On optimal potentials problems for bounded and unbounded domains

Faustino Maestre

Departamento de Ecuaciones Diferenciales y Análisis Numérico
Universidad de Sevilla. Spain.

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Joint work: Giuseppe Buttazzo, Università di Pisa
Juan Casado-Díaz, Universidad de Sevilla
Bozhidar Velichkov, Université Grenoble Alpes.

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    - Capacitary measures
2. Existence results
    - The admissible class of potentials and its relaxation
    - Optimality conditions
    - Saturation of the Constraint
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We consider $D \subset \mathbb{R}^d$ a fixed open bounded set. We are interested in the optimization problem:

$$\min_{V \in \mathcal{V}} \int_D g(x)u(x)\, dx$$

subject to

$$\begin{cases} -\Delta u + V\, u = f & \text{in} \quad D, \\ u = 0 & \text{on} \quad \partial D, \end{cases}$$

where,

$$\mathcal{V} = \left\{ V : D \to [0, +\infty] \ : \ V \text{ Lebesgue measurable, } \int_D \psi(V(x))\, dx \leq 1 \right\}$$

and ψ satisfying some appropriate qualitative conditions.

The function $\psi : [0, +\infty] \to [0, +\infty]$ we assume that:

(i) ψ is strictly decreasing;

(ii) there exist $p > 1$ such that the function $s \mapsto \psi^{-1}(s^p)$ is convex.

For instance the following functions:

1. $\psi(s) = s^{-p}$, for any $p > 0$,
2. $\psi(s) = e^{-\alpha s}$, for any $\alpha > 0$,

The choice $\psi(s) = e^{-\alpha s}$ was proposed in [Buttazzo et al ,2014], in order to approximate shape optimization problems with Dirichlet condition on the free boundary.
Moreover, as $\alpha \to 0$ the problems with the parameter α were shown to Γ-converge to the shape optimization problem with a volume constraint $|\Omega| \leq 1$ being Ω the shape variable.

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Existence results:

- $f \geq 0$ and $g \leq 0$ (o reverse case), maximum principle, cost is monotonically increasing, and volume constraint saturated ([Buttazzo et al., 2014])
- Optimal domains with f and g are allowed to change sign ([Buttazzo and Velichkov, 2018])
- We analyze the existence of optimal potentials when f and g are allowed to change sign. We expect no saturation of volume constraint.

Main assumptions:

- linear cost (otherwise simple examples show optimal solution only exists in a relaxed sense).
- D bounded (characterization of the relaxed formulation).

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A *capacitary measure* μ is a nonnegative Borel measure on D, possibly taking the value $+\infty$, that vanishes on all sets of capacity zero. Notation $\mu \in \mathcal{M}_{cap}$.
Capacity is intended with respect to the H^1 norm

$$cap(E, D) = \inf \left\{ \int_D |\nabla u|^2 \, dx + \int_D u^2 \, dx : u \in H_0^1(D), \right. \\ \left. u \geq 1 \text{ in a neigborhood of } E \right\}$$

We consider the Hilbert space $H_0^1(D) \cap L^2(\mu)$ endowed with norm:

$$\|u\| = \left(\|\nabla u\|^2_{L^2(D)} + \|u\|^2_{L^2(\mu)} \right)^{1/2}.$$

We say that $u \in H_0^1(D) \cap L^2(\mu)$ is a solution of the problem

$$-\Delta u + \mu u = f, \qquad \text{for a function } f \in L^2(D),$$

if

$$\int_D \nabla u \nabla \phi \, dx + \int_D u\phi \, d\mu = \int_D f\phi \, dx \qquad \forall \phi \in H_0^1(D) \cap L^2(\mu),$$

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For $V \in \mathcal{V}$ the state equation

$$-\Delta u + Vu = f, \qquad u \in H_0^1(D) \cap L^2(V).$$

The capacitary measure μ associated to V is defined as:

$$\mu(A) = \begin{cases} \int_A V(x)\,dx & \text{if } \operatorname{cap}\left(A \cap \{V = +\infty\}\right) = 0 \\ +\infty & \text{if } \operatorname{cap}\left(A \cap \{V = +\infty\}\right) > 0, \end{cases}$$

which implies $u = 0$ quasi-everywhere on the set $\{V = +\infty\}$.
Abusing the terminology we will identify μ and V.

Relaxed problem

We put $\overline{\mathcal{V}}$ the family of capacitary measures μ obtained as limits of sequences (V_n) of potentials in $\mathcal{V}$. Relaxed problem:

$$\min_{\mu \in \overline{\mathcal{V}}} \int_D g(x)u(x)\,dx$$

subject to

$$\begin{cases} u \in H_0^1(D) \cap L^2(\mu) \\ -\Delta u + \mu\, u = f \qquad \text{in} \quad D, \end{cases}$$

Theorem

Let $D \subset \mathbb{R}^d$ be a bounded open set and let ψ satisfy the assumptions i) and ii) above. Then, for every $f, g \in L^2(D)$, the original optimization problem has a solution.

Sketch of the proof:

- $\mu \in \overline{\mathcal{V}}$ be solution of the relaxed problem $\Longrightarrow \mu = V + \mu^s$, μ^s singular respect to the Lebesgue measure and $\mu^s(\{V = +\infty\}) = 0$.
- $\mu^s = +\infty_K$, with K quasi-closed set $\Longrightarrow$ $\mu = V + \mu^s = V + (+\infty_K) \in \mathcal{V}$.

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We consider $u, p \in H_0^1(D) \cap L^2(\mu)$ solutions of:

$$\begin{cases} -\Delta u + \mu\, u = f & \text{in} \quad D, \\ u = 0 & \text{on} \quad \partial D, \end{cases} \qquad \begin{cases} -\Delta p + \mu\, p = g & \text{in} \quad D, \\ p = 0 & \text{on} \quad \partial D, \end{cases}$$

Proposition

Suppose that μ is a solution of the relaxed optimization problem on the bounded domain $D \subset \mathbb{R}^d$. Then

$$u\, p \leq 0 \quad \text{a.e. on } D.$$

Moreover, the above inequality holds quasi-everywhere on D.

Remark

In fact, for a potential V solution of the original optimization problem, (whose existence follows from previous Theorem), we have

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One cannot expect that the constraint $\int_D \psi(V)dx \leq 1$ is saturated.

- $\psi(0) = +\infty$ (for instance, $\psi(s) = s^{-p}, p > 0$):
  - If $\int_D \psi(+\infty)dx < 1 \Rightarrow V_{opt} = +\infty$ (i.e. $\Omega = \emptyset$).
  - If $\int_D \psi(+\infty)dx \geq 1 \Rightarrow \int_D \psi(V_{opt})dx = 1$.
- $\psi(0) < +\infty$ (for instance, $\psi(s) = e^{-\alpha s}, \alpha > 0$):
  - $\exists V_{opt}$ s.t. $\int_D \psi(V_{opt})dx = 1$.
  - If $\int_D \psi(V_{opt})dx < 1 \Rightarrow V_{opt} = \begin{cases} 0 & \text{on } \Omega \\ +\infty & \text{on } \Omega^c \end{cases}$.

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Numerical Analysis

We propose the numerical analysis of the following problem. We consider $D = [0,1]^2$:

$$\min_{V \in \mathcal{V}} \int_D g(x)u(x)\, dx$$

subject to

$$\begin{cases} -\Delta u + V\, u = f & \text{in} \quad D, \\ u = 0 & \text{on} \quad \partial D, \end{cases}$$

where,

$$\mathcal{V} = \left\{ V : D \to [0, +\infty] \ :\ V \text{ Lebesgue measurable, } \int_D \psi(V(x))\, dx \leq 1 \right\}$$

and $\psi(s) = \frac{1}{m} e^{-\alpha s}$

We use Method of Moving Asymptotes (MMA).
The structure of the algorithm is as follows.

- Initialization of the potential $V^0 \in \mathcal{V}$;
- for $k \geq 0$, iteration until convergence as follows:
  - compute the state u_{V^k} and then the co-state p_{V^k},
  - compute the descent direction $\tilde{V}^k(x) = -u_{V^k} \cdot p_{V^k}$
  - update the potential V^k in D:

$$V^{k+1} = V^k + \ell_k \tilde{V}^k,$$

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We use `FreeFem++ v 3.50` completed with the library `NLopt`. The main required data:

- Initialization V^0
- routines for cost function and associated gradient through the adjoint state
- lower and upper bounds for $V \in [0, V_{max}]$ (V_{max} large enough)
- stopping criteria

In the following, we take $g \equiv 1$ and consider different choices for f and parameters α and m.

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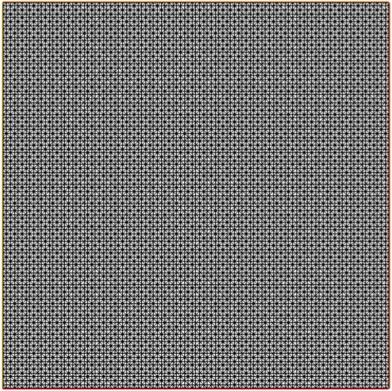

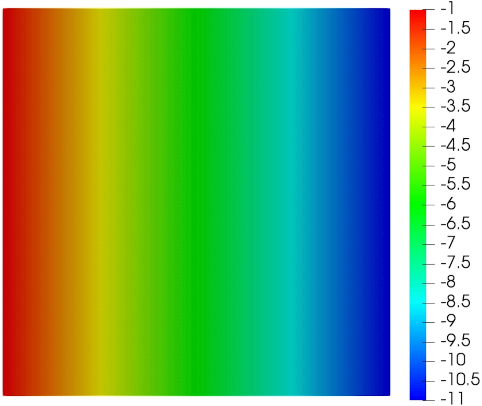

Figure: To the left: the domain D and its triangulation; number of nodes: 40401; number of triangles: 80000. To the right: the right-hand side function $f(x, y) = -(1 + 10x)$.

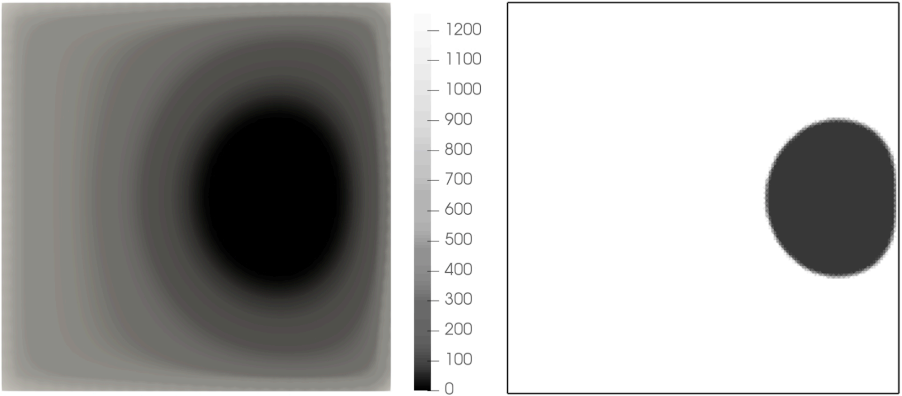

Figure: The optimal potential V_{opt} for volume constraint $m = 0.2 = m_{opt}$. Case $\alpha = 10^{-2}$ (left) and $\alpha = 10^{-4}$ (right).

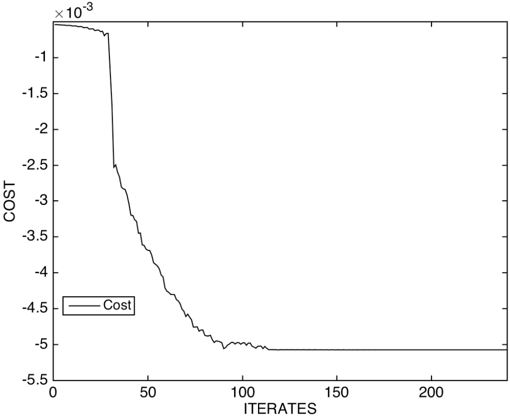

Figure: Cost evolution for the example from previous Figure, case $\alpha = 10^{-4}$.

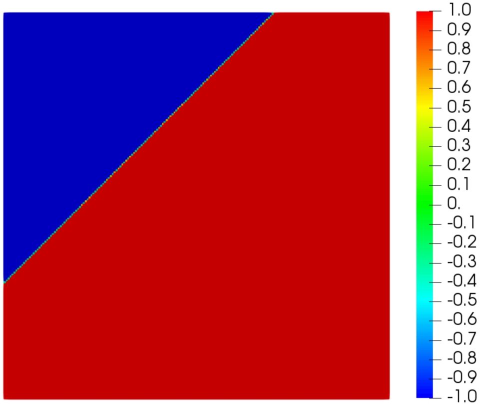

Figure: The right-hand side function f is given by $f(x,y) = -1$, if $y - 1.4x \geq 0.3$, and $f(x,y) = 1$, if $y - 1.4x < 0.3$

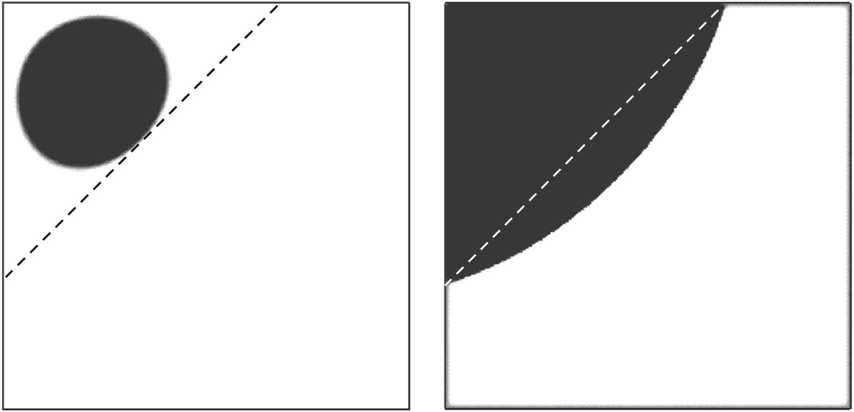

Figure: Optimal potential V_{opt} for $m = 0.2$ (left) and $m = 0.45$ (right). The occupied volume on the right is 0.33276.

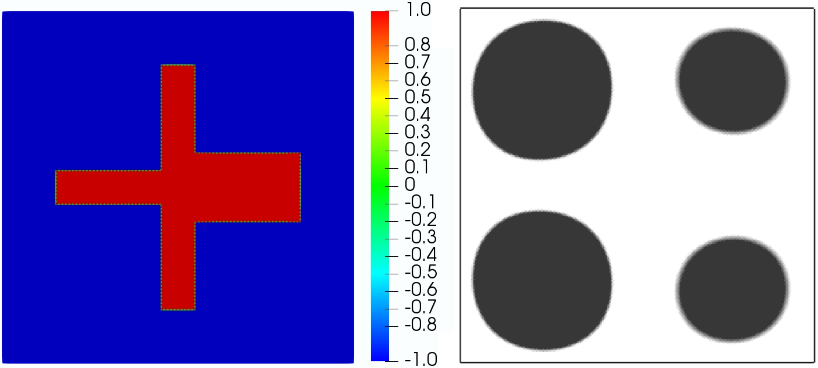

Figure: The right-hand side function f (left) and the optimal potential V_{opt} (right). The volume $m = 0.45$ is entirely occupied.

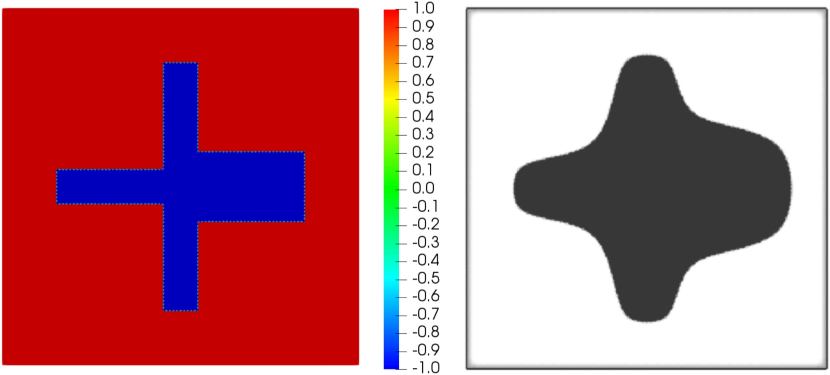

Figure: The right-hand side function f (left) and the optimal potential V_{opt} (right). The occupied volume is 0.378404 of the total available $m = 0.5$.

Homogenization, Spectral problems and other topics in PDE's

1. Introduction
   - Statement of the problem
   - Capacitary measures
2. Existence results
   - The admissible class of potentials and its relaxation
   - Optimality conditions
   - Saturation of the Constraint
3. Numerical experiments
4. Unbounded domain

Unbounded domain

We consider $D = \mathbb{R}^d$. We are interested in the optimization problem:

$$\min_{\mu \in \mathcal{M}_{cap}} \int_{\mathbb{R}^d} j(x, u(x), \nabla u(x))\, dx$$

subject to

$$-\Delta u + \mu\, u = f \quad \text{in} \quad \mathbb{R}^d,$$

and

$$\Psi(\mu) \leq 1,$$

Work in progress with: G. Buttazzo and J. Casado-Díaz.

some dificulties

- Which is good space X for the solutions of $-\Delta u + \mu u = f$, ???

$$\int_{\mathbb{R}^d} \nabla u \cdot \nabla v \, dx + \int_{\mathbb{R}^d} u \, v \, d\mu = \int_{\mathbb{R}^d} fv, \quad v \in X,$$

We consider:

$$W(x) = \frac{1}{1+|x|} \text{ if } d \neq 2, \quad W(x) = \frac{1}{(1+|x|)\log(2+|x|)} \text{ if } d = 2.$$

and we put:

$$L = \{u : \mathbb{R}^d \to \mathbb{R} : Wu \in L^2(\mathbb{R}^d)\}$$

$$H = \{u \in L \cap H^1_{loc}(\mathbb{R}^d) : \nabla u \in L^2(\mathbb{R}^d)^d\}$$

and one gets

$$\|u\|_L \leq C\left(\|\nabla u\|_{L^2(\mathbb{R}^d)} + \|u\|_{L^2_\mu}\right)$$

We take

$$X = H \cap L^2_\mu$$

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We take

$$X = H \cap L^2_\mu$$

some dificulties

- How is defined $\Psi(\mu)$???.

  We decompose $\mu = \mu^a + \mu^s + \infty_K$, and consider:

  $$\Psi(\mu) = \int_{\mathbb{R}^d} \psi(\mu^a)\, dx + C_\psi \mu^s(\mathbb{R}^d) + \psi(\infty) cap(K),$$

  where $\psi : \mathbb{R}^+ \to [0, +\infty]$ is a convex and lower semicontinuous function and

  $$C_\psi = \lim_{t \to +\infty} \frac{\psi(t)}{t}$$

Existence result

Theorem

We consider j $: \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ *measurable in* $x \in \mathbb{R}$, *lower semincontinuous in* (s, ξ) *and some growth conditions in* (s, ξ). *We consider* $\psi : \mathbb{R} \to [0, +\infty]$ *convex and lower semicontinuous and a measure* $\nu \in \mathcal{M}_{cap}$ *such that there exists* $\hat{\mu} \in \mathcal{M}_{cap}$ *satisfying*

$$\hat{\mu} \geq \nu, \quad \Psi(\hat{\mu}) \leq 1.$$

Moreover, if $d = 1, 2$ *we assume that:*

either $\psi(0) > 0$ *or* ν *is not the null measure*

Then, for every $f \in H'$ *the optimization problem has at least one solution.*

THANKS

FOR YOUR

ATTENTION!!!