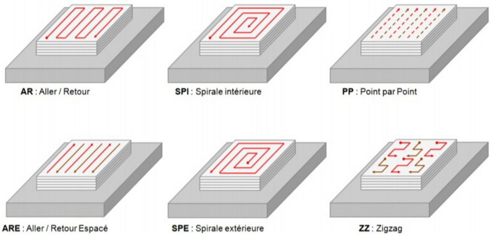
- Thermal gradients, residual stresses, cracks and deformations has a strong dependence on laser scan strategy.

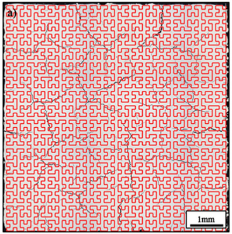
- Simulation results in literature show that thermal gradients are very disparate from one type of trajectory to another one.

A. Haidar et al., Materials Science and Engineering A. (2018).

B. Cheng et al., Additive Manufacturing. (2016).

Different type of trajectories studied in the literature





L. Van Belle, Thesis. (2013).

S. Catchpole-Smith et al., Additive Manufacturing. (2017).

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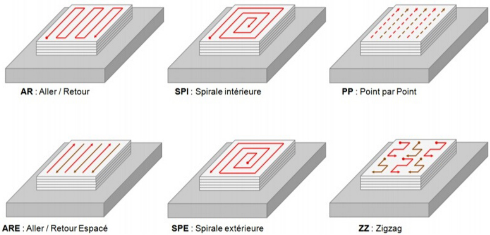

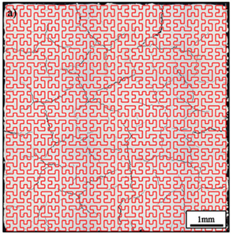

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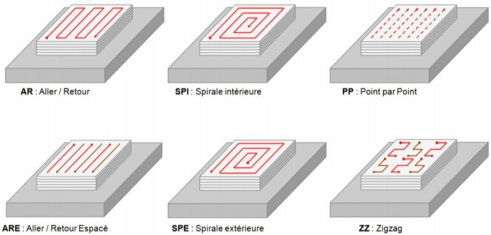

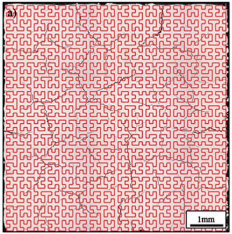

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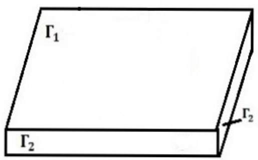
Main objective : Develop a mathematical optimization model to find an optimal trajectory minimizing thermal gradients

Content

One layer model :

$$(\mathcal{P}_\gamma)\quad \begin{cases} \rho\, c\, \partial_t y - \kappa\, \Delta y = 0 & \text{in} \quad Q = \Omega \times (0, T)\,, \\ -\kappa \dfrac{\partial y}{\partial \nu} = h\, y - g_\gamma & \text{on} \quad \Sigma_1 = \Gamma_1 \times (0, T)\,, \\ -\kappa \dfrac{\partial y}{\partial \nu} = h\, y & \text{on} \quad \Sigma_2 = \Gamma_2 \times (0, T)\,, \\ y = 0 & \text{on} \quad \Sigma_3 = \Gamma_3 \times (0, T)\,, \\ y(x, 0) = y_0(x) & \text{for} \quad x \in \Omega. \end{cases}$$



- $\Omega \subset \mathbb{R}^3$ bounded lipschitz domain
- $\partial\Omega = \overline{\Gamma_1} \cup \overline{\Gamma_2} \cup \overline{\Gamma_3}$
- $y_0 \in L^2(\Omega)$

- ν outward normal vector
- ρ, c, κ and h positive constants

- $$g_\gamma(x, t) := \alpha \frac{2P}{\pi r^2} \exp\left(-2 \frac{|\, x - \gamma(t)\,|^2}{r^2} \right), \quad \forall (x, t) \in \Sigma_1$$
  - $\gamma : t \in [0, T] \to \Gamma_1$ the laser path which represents the displacement of the laser beam center on Γ_1 with respect to time
  - r radius of the laser spot
  - α and P positive constants

The state equation

Theorem

For all $\gamma \in H^1(0,T;\mathbb{R}^2)$ the state equation $(\mathcal{P}_\gamma)$ has a unique weak solution $y \in W(0,T) \cap L^\infty(0,T;L^2(\Omega))$.

$$H^1_{\Gamma_3}(\Omega) := \{y \in H^1(\Omega) \text{ such that } y_{|_{\Gamma_3}} = 0\},$$

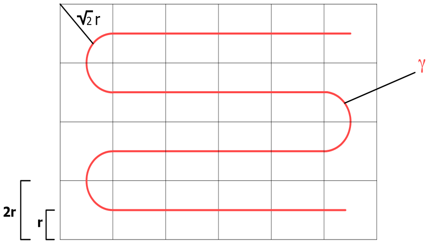
$$W(0,T) := \{y \in L^2(0,T;H^1_{\Gamma_3}(\Omega)) \text{ such that } \frac{dy}{dt} \in L^2(0,T;(H^1_{\Gamma_3}(\Omega))^*)\}.$$

Remark

If $y_0 \in C(\bar{\Omega})$ and $y_0 = 0$ on Γ_3, then $(\mathcal{P}_\gamma)$ has a unique solution in $W(0,T) \cap C(\bar{Q})$.

F.Tröeltzsch, Graduate studies in Mathematics, AMS. (2010)

The control γ has a space-filling curve property



- $R(\gamma) := \gamma([0, T])$
- $\epsilon \geq r$
- $\Gamma_{1,-\epsilon} := \{x \in \Gamma_1; dist\,(x, \partial\Gamma_1) \geq \epsilon\}$
- $R_{\sqrt{2}\epsilon}(\gamma) := \{x \in \Gamma_1;\ dist\,(x, R(\gamma)) \leq \sqrt{2}\epsilon\}$

The control γ satisfy :

$$R(\gamma) \subset \Gamma_{1,-\epsilon} \text{ and } R_{\sqrt{2}\epsilon}(\gamma) = \Gamma_1$$

PDE-constraints optimization problem

Minimize $\quad J(y, \gamma) := \frac{1}{2} \| \nabla y \|^2_{L^2(Q)} + \frac{\lambda_Q}{2} \| y - y_Q \|^2_{L^2(Q)} + \frac{\lambda_\gamma}{2} \| \gamma \|^2_{H^1(0,T;\mathbb{R}^2)}$

subject to $\quad (\mathcal{P}_\gamma) \quad$ PDE-constraint

$R(\gamma) \subset \Gamma_{1,-\epsilon} \quad R_{\sqrt{2}\epsilon}(\gamma) = \Gamma_1$ non convex constraints describing the **space filling curve** property of the control γ

and $\quad | \gamma'(t) | \leq c \;$ a.e. $\; t \in [0, T] \quad$ convex constraint

($\lambda_Q \geq 0$, $\lambda_\gamma > 0$, $c > 0$ fixed)

Two sets of optimization variables

$\gamma \in H^1(0, T; \mathbb{R}^2)$ control variables

$y \in L^2(0, T; H^1_{\Gamma_3}(\Omega))$ state variables

coupled through the PDE ($\mathcal{P}_\gamma$).

PDE-constraints optimization problem

$\rightsquigarrow$ The set of admissible controls :

$$U_{ad} := \{\gamma \in H^1(0,T\,;\mathbb{R}^2)\,;\ R(\gamma) \subset \Gamma_{1,-\epsilon},\ R_{\sqrt{2}\epsilon}(\gamma) = \Gamma_1$$
$$\text{and } \exists\, c > 0 \text{ s.t } \mid \gamma'(t) \mid \leq c \text{ a.e. } t \in [0,T]\}$$

Proposition

U_{ad} is a weakly sequentially closed subset of $H^1(0,T;\mathbb{R}^2)$.

- $H^1(0,T;\mathbb{R}^2) \overset{c}{\hookrightarrow} C([0,T];\mathbb{R}^2)$

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Non linear control

$$g_\gamma(x,t) := \alpha \frac{2P}{\pi r^2} \exp\left(-2\frac{\mid x-\gamma(t) \mid^2}{r^2}\right),\ \forall (x,t) \in \Sigma_1$$

PDE-constraints optimization problem

AIM :

We want to prove that the optimal control problem

$$\text{(OC)} \qquad \min_{\gamma \in U_{ad}} J(y(\gamma), \gamma)$$

admits at least one optimal control $\bar{\gamma} \in U_{ad}$.

$\rightsquigarrow$ $y(\gamma)$ denotes the solution of $(\mathcal{P}_\gamma)$ associated with the control γ.

Analysis of the control-to-state mapping

Proposition

The control-to-state mapping $G : \gamma \in U_{ad} \longmapsto y(\gamma) \in W(0,T)$ *is weakly sequentially continuous.*

Sketch of the proof :

- Let $(\gamma_n)_{n\in\mathbb{N}} \subset U_{ad}$ be a weakly convergent sequence in $H^1(0,T;\mathbb{R}^2)$ to some $\gamma \in U_{ad}$,

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* $y = y(\gamma)$?
* For $\frac{1}{2} < s < 1$, $W(0,T) \overset{c}{\hookrightarrow} L^2(0,T;H^s(\Omega)) \to L^2(0,T;H^{s-1/2}(\Gamma)) \hookrightarrow L^2(\Sigma) := L^2(0,T;L^2(\Gamma))$,

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* The traces on $\Sigma := \Sigma_1 \cup \Sigma_2 \cup \Sigma_3$ of $(y(\gamma_{n_j}))_{j\in\mathbb{N}}$ strongly converge to y in $L^2(\Sigma)$,

Analysis of the control-to-state mapping

* we can pass to the limit in

$$
\begin{aligned}
&\rho\, c \int_0^T \langle \frac{dy}{dt}(\gamma_{n_j})(.,t)\,,\ v\rangle_{(H^1_{\Gamma_3}(\Omega))^*,H^1_{\Gamma_3}(\Omega)}\,\varphi(t)dt \\
&+ h \int_0^T \int_{\Gamma_1\cup\Gamma_2} y(\gamma_{n_j})(x,t)\,v(x)\,\varphi(t)\,dS(x)\,dt + \kappa \int_0^T \int_\Omega \nabla y(\gamma_{n_j})(x,t)\cdot\nabla v(x)\varphi(t)\,dx\,dt \\
&- \alpha \frac{2P}{\pi R^2} \int_0^T \int_{\Gamma_1} \exp\left(-2\frac{\mid x-\gamma_{n_j}(t)\mid^2}{R^2}\right) v(x)\,\varphi(t)\,dS(x)\,dt = 0,
\end{aligned}
$$

$\forall v \in H^1_{\Gamma_3}(\Omega)$, $\forall \varphi \in L^2(0,T)$.

* Any subsequence of $(y(\gamma_n))_{n\in\mathbb{N}}$ contains a further subsequence which converges weakly to $y(\gamma)$ in $W(0,T)$.

* Thus $(y(\gamma_n))_{n\in\mathbb{N}}$ itself converges weakly to $y(\gamma)$ in $W(0,T)$.

Existence of an optimal control

The reduced cost functional is defined by

$$\begin{array}{rccl} \hat{J}: & U_{ad} & \longrightarrow & L^2(0,T;H^1_{\Gamma_3}(\Omega)) \\ & \gamma & \longmapsto & J(G(\gamma),\gamma). \end{array}$$

Theorem

The optimal control problem (OC) *admits at least one optimal control* $\bar{\gamma} \in U_{ad}$.

$\rightsquigarrow$ $\bar{\gamma}$ denote a local solution and $\bar{y} := G(\bar{\gamma})$ the associated state.

Sketch of the proof

* $\hat{J}(\gamma) \geq 0$, the infimum

$$L := \inf_{\gamma \in U_{ad}} \hat{J}(\gamma),$$

exists.

* There exists a minimizing sequence $(\gamma_n)_{n \in N} \subset U_{ad}$ such that $\hat{J}(\gamma_n) \to L$ as $n \to \infty$,
* $(\gamma_n)_{n \in \mathbb{N}}$ being bounded in $H^1(0, T; \mathbb{R}^2)$, possesses a subsequence $(\gamma_{n_j})_{j \in \mathbb{N}}$ weakly convergent to some element $\bar{\gamma} \in U_{ad}$,
* $\hat{J}(\bar{\gamma}) \leq \lim_{j \to \infty} \inf \hat{J}(\gamma_{n_j}) = L$,
* $L \leq \hat{J}(\bar{\gamma})$,
* since $\bar{\gamma} \in U_{ad}$, $L = \hat{J}(\bar{\gamma})$.

Necessary conditions of optimality

It is well known that an optimal control $\bar{\gamma}$ minimizing $\hat{J}$ in U_{ad} has to obey the variational inequality

$$\hat{J}'(\bar{\gamma})(\gamma - \bar{\gamma}) \geq 0 \text{ for all } \gamma \in U_{ad},$$

provided that $\hat{J}$ is **Gâteaux differentiable** at $\bar{\gamma}$ and U_{ad} **convex**.

In our case :

- $\hat{J}$ is Fréchet differentiable

Proposition

The control-to-state mapping
$G : \gamma \in H^1(0,T;\mathbb{R}^2) \longmapsto y(\gamma) \in L^2(0,T;H^1_{\Gamma_3}(\Omega))$ *is Fréchet differentiable.*

Proposition

The mapping $\gamma \in H^1(0,T;\mathbb{R}^2) \mapsto \hat{J}(\gamma) := J(G(\gamma),\gamma) \in \mathbb{R}$ *is Fréchet differentiable.*

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Proposition

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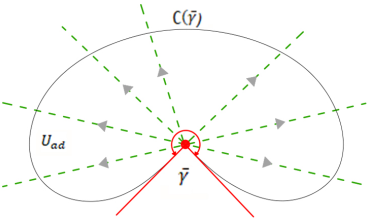
- **But U_{ad} is not convex $\Rightarrow$ we must introduce the cone of amissible directions at the point $\bar{\gamma}$**

Necessary conditions of optimality

Theorem (necessary conditions)

Let $\bar{\gamma}$ *be the solution of the optimal control problem* (OC) *then*

$$\hat{J}'(\bar{\gamma}) \cdot \delta\gamma \geq 0 \qquad \forall \delta\gamma \in C(\bar{\gamma}).$$



$C(\bar{\gamma})$ is the cone of admissible directions at $\bar{\gamma}$.

J. Jahn, Introduction to the theory of nonlinear optimization (2007).

Necessary conditions of optimality

The adjoint system :

$$
\text{(ad)} \qquad \begin{cases}
\rho\, c\, \partial_t p + \kappa\, \Delta p = \Delta \bar{y} - \lambda_d(\bar{y} - y_d) & \text{in} \quad Q, \\
\kappa \frac{\partial p}{\partial \nu} + h\, p = \frac{\partial \bar{y}}{\partial \nu} & \text{on} \quad \Sigma_1, \\
\kappa \frac{\partial p}{\partial \nu} + h\, p = \frac{\partial \bar{y}}{\partial \nu} & \text{on} \quad \Sigma_2, \\
p = 0 & \text{on} \quad \Sigma_3, \\
p(., T) = 0 & \text{in} \quad \Omega,
\end{cases}
$$

Theorem

The adjoint system (ad) *has a unique weak solution in* $W(0, T)$.

Necessary conditions of optimality

Theorem

If $\bar{\gamma} \in U_{ad}$ *is an optimal control with associated state* $\bar{y}$*, and* $p \in W(0, T)$ *the corresponding adjoint state that solves* (ad)*, then the variational inequality*

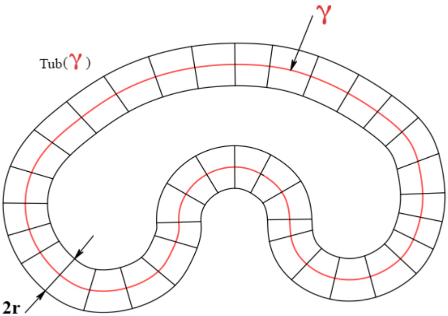
$$\lambda_\gamma \int_0^T \bar{\gamma}(t) \cdot (\gamma - \bar{\gamma})(t)dt + \lambda_\gamma \int_0^T \bar{\gamma}(t) \cdot (\gamma - \bar{\gamma})(t)dt$$
$$+2ac_R \int \int_{\Sigma_1} \exp(w(\bar{\gamma})(x,t))\tilde{\gamma}(x,t) \cdot (\gamma - \bar{\gamma})(t)p(x,t)dS(x)dt \geq 0$$

holds for $\gamma \in \{\bar{\gamma} + C(\bar{\gamma})\}$.

T-M. A., S. Nicaise and L. Paquet, submitted. (2019).

The penalized control problem

- Relax the non convex constraints $R(\gamma) \subset \Gamma_{1,-\epsilon}$ and $R_{\sqrt{2}\epsilon}(\gamma) = \Gamma_1$ by adding a penalization term in the cost functional



- If γ is a submanifold of Γ_1 then $2r \times length(\gamma) - area(Tub(\gamma)) = 0$

$$2r \int_0^T \mid \gamma'(t) \mid dt - \mid \Gamma_1 \mid = 0$$

The penalized optimal control problem

$$\underset{\gamma \in U_{ad}^{p}}{\text{Minimize}} \quad J^{\delta}(y(\gamma),\gamma) := \frac{1}{2}\,\|\nabla y(\gamma)\|^2_{L^2(Q)} + \frac{\lambda_Q}{2}\,\|y(\gamma) - y_Q\|^2_{L^2(Q)}$$

$$+\tfrac{\lambda_\gamma}{2}\,\|\gamma\|^2_{H^2(0,T;\mathbb{R}^2)} + \frac{1}{\delta^2}\left(2r\int_0^T \sqrt{|\gamma'(t)|^2 + \delta^2}\,dt - |\Gamma_1|\right)^2$$

subject to $(\mathcal{P}_\gamma)$ PDE-constraint

$$U_{ad}^{p} = \{\gamma \in H^2(0,T;\Gamma_1)\,;\ \text{exists } c > 0 \text{ s.t } |\gamma'(t)| \leq c \text{ a.e. } t \in [0,T]$$
$$\text{and } 2r\int_0^T |\gamma'(t)| \leq |\Gamma_1| + 2diam(\Gamma_1)r\}$$

$\rightsquigarrow$ U_{ad}^{p} is closed and convex $\Rightarrow$ weakly closed

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$$\text{and}\ 2r\int_0^T |\gamma'(t)| \leq |\Gamma_1| + 2diam(\Gamma_1)r\}$$

$\rightsquigarrow$ U_{ad}^p is closed and convex $\Rightarrow$ weakly closed

Theorem

The penalized optimal control problem admits at least one optimal control $\bar{\gamma}^\delta \in U_{ad}^p$.

The penalized control problem

Theorem (A.-Nicaise-Paquet, 2019)

Let $\bar{\gamma}^{\delta}$ *be an optimal control of the penalized control problem. If there exists* $\gamma_1 \in U_{ad}^{p}$ *and a constant* c *independant of* δ *such that* $\hat{J}^{\delta}(\gamma_1) \leq c$*, then there is a subsequence* $(\bar{\gamma}^{\delta_j})_{j\in\mathbb{N}}$ *such that* $\bar{\gamma}^{\delta_j}$ *converges strongly to some* $\gamma \in H^1(0,T;\mathbb{R}^2)$ *as* $j \to +\infty$ *and*

$$2r\int_0^T \mid \gamma'(t) \mid dt = \mid \Gamma_1 \mid .$$

Conclusion

- Work in progress :
  - Numerical tests using the projected gradient method
  - convergence analysis
- Further work :
  - Complex geometries

Thank you for your attention