Exact controllability for networks of nonlinear strings with nonlocal coupling conditions

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Joint work with
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PDE, Design and Numerics Benasque VIII

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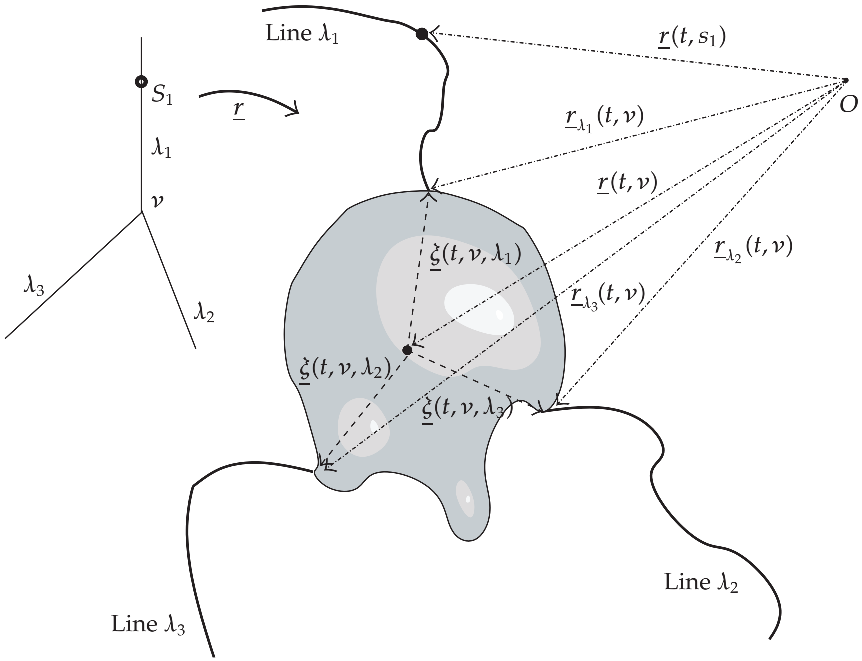
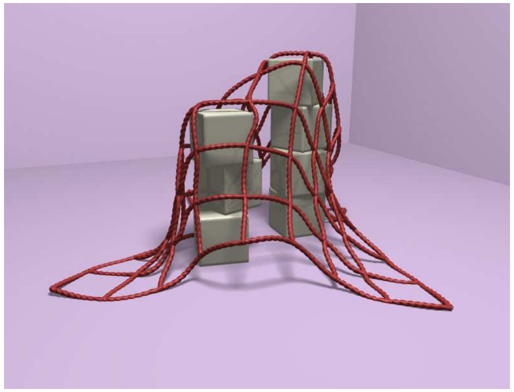
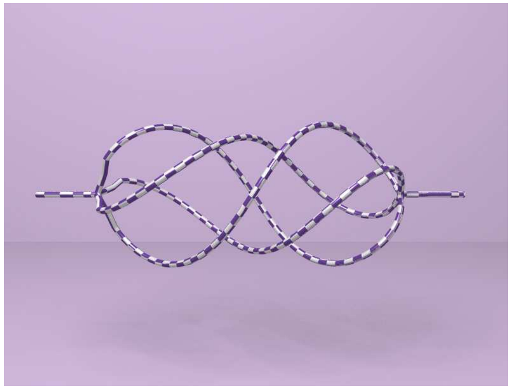
CONTINUOUS OPTIMIZATION

Networks of strings and masses







Cf. Teschner

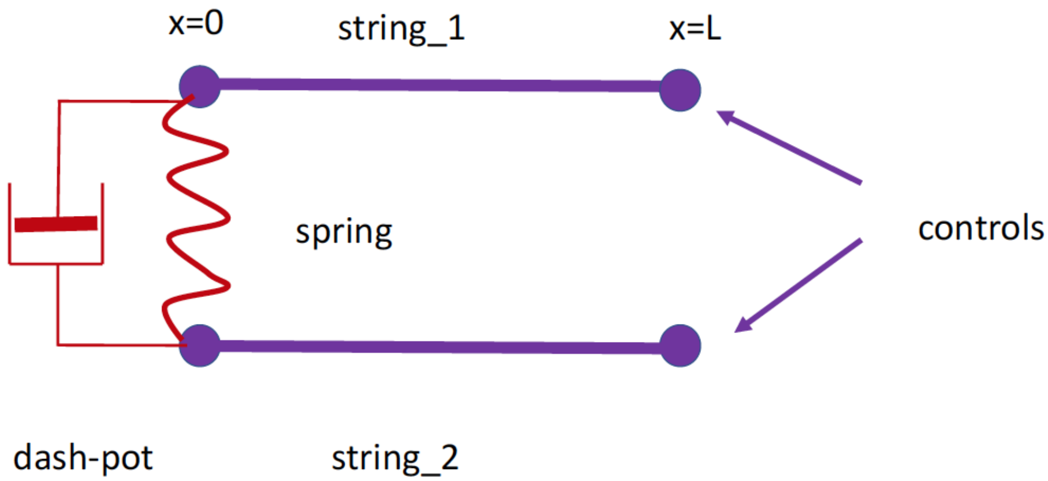
Cosserat theory of (strings, beams) rods
Antman et al., Grautus et al.........
No control theory so far!
See, however, Ch. Strohmeyer PhD 2017 at FAU
And Charlotte Rodriguez current PhD student in ConFlex see Wednesday morning

Linear strings and masses: Hansen Zuazua 1995 -> asymmetric spaces

Two strings coupled via springs

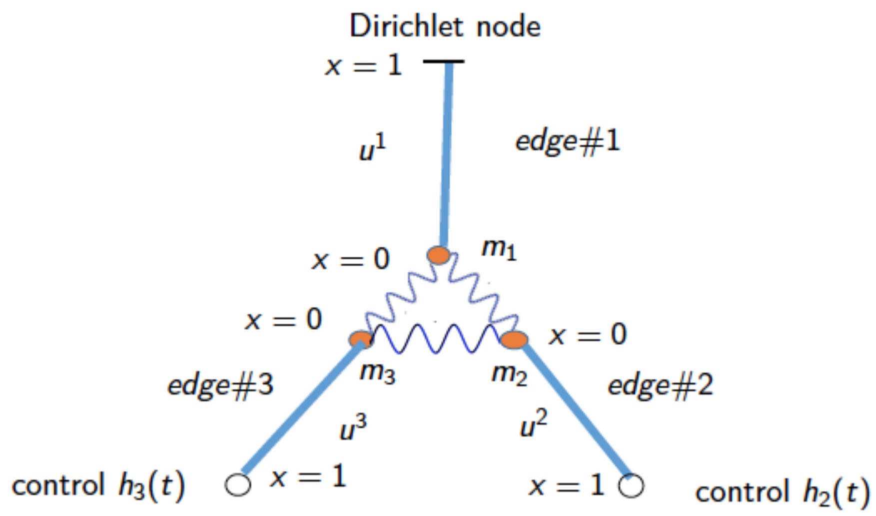
$$\begin{cases} u^i_{tt} - K_i(u^i_x)_x = 0, \quad 0 < x < L, & i = 1, 2, \\ m_0 u^1_{tt}(0,t) = K_1(u^1_x)(0,t) - \kappa(u^1(0,t) - u^2(0,t)), \\ m_0 u^2_{tt}(0,t) = K_2(u^2_x)(0,t) + \kappa(u^1(0,t) - u^2(0,t)), \\ m_L u^1_{tt}(L,t) = -K_1(u^1_x((L,t) + h^1(t), \\ m_L u^2_{tt}(L,t) = -K_2(u^2_x)(L,T) + h^2(t), \\ u^i(x,0) = \phi^i_0(x), u^i_t(x,0) = \psi^i_1(x), 0 < x < L, i = 1, 2. \end{cases}$$

Two strings coupled via a Maxwell element



$$
\begin{cases}
u^i_{tt} - K_i(u^i_x)_x = 0, \quad 0 < x < L, \\
m_0 u^1_{tt}(0,t) = K_1(u^1_x)(0,t) - \kappa(u^1(0,t) - u^2(0,t)) - \tau(u^1_t(0,t) - u^2_t(0,t)), \\
m_0 u^2_{tt}(0,t) = K_2(u^2_x)(0,t) + \kappa(u^1(0,t) - u^2(0,t)) + \tau(u^1_t(0,t) - u^2_t(0,t)), \\
m_L u^1_{tt} = -K_1(u^1_x)(L,t) + h^1(t), \\
m_L u^2_{tt} = -K_2(u^2_x)(L,t) + h^2(t), \\
u^i(x,0) = \phi^i_0(x), u^i_t(x,0) = \psi^i_1(x), 0 < x < L, i = 1,2.
\end{cases}
$$

Mutliple string-spring system



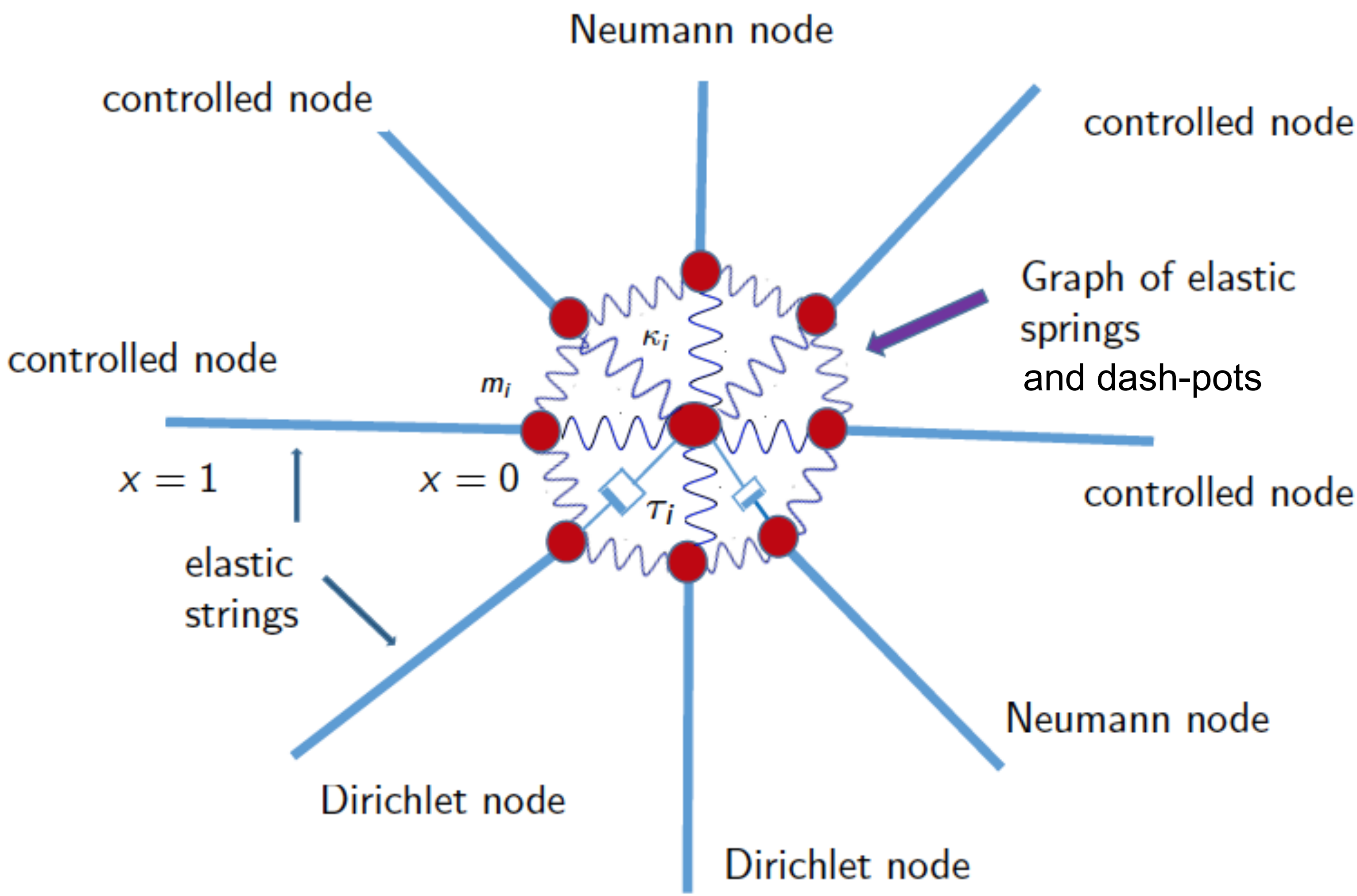
$$u^i_{tt}(x,t) - K_i(u^i_x)_x = 0,\ i = 1,2,3, (x,t) \in (0,1) \times (0,T)$$

$$u^1(1,t) = 0,\ u^i(1,t) = h^i(t),\ i = 2,3,\ t \in (0,T)$$

$$-K_i(u^i x)(0,t) + m_i u^i_{tt}(0,t) + \kappa_i \left(3u^i(0,t) - \sum_{j=1}^{3} u^j(0,t) \right) = 0,\ i = 1,2,3,$$

$$u^i(x,0) = \phi^i(x),\ u^i_t t(x,0) = \psi^i(x),\ x \in (0,1).$$

Coupling via a general viscoelastic graph



A general viscoelastic coupling

see also Rivera and Andrade Math. Meth. Appl. Sci., 23, 41-61 (2000)

$$
\begin{cases}
u^i_{tt} - K_i(u^i_x)_x = 0, \quad 0 < x < L, \quad i = 1, 2, \\
x = 0 : m_0 u^1_{tt}(0,t) = K_1(u^1_x)(0,t) - \kappa(u^1(0,t) - u^2(0,t)) \\
\qquad - \dfrac{\partial}{\partial t} \displaystyle\int_0^t a(t-s)(u^1(0,s) - u^2(0,s))ds, \\
\qquad m_0 u^2_{tt}(0,t) = K_2(u^2_x)(0,t) + \kappa(u^1(0,t) - u^2(0,t)) \\
\qquad + \dfrac{\partial}{\partial t} \displaystyle\int_0^t a(t-s)(u^1(0,s) - u^2(0,s))ds, \\
x = L : m_L u^1_{tt} = -K_1(u^1_x)(L,t) + h^1(t), \\
\qquad m_L u^2_{tt} = -K_2(u^2_x)(L,t) + h^2(t), \\
t = 0 : u^i(x,0) = \phi^i(x), u^i_t(x,0) = \psi^i(x), u^i(x,s) = 0, \\
\qquad s < 0, 0 < x < L, i = 1, 2.
\end{cases}
$$

Integration of the nodal conditions....

$$
\begin{cases}
\qquad\qquad u^i_{tt} - K_i(u^i_x)_x = 0, \quad 0 < x < L, \quad i = 1, 2, \\
x = 0: \;\; m_0 u^1_t(0,t) = G_{11}(\psi^1(0), \phi^1(0), \phi^2(0)) + G_{21}(u^1(0,t), u^2(0,t)) \\
\qquad\qquad + \int\limits_0^t G_{31}(t, s, u^1(0,s), u^2(0,s), u^1_x(0,s))ds, \\
\qquad\quad m_0 u^2_t(0,t) = G_{12}(\psi^2(0), \phi^1(0), \phi^2(0)) + G_{22}(u^1(0,t), u^2(0,t)) \\
\qquad\qquad + \int\limits_0^t G_{32}(t, s, u^1(0,s), u^2(0,s), u^2_x(0,s))ds \\
x = L: \;\; m_L u^1_t(L,t) = m_L \psi^1(L) + \int\limits_0^t \bar{G}_{21}(u^1_x)(L,s)ds + \int\limits_0^t h^1(s)ds, \\
\qquad\quad m_L u^2_t(L,t) = m_L \psi^2(L) + \int\limits_0^t \bar{G}_{22}(u^2_x)(L,s)ds + \int\limits_0^t h^2(s)ds, \\
t = 0: \;\; u^i(x,0) = \phi^i_0(x), u^i_t(x,0) = \psi^i_1(x), 0 < x < L, i = 1, 2.
\end{cases}
$$

Standard format for a scalar problem

Y.Wang, G.L. T.Li :Math. Meth. Appl. Sci. 40(10), 3808-3820, 2017

$$\begin{cases} u_{tt} - K(u, u_x)_x = F(u, u_x, u_t), \quad 0 < x < L, \\ x = 0 : m_0 u_{tt}(0, t) = G(t, u, u_x, u_t)(0, t) + h(t) \\ x = L : m_L u_{tt}(L, t) = \bar{G}(t, u, u_x, u_t)(L, t) + \bar{h}(t), \quad 0 \le t \le T \\ t = 0 : (u(x, 0), u_t(x, 0))^{\mathrm{T}} = (\phi(x), \psi(x))^{\mathrm{T}}, \quad 0 \le x \le L \end{cases}$$

where we assume:

$$K_v(u, v) > 0, \quad F(0, 0, 0) = 0, \qquad K(0, 0) = 0$$

$$G(t, 0, 0, 0) = 0, \quad \bar{G}(t, 0, 0, 0) = 0, \qquad T > \frac{L}{\sqrt{K_v(0, 0)}}$$

integrated version:

$$x = 0 : m_0 u_t(0, t) = m_0 \psi(0) + \int_0^t G(t, u, u_x, u_t)(0, s) ds + \int_0^t h(s) ds$$

$$x = L : m_L u_t(L, t) = m_L \psi(l) + \int_0^t \bar{G}(t, u, u_x, u_t)(L, s) + \int_0^t \bar{h}(s) ds$$

Essential result on first order systems

Indeed, we look into a more general set-up:

$$\partial_t u + A(u)\partial_x u = F(u),$$

where $F(0) = 0$ and $A(u)$ has the following structure

$$\begin{aligned} l_i(u)A(u) &= \lambda_i(u)l_i(u),\ A(u)r_i(u) = \lambda_i(u)r_i(u),\ i = 1, \dots, n \\ l_i(u)r_j(u) &= \delta_{ij},\ r_j(u)r_i(u) = 1,\ i, j = 1, \dots, n \\ \lambda_p(u) &< \lambda_q(u) = 0 < \lambda_r(u), \\ 1 \leq p \leq l, l+1 &\leq q \leq m,\ m+1 \leq r \leq n \end{aligned}$$

We define $v_i := l_i(u)u$ and then reformulate the system as

Equation in characteristic form

$$l_i(u)\left(\partial_t u_j + \lambda_i(u)\partial_x u_j\right) = l_i(u)F(u) =: f_i(u),\ i = 1, \dots, n$$

$$x = 0 : v_r = G_r(t, v_1, .., v_l, v_{l+1}, .., v_m) + \int_0^t H_r(s, v)ds + h_r(t),\ r = m+1, .., n$$

$$x = L : v_p = G_p(t, v_{l+1}, .., v_m, v_{m+1}, .., v_n) + \int_0^t H_p(s, v)ds + h_p(t),\ p = 1, .., l$$

$$t = 0 : u = \phi(x),\ 0 \leq x \leq L$$

We require additionally:

$$G_r(t, 0, \dots, 0) = 0, H_r(t, 0) = 0,\ r = m+1, \dots, n$$
$$G_p(t, 0 \dots, 0) = 0, H_p(t, 0) = 0, p = 1, \dots, l$$

and C^1 compatibility conditions.

Two-sided exact controllability

Theorem (two-sided controllability)
Let $T > \frac{L}{\sqrt{K_v(0,0)}}$. For any given initial data (ϕ, ψ) and final data (Φ, Ψ) with small norms $\|(\phi, \psi)\|_{C^2 \times C^1}$, $\|(\Phi, \Psi)\|_{C^2 \times C1}$, there exist boundary controls $h(t)$ and $\tilde{h}(t)$ with small $C(0, T)$-norms such that the IBVP with initial data

$$t = 0 : \quad u = \phi(x),\ \partial_t u = \psi(x),\ 0 \leq x \leq L$$

admits a C^2-solution u with small C^2 norm on $R(T) := \{(t, x) | 0 \leq t \leq T, 0 \leq x \leq L\}$ such that the final conditions

$$t = T : \quad u = \Phi(x),\ \partial_t u = \Psi(x),\ 0 \leq x \leq L.$$

One-sided exact controllability

Theorem (one-sided controllability)
Let $T > \frac{2L}{\sqrt{K_v(0,0)}}$ and assume

$$\frac{\partial G(t,0,0,0)}{\partial v} \neq 0, t \in [0, T].$$

For any given initial data (ϕ, ψ) and final data (Φ, Ψ) with small norms $\|(\phi, \psi)\|_{C^2 \times C^1}$, $\|(\Phi, \Psi)\|_{C^2 \times C1}$, with $h(t) \equiv 0$ there exist a boundary control $\tilde{h}(t)$ with small $C(0, T)$-norms such that the IBVP with initial data

$$t = 0: \quad u = \phi(x),\ \partial_t u = \psi(x),\ 0 \leq x \leq L$$

admits a C^2-solution u with small C^2 norm on $R(T) := \{(t,x)|0 \leq t \leq T, 0 \leq x \leq L\}$ such that the final conditions

$$t = T: \quad u = \Phi(x),\ \partial_t u = \Psi(x),\ 0 \leq x \leq L.$$

Results for system Wang, G.L., Li
Nonlinear Analysis 49 pp. 71-89 to be published: OCT 2019

Consider the following coupled system of 1-D quasilinear wave equations:

$$u^i_{tt} - (K^i(u^i, u^i_x))_x = F^i(\mathbf{u}, \mathbf{u}_x, \mathbf{u}_t) \quad (i = 1, \ldots, n),$$

where $\mathbf{u} = (u^1, \ldots, u^n)^{\mathrm{T}}$ is an unknown vector function of (t, x), $\mathbf{u}_x = (u^1_x, \ldots, u^n_x)^{\mathrm{T}}, \mathbf{u}_t = (u^1_t, \ldots, u^n_t)^{\mathrm{T}}, K^i = K^i(u^i, v^i)$ are given C^2 functions of u^i, v^i, such that

$$K^i_{v^i}(u^i, v^i) > 0 \qquad (i = 1, \ldots, n),$$

$F^i = F^i(\mathbf{u}, \mathbf{v}, \mathbf{w})$ are given C^1 functions of $\mathbf{u}, \mathbf{v}, \mathbf{w}$, such that

$$F^i(\mathbf{0}, \mathbf{0}, \mathbf{0}) = 0 \qquad (i = 1, \ldots, n).$$

Without loss of generality, we may assume that

$$K^i(0, 0) = 0 \qquad (i = 1, \ldots, n).$$

Nonlinear systems with masses

At the end $x = 0$, we prescribe the following coupled nonlinear dynamical boundary conditions:

$$x = 0: \quad u^i_{tt} = G^i(t, \mathbf{u}, \mathbf{u}_x, \mathbf{u}_t) + h^i(t) \qquad (i = 1, \ldots, n),$$

where $G^i = G^i(t, \mathbf{u}, \mathbf{u}_x, \mathbf{u}_t)$ are given C^1 functions of its arguments, $h^i(t)$ are C^0 functions of t for $(i = 1, \ldots, n)$.
Similarly, at other end $x = L$, the boundary conditions are given by

$$x = L: \quad u^i_{tt} = \bar{G}^i(t, \mathbf{u}, \mathbf{u}_x, \mathbf{u}_t) + \bar{h}^i(t) \qquad (i = 1, \ldots, n),$$

where $\bar{G}^i = \bar{G}^i(t, \mathbf{u}, \mathbf{u}_x, \mathbf{u}_t)$ are given C^1 functions of its arguments and $\bar{h}^i(t)$ are C^0 functions of t for $(i = 1, \ldots, n)$, respectively.
Here, without loss of generality, we may assume that

$$G^i(t, \mathbf{0}, \mathbf{0}, \mathbf{0}) \equiv \bar{G}^i(t, \mathbf{0}, \mathbf{0}, \mathbf{0}) \equiv 0 \qquad (i = 1, \ldots, n).$$

Initial and final data

The initial condition is given by

$$t = 0 : \quad (\mathbf{u}, \mathbf{u}_t) = (\phi, \psi), \quad 0 \le x \le L,$$

where $\phi = (\phi_1, ..., \phi_n)^{\mathrm{T}}$ is C^2 a vector-valued function of x with small $C^2[0, L]$ norm, $\psi = (\psi_1, ..., \psi_n)^{\mathrm{T}}$ is C^1 a vector-valued function of x with small C^1 norm, such that the conditions of C^2 compatibility at the points $(t, x) = (0, 0)$ and $(0, L)$ are satisfied, respectively.
The final condition is given by

$$t = T : \quad (\mathbf{u}, \mathbf{u}_t) = (\Phi, \Psi), \quad 0 \le x \le L,$$

where $\Phi = (\Phi_1, ..., \Phi_n)^{\mathrm{T}}$ is a C^2 vector-valued function of x with small $C^2[0, L]$ norm, $\Psi = (\Psi_1, ..., \Psi_n)^{\mathrm{T}}$ is a $C^1[]$ vector-valued function of x with small $C^1[0, L]$ norm, such that the conditions of C^2 compatibility at the points $(t, x) = (T, 0)$ and (T, L) are satisfied, respectively.

Semi-global classical solutions

Theorem: Existence of semi-global classical solutions:

Under the assumptions above, for any given $T > 0$, suppose that $\|(\phi, \psi)\|_{(C^2[0,L])^n \times (C^1[0,L])^n}$, $\|h\|_{(C^0[0,T])^n}$ and $\|\bar{h}[0, T]\|_{(C^0[0,T])^n}$ are small enough (depending on T), and the conditions of C^2 compatibility are satisfied at the points $(t, x) = (0, 0)$ and $(0, L)$, respectively. Then, the forward mixed initial-boundary value problem admits a unique semi-global C^2 solution $\mathbf{u} = \mathbf{u}(t, x)$ with small C^2 norm on the domain $\mathcal{R}(T) = \{(t, x)|0 \le t \le T, 0 \le x \le L\}$.

[Hidden Regularity] For the semi-global C^2 solution $\mathbf{u} = \mathbf{u}(t, x)$ given in the last Theorem, if $h^i(t) \equiv 0(i = 1, ..., n)$, or more generally, $h^i(t) \in C^1[0, T]$ with small $C^1[0, T]$ norm, there is a hidden regularity on $x = 0$ that $u^i(t, 0) \in C^3[0, T](i = 1, ..., n)$ with small C^3 norm.

Controllability

Theorem: Two-sided controllability Let

$$T > L \max_{i=1,\dots,n} \left(\frac{1}{\sqrt{K^i_{v^i}(0,0)}} \right).$$

For any given initial data (ϕ, ψ) and final data (Φ, Ψ) with small norms $\|(\phi, \psi)\|_{(C^2[0,L])^n \times (C^1[0,L])^n}$ and $\|(\Phi, \Psi)\|_{(C^2[0,L])^n \times (C^1[0,L])^n}$, there exist boundary controls $H = (h^1, \dots, h^n)$ and $\overline{H} = (\bar{h}^1, \dots, \bar{h}^n)$ with small norms $\|h^i\|_{C^0[0,T]}$ and $\|\bar{h}^i\|_{C^0[0,T]}(i = 1, \dots, n)$, such that the mixed initial-boundary value problem above admits a unique C^2 solution $\mathbf{u} = \mathbf{u}(t, x)$ with small C^2 norm on the domain $\mathcal{R}(T) = \{(t, x) | 0 \le t \le T, 0 \le x \le L\}$, which exactly satisfies the final condition

Controllability

Theorem: One-sided controllability Let

$$T > 2L \max_{i=1,\dots,n} \left(\frac{1}{\sqrt{K^i_{v^i}(0,0)}} \right).$$

For any given initial data (ϕ, ψ) and final data (Φ, Ψ) with small norms $\|(\phi, \psi)\|_{(C^2[0,L])^n \times (C^1[0,L])^n}$ and $\|(\Phi, \Psi)\|_{(C^2[0,L])^n \times (C^1[0,L])^n}$, and for any given boundary condition with $h^i \equiv 0 (i = 1, \dots, n)$, such that the conditions of C^2 compatibility are satisfied at the points $(t, x) = (0, 0)$ and $(T, 0)$, respectively. Suppose furthermore that

$$det \left(\frac{\partial G^i(t, \mathbf{0}, \mathbf{0}, \mathbf{0})}{\partial v_j} \right)_{n \times n} \neq 0,$$

Then, there exist boundary controls $\overline{H} = (\bar{h}^1, \dots, \bar{h}^n)$ with small norm $\|\overline{H}\|_{(C^0[0,T])^n}$ on $x = L$, such that the mixed initial-boundary value problem admits a unique C^2 solution $\mathbf{u} = \mathbf{u}(t, x)$ with small C^2 norm on the domain $\mathcal{R}(T) = \{(t, x) | 0 \leq t \leq T, 0 \leq x \leq L\}$, which exactly satisfies the final condition

3-d string-spring networks:Energies

Let ρ_i be the constant density of the corresponding string. Then the kinetic energy of a single string, labeled by i, at time t is given by

$$\mathcal{K}^i(\mathbf{R}^i(\cdot,t)) := \tfrac{1}{2}\int_0^{L_i} \rho_i |\mathbf{R}^i_t(x,t)|^2\,dx + \frac{1}{2}\sum_{k,j:i\in\mathcal{I}^j\cap\mathcal{I}^k} m^j_k |\mathbf{R}^i_t(x_{ij},t)|^2.$$

We shall assume that the potential energy of the same string is of the form

$$\begin{aligned}\mathcal{V}^i(\mathbf{R}^i(\cdot,t)) : = &\int_0^{L_i} \left[V^i(|\mathbf{R}^i_x(x,t)|) + \rho_i g \mathbf{R}(x,t)^i\cdot\mathbf{e}\right]\,dx\\ &+\frac{1}{2}\sum_{k>i}\kappa_j a^j_{ik}|\mathbf{R}^i(x_{ij},t)-\mathbf{R}^k(x_{kj},t)|^2\end{aligned}$$

Total energies for network

- $V^i(s)$ is a twice continuously differentiable, convex real valued function defined on an open subinterval $I^i = (a^i, b^i)$ of the positive real axis, with $a^i < 1 < b^i$, satisfying $V^i_{ss}(s) > 0$ and $V^i(1) = V^i_s(1) = 0$;
- $\mathbf{e}$ is the vertical unit vector and g is the gravitational constant.
- As for the total kinetic energy and total potential energy, we define

$$\mathcal{K}(\mathbf{R}(\cdot, t)) := \sum_{i \in \mathcal{I}} \mathcal{K}^i(\mathbf{R}^i)$$

and

$$\mathcal{V}(\mathbf{R}(\cdot, t)) := \sum_{i \in \mathcal{I}} \mathcal{V}^i(\mathbf{R}^i)$$

respectively.

Lagrangian

We now apply Hamilton's principle to the Lagrangian functional defined for fixed $T > 0$ by

$$\mathcal{L}(\mathbf{R}) := \int_0^T \Bigg\{ \sum_{i\in\mathcal{I}} \int_0^{L_i} \left[\tfrac{1}{2}\rho_i |\mathbf{R}_t^i(x,t)|^2 dx - V^i(|\mathbf{R}_x^i(x,t)|) - \rho_i g \mathbf{R}^i(x,t)\cdot\mathbf{e}\right] dx$$

$$+\frac{1}{2}\sum_{k,j:i\in\mathcal{I}^j\cap\mathcal{I}^k} m_k^j |\mathbf{R}_t^i(x_{ij},t)|^2 - \frac{1}{2}\sum_{k>i} \kappa_j a_{ik}^j |\mathbf{R}^i(x_{ij},t) - \mathbf{R}^k(x_{kj},t)|^2 \Bigg\} dt$$

The domain of $\mathcal{L}$ consists of $\mathbf{R}$ with $\mathbf{R}^i$ in $\mathcal{O}^i := \left\{\mathbf{R}^i \in C^2([0,L_i]\times[0,T];R^3)\right\}$ and

$$\mathbf{R}^i(\cdot,0) = \mathbf{R}^{0,i} \quad \text{and} \quad \mathbf{R}_t^i(\cdot,0) = \mathbf{R}^{1,i},$$

as well as *prescribed Dirichlet boundary conditions* at simple nodes

$$\mathbf{R}^{i_j}(x_{i_j j},t) = \mathbf{U}^j(t) \quad \text{for } j\in\mathcal{J}^D \quad \text{and for } t\in[0,T].$$

Euler-Lagrange equations

$$\rho_i \mathbf{R}^i_{tt}(x,t) = \mathbf{G}^i(\mathbf{R}^i_x(x,t))_x - \rho_i g \mathbf{e} \quad \text{for each } i \in \mathcal{I}$$

with $\mathbf{G}^i : \boldsymbol{R}^3 \mapsto \boldsymbol{R}^3$ defined by

$$\mathbf{G}^i(\mathbf{v}) := V^i_s(|\mathbf{v}|)\frac{\mathbf{v}}{|\mathbf{v}|}.$$

Next, for $j \in \mathcal{J}^M$ we choose perturbations with $\mathbf{r}^i = \mathbf{0}$ for $i \notin \mathcal{I}^j$ and with support in a small neighbourhood of x_{ij} for $i \in \mathcal{I}^j$. As there is no continuity condition across the joints, we are led to the multiple node condition

$$\begin{aligned} &\epsilon_{ij}\mathbf{G}^i(\mathbf{R}^i_x(x_{ij},t)) + m^j_i \mathbf{R}^i_{tt}(x_{ij},t) \\ &+ \kappa_j \left\{ \left(\sum_{k=1}^{d_j} a^j_{ik} \right) \mathbf{R}^i(x_{ij},t) - \sum_{k=1}^{d_j} a^j_{ik} \mathbf{R}^k(x_{kj},t) \right\} = 0 \quad \text{for each } j \in \mathcal{J}^M. \end{aligned}$$

The full network system

We collect the equations and nodal conditions and write down the entire system as follows:

$$
\begin{cases}
\rho_i \mathbf{R}^i_{tt}(x,t) = \mathbf{G}^i(\mathbf{R}^i_x(x,t))_x - \rho_i g \mathbf{e}, x \in [0, L_i], t \in [0,T], \quad i \in \mathcal{I} \\
\mathbf{R}^{i_j}(x_{i_j j}, t) = \mathbf{U}^j(t), \quad x \in [0, L_i], t \in [0,T], \ j \in \mathcal{J}^D \\
\epsilon_{i_j j} \mathbf{G}^{i_j}(\mathbf{R}^{i_j}_x(x_{i_j j}, t)) = \mathbf{U}^j(t), \ t \in [0,T], \ j \in \mathcal{J}^N \\
\epsilon_{ij} \mathbf{G}^i(\mathbf{R}^i_x(x_{ij}, t)) + m^j_i \mathbf{R}^i_{tt}(x_{ij}, t) \\
\qquad + \kappa_j \left\{ \left(\sum_{k=1}^{d_j} a^j_{ik} \right) \mathbf{R}^i(x_{ij}, t) - \sum_{k=1}^{d_j} a^j_{ik} \mathbf{R}^k(x_{kj}, t) \right\} = 0 \quad t \in [0,T], j \in \mathcal{J}^M \\
\mathbf{R}^i(x,0) = \mathbf{R}^{0,i} \quad \mathbf{R}^i_t(x,0) = \mathbf{R}^{1,i}, \ x \in [0, L_i], i \in \mathcal{I}.
\end{cases}
\tag{E}
$$

Limiting model as spring stifness tends to infinity

Notice that

$$\sum_{i\in\mathcal{I}_j} \epsilon_{ij}\mathbf{G}^i(\mathbf{R}^i_x(x_{ij},t)) + \sum_{i\in\mathcal{I}_j} m^j_i \mathbf{R}^i_{tt}(x_{ij},t) = 0$$

which provides a balance of elastic forces at node n_j. if, moreover κ_j at the node n_j tends to inifinity, then we obtain

$$\mathbf{R}^i(x_{ij},t) = \mathbf{R}^k(x_{kj},t),\ \forall i,k\in\mathcal{I}_j,\ t\in[0,T].$$

If then the masses $m^j_i = m^j$, one gets the classical Kirchhoff condition including the mass m^j, or if $m = 0$ the classical transmission conditions used in

G.L. and E.J.P.G. Schmidt: On Exact Controllability of Networks of Nonlinear Elastic Strings in 3-Dimensional Space, Chinese Annals of Mathematics 32B(6), 1-28 (2011).

Deviations from equlibria

We assume a stretched equilibrium $\mathbf{R}^e = \{\mathbf{R}^{e,i}\}_{i\in\mathcal{I}}$ and introduce perturbations away from the given equilibrium by setting

$$\mathbf{r}^i(x,t) := \mathbf{R}^i(x,t) - \mathbf{R}^{e,i}(x).$$

Noting that the $\mathbf{R}^{e,i}$ do not depend on t, the system (**??**) is equivalent to

$$\begin{cases} \rho_i \mathbf{r}^i_{tt}(x,t) = [\mathbf{G}^i(\mathbf{R}^{e,i}_x(x) + \mathbf{r}^i_x(x,t))]_x - \rho_i g\mathbf{e}, \quad \text{for } i \in \mathcal{I}, \\ \mathbf{r}^i(x,0) = \mathbf{r}^{0,i}(x), \quad \mathbf{r}^i_t(x,0) = \mathbf{r}^{1,i}(x) \quad \text{for } i \in \mathcal{I}, \\ \mathbf{r}^{i_j}(x_{i_j j},t) = \mathbf{U}^j(t) \text{ for } j \in \mathcal{J}^S, \\ \epsilon_{ij}\mathbf{G}^i(\mathbf{R}^{e,i}_x(x_{ij}) + \mathbf{r}^i_x(x_{ij},t)) + m^j_i \mathbf{r}^i_{tt}(x_{ij},t) \\ \qquad + \kappa_j \left\{ \left(\sum_{k=1}^{d_j} a^j_{ik} \right) (\mathbf{R}^{e,i}(x_{ij}) + \mathbf{r}^i(x_{ij},t)) \right. \\ \qquad\quad \left. - \sum_{k=1}^{d_j} a^j_{ik} (\mathbf{R}^{e,k}(x_{kj}) + \mathbf{r}^k(x_{kj},t)) \right\} = 0 \quad \text{for } j \in \mathcal{J}^M \end{cases} \tag{E''}$$

Examples

We assume $x_{ij} = 0$ for all edges adjacent to N^j. At this node we now have a graph $G^j = (V^j, E^j)$ of springs with adjacency structure given by $A^j = (a^j_{ik})$. Let $D^j := \text{diag}((\sum_{k=1}^{d_j} a^j_{ik}), i = 1, \dots, d^j)$. Then weighted discrete Laplacean of G^j is given by $\mathbf{L}^j := D^j - A^j$. Devote the diagonal mass matrix $\mathbf{M}^j :=$ $\text{diag}(m^j_i, i = 1, \dots, d^j)$ and the strain matrix

$$K^j(\mathbf{r}_x(0,t)) := \text{diag}(G^i(\mathbf{R}^{e,i}(0) + \mathbf{r}^i_x(0,t)), i = 1, \dots, d^j),$$

with $\mathbf{r}_x := (\mathbf{r}^1_x, \dots, \mathbf{r}^{d^j}_x)^T$. Then with $\mathbf{r}_{tt} := (\mathbf{r}^1_{tt}, \dots, \mathbf{r}^{d^j}_x)^T$, $\mathbf{r} := (\mathbf{r}^1, \dots, \mathbf{r}^{d^j})^T$. Then the nodal condition coupling the adjacent strings to the springs can be written as

$$\mathbf{M}^j \mathbf{r}_{tt}(0,t) = \mathbf{K}^j(\mathbf{r}_x(0,t)) - \kappa_j \mathbf{L}^j \mathbf{r}(0,t).$$

The ring

For a ring (cycle) with springs connecting the ends of the strings, we arrive at the new coupling matrix

$$\mathbf{L}_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \dots & -1 \\ 0 & 1 & -1 & 0 & 0 & \dots & 0 \\ 0 & -1 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & -1 & \dots & 0 \\ 0 & 0 & 0 & -1 & 1 & \dots & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ -1 & 0 & 0 & 0 & 0 & \dots & 1 \end{pmatrix}$$

$$\begin{aligned} \mathbf{r}_{tt}(x,t) - (\mathbf{K}(\mathbf{r}_x(x,t)))_x &= 0, x \in [0,1], \quad t \in [0,T] \\ \mathbf{M}_0\mathbf{r}_{tt}(0,t) &= \mathbf{K}_1(\mathbf{r}_x(0,t)) - \kappa_0\mathbf{L}_0\mathbf{r}(0,t), \quad t \in [0,T] \\ \mathbf{M}_1\mathbf{r}_{tt}(L,t) &= -\mathbf{K}_1(\mathbf{r}_x(L,t)) - \kappa_1\mathbf{L}_1\mathbf{r}(0,t) + \mathbf{H}^1(t), \quad t \in [0,T] \\ \mathbf{r}(x,0) = \mathbf{r}_0(x), \mathbf{r}_t(x,0) &= \mathbf{r}_{10}(x), \quad x \in [0,1]. \end{aligned}$$

The star

For a star-graph, we obtain accordingly

$$
\begin{aligned}
\mathbf{r}_{tt}(x,t) - (\mathbf{K}(\mathbf{r}_x(x,t)))_x = 0, x \in [0,1], &\quad t \in [0,T] \\
\mathbf{M}\mathbf{r}_{tt}(0,t) = \mathbf{K}(\mathbf{r}_x(0,t)) - \kappa_j \mathbf{L}\mathbf{r}(0,t), &\quad t \in [0,T] \\
\mathbf{r}(1,t) = u(t), &\quad t \in [0,T] \\
\mathbf{r}(x,0) = \mathbf{r}_0(x), \mathbf{r}_t(x,0) = \mathbf{r}_{10}(x), &\quad x \in [0,1]
\end{aligned}
$$

Semi-global classical solutions

Consider a tree like network. Let $\mathbf{R}^e$ be a given stretched equilibrium. For a specified value of $T > 0$ there exist constants c_0 and c_T such that if the initial data

$$\mathbf{w}^{0,i} = (\mathbf{w}_1^{0,i}, \mathbf{w}_2^{0,i}, \mathbf{w}_3^{i,0}) \in C^1([0, L_i] \times [0, T]; \boldsymbol{R}^3)^3$$

and the boundary data

$$\mathbf{v}^j(t) \in C_0^1([0, T]; \boldsymbol{R}^3)$$

satisfy the C^2-compatibility conditions and satisfy

$$\max\left\{\|\mathbf{w}_1^{0,i}\|_1, \|\mathbf{w}_2^{0,i}\|_1, |\mathbf{w}_3^{0,i}|, \|\mathbf{v}^j\|_1\right\}_{i\in\mathcal{I},\ j\in\mathcal{J}^S} < c_0$$

there exists a unique small solution semi-global solution

$$\mathbf{w} \in \prod_{i\in\mathcal{I}} C^2([0, L_i] \times [0, T]; \boldsymbol{R}^3)^3$$

Exact controllability and more....open problems

- Exact controllability of a star with controls at all ends (ready)

- Open: Keep one simple node fixed. Then we expect to control in asymmetric spaces as in the work of Zuazua and Hansen: Exact controllability and stabilization of a vibrating string with an interior point mass. SIAM J. Control Optim. 33 (1995), no. 5, 1357–1391.
Avdonin and Julian: Controllability for a string with attached masses and Riesz bases for asymmetric spaces. Math. Control Relat. Fields 9 (2019), no. 3, 453–494.
We expect interesting smoothing pattern across the spring-mass-graphs between the controlled and fixed nodes

- Work in progress with (nonlinear) viscoelastic springs

- Stabilization issues in the general case (M.Gugat for linear strings see Wednesday

- Exact Profil nodal controllability (see Wednesday)

- Geometrically exact beams...(Ch. Rodriguez see Wednesday)

Open problems: sustainable controls

The spring stiffness may deteriorate once large deflections are involved. The Springs undergo damage which can be represented by the time evolution of a damage variable.

$$\left\{\begin{array}{c} y^i_{tt} - K_i(y^i_x)_x = 0 \quad \text{in} \ \ (0,T)\times I^i, \\ \zeta_t = \Phi(\zeta, y^1(t,x_0), y^2(t,x_0)) \quad \text{in} \ \ (0,T), \\ K_i(y^i_x)(x_0,t) = \zeta(t)\left(y^1(t,x_0) - y^2(t,x_0)\right), i = 1,2 \\ K_i(y^2_x)(t,L) = u(t), \quad y^1(t,0) = 0, \quad t \in (0,T), \\ y^i(0,x) = y^i_0(x), \quad y^i_t(0,x) = y^i_1(x), \quad x \in I^i, \\ u \in U_{ad}, \ 0 \le \zeta(t) \le 1, \ t \in L^2(0,T), \zeta(0) = 1 \end{array}\right. \qquad \text{(Damage)}$$

We then ask to minimize

$$\text{Minimize} \ \ J(u,y) = I(u,y) + \frac{1}{2}\int_0^T |\zeta(t) - 1|^2 \, dt$$

subject to system (damage). The optimal, damage-avoiding control is then called *sustainable control*.

Inspired by the joint work with F. Alabau and P. Cannarsa (SICON 2017) and in coll. also with P. Kogut

$$\text{Minimize} \quad J(u,y) = I(u,y) + \frac{1}{p}\int_0^T \int_0^L |\zeta(t,x) - 1|^p \, dx$$

subject to the constraints

$$\begin{cases} y^i_{tt} - \left(\zeta^i(t,x) K_i(y_x)\right)_x = 0 \quad \text{in} \quad (0,T)\times(0,L), i \in \mathcal{I} \\ \zeta^i_t - \left(|\zeta^i_x|^{p-2}\zeta^i_x\right)_x = \Phi_i(\zeta^i, y^i) \quad \text{in} \quad \times (0,L), i \in \mathcal{I} \\ y^i_x(t, x_{ij}) = u_j(t), i \in \mathcal{I}_j, j \in \mathcal{J}^S \\ y^i(t,x_{ij}) = y^k(t,x_{kj}), i,k \in \mathcal{I}_j, j \in \mathcal{J}^M, \quad t \in (0,T), \\ \sum_{i\in\mathcal{I}_j} d_{ij} \lim_{x\to x_{ij}} \left(\zeta(x) K_i(y^i)(x)\right) = 0, \; j \in \mathcal{J}^M, t \in (0,T), \\ y^i(0,x) = y^i_0(x), \quad y^i_t(0,x) = y^i_1(x), \quad \zeta^i(0,x) = 1 \quad x \in (0,L), \\ u_i \in U_{ad} \in L^2(0,T), \quad \zeta^i(t,x) \in [0,1] \quad \text{in} \quad (0,T)\times(0,L), \\ \zeta^i(t,x) = 1 \text{ in } (0,L)\setminus E, i \in \mathcal{I}, t \in (0,T). \end{cases}$$

Thank you for your attention!