Control of incomplete data problems. Application to an ecology problem

Abdennebi Omrane
(joint work with L. Louison)

UMR 42 Ecologie des Forêts de Guyane (EcoFoG)
University of French Guiana,
Campus of TrouBiran, Cayena
aomrane@gmail.com

Benasque (Spain), August 27^{th}, 2019

8^{th} Conference - PDE's, Optimal Design and Numerics

Contents

Biological aspect of nutrient uptake

- **Photosynthesis** :
  — carbon (C)
  from carbon dioxide (CO_2).
- **Root absorption** :
  - magnesium (Mg^{2+}),
  - calcium (Ca^{2+}),
  - potassium (K^{+}),
  - nitrogen (NO_3^{-}),
  - phosphorus (P),
  - water (H_2O).
- Absorption of nutrients by roots
  - small absorption zone
  - need of more nutrients !

Biological aspect of nutrient uptake

- **Photosynthesis** :

  — carbon (C)

  from carbon dioxide (CO_2).

- **Root absorption** :

  - magnesium (Mg^{2+}),
  - calcium (Ca^{2+}),
  - potassium (K^{+}),
  - nitrogen (NO_3^{-}),
  - phosphorus (P),
  - water (H_2O).

- Absorption of nutrients by roots
  - small absorption zone
  - need of more nutrients !

Biological aspect of nutrient uptake

- **Photosynthesis** :

  – carbon (C)

  from carbon dioxide (CO_2).

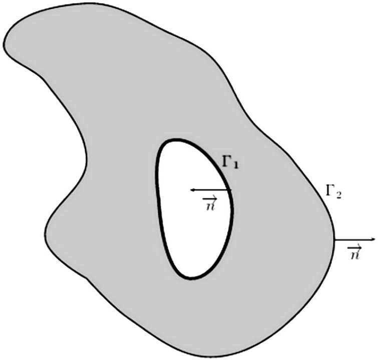
- **Root absorption** :

  - magnesium (Mg^{2+}),
  - calcium (Ca^{2+}),
  - potassium (K^{+}),
  - nitrogen (NO_3^{-}),
  - phosphorus (P),
  - water (H_2O).

- Absorption of nutrients by roots

  - small absorption zone
  - need of more nutrients !

Description of the domain of study

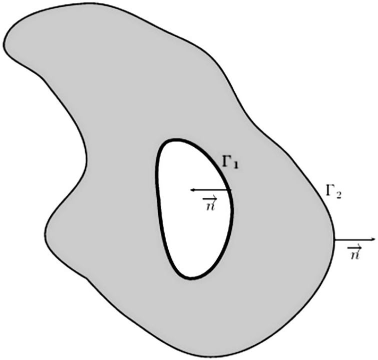


The figure, above, shows the domain of study Ω, an open bounded set of $\mathbb{R}^2$ of boundary Γ.

- Γ_1 : the root surface,
- Γ_2 : the boundary between a piece of observed soil and the rest of soil,

where $\Gamma := \Gamma_1 \cup \Gamma_2$ et $\Gamma_1 \cap \Gamma_2 = \emptyset$.

Description of the domain of study

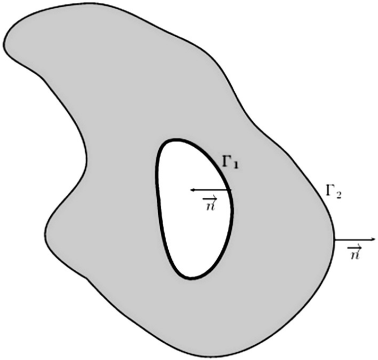


The figure, above, shows the domain of study Ω, an open bounded set of $\mathbb{R}^2$ of boundary Γ.

- Γ_1 : the root surface,
- Γ_2 : the boundary between a piece of observed soil and the rest of soil,

where $\Gamma := \Gamma_1 \cup \Gamma_2$ et $\Gamma_1 \cap \Gamma_2 = \emptyset$. For the study of uptake problem, we consider :

- $Q :=]0, T[\times\Omega$,
- $\Sigma_1 :=]0, T[\times\Gamma_1$,
- $\Sigma_2 :=]0, T[\times\Gamma_2$, complementary of Σ_1.



The figure, above, shows the domain of study Ω, an open bounded set of $\mathbb{R}^2$ of boundary Γ.

- Γ_1 : the root surface,
- Γ_2 : the boundary between a piece of observed soil and the rest of soil,

where $\Gamma := \Gamma_1 \cup \Gamma_2$ et $\Gamma_1 \cap \Gamma_2 = \emptyset$. For the study of uptake problem, we consider :

- $Q :=]0, T[\times\Omega$,
- $\Sigma_1 :=]0, T[\times\Gamma_1$,
- $\Sigma_2 :=]0, T[\times\Gamma_2$, complementary of Σ_1.

Absorption mechanisms

- root interception of nutrients, mass flow .. $< 5\%$
- diffusion : 93% of phosphorus, 80% of potassium.

The Nutrient uptake model

The Nye-Tinker-Barber (NTB) system (1980's) :

Let the function $c = c(t,x)$ represents the concentration of nutrient,

$$\begin{cases} \alpha \dfrac{\partial c}{\partial t} + q\nabla c - D\Delta c & = & 0 & \text{in} & Q, \\ (D\nabla c - qc).\overrightarrow{n} & = & h(c) & \text{on} & \Sigma_1, \\ (D\nabla c - qc).\overrightarrow{n} & = & 0 & \text{on} & \Sigma_2, \\ c(0,x) & = & c_0(x) & \text{in} & \Omega. \end{cases} \tag{1}$$

Description of the NTB System :

- $\alpha = b + \theta$ with b : the buffer power and θ : the liquid saturation.
- $q\nabla$ represents the spatial convection with q : the Darcy flux, with **div** $q = 0$.
- $D\Delta$ the spatial diffusion with D the diffusion coefficient.
- $h(c) = \frac{I\,c}{K+c}$ the Michaelis-Menten function : nutrient absorption function at the root surface. The linear version of h is $h(c) = \frac{I\,c}{K}$ when $K >> c$.
- I : the maximum uptake constant and K : the Michaelis-Menten constant.

The Nutrient uptake model

The Nye-Tinker-Barber (NTB) system (1980's) :

Let the function $c = c(t, x)$ represents the concentration of nutrient,

$$\begin{cases} \alpha \dfrac{\partial c}{\partial t} + q\nabla c - D\Delta c & = & 0 & \text{in} & Q, \\ (D\nabla c - qc).\overrightarrow{n} & = & h(c) & \text{on} & \Sigma_1, \\ (D\nabla c - qc).\overrightarrow{n} & = & 0 & \text{on} & \Sigma_2, \\ c(0, x) & = & c_0(x) & \text{in} & \Omega. \end{cases} \quad (1)$$

Description of the NTB System :

- $\alpha = b + \theta$ with b : the buffer power and θ : the liquid saturation.
- $q\nabla$ represents the spatial convection with q : the Darcy flux, with **div** $q = 0$.
- $D\Delta$ the spatial diffusion with D the diffusion coefficient.
- $h(c) = \frac{I\,c}{K+c}$ the Michaelis-Menten function : nutrient absorption function at the root surface. The linear version of h is $h(c) = \frac{I\,c}{K}$ when $K >> c$.
- I : the maximum uptake constant and K : the Michaelis-Menten constant.

The Nutrient uptake model

The Nye-Tinker-Barber (NTB) system (1980's) :

Let the function $c = c(t,x)$ represents the concentration of nutrient,

$$\begin{cases} \alpha \dfrac{\partial c}{\partial t} + q\nabla c - D\Delta c & = & 0 & \text{in} & Q, \\ (D\nabla c - qc).\overrightarrow{n} & = & h(c) & \text{on} & \Sigma_1, \\ (D\nabla c - qc).\overrightarrow{n} & = & 0 & \text{on} & \Sigma_2, \\ c(0,x) & = & c_0(x) & \text{in} & \Omega. \end{cases} \quad (1)$$

Description of the NTB System :

- $\alpha = b + \theta$ with b : the buffer power and θ : the liquid saturation.
- $q\nabla$ represents the spatial convection with q : the Darcy flux, with **div** $q = 0$.
- $D\Delta$ the spatial diffusion with D the diffusion coefficient.
- $h(c) = \frac{I\,c}{K+c}$ the Michaelis-Menten function : nutrient absorption function at the root surface. The linear version of h is $h(c) = \frac{I\,c}{K}$ when $K >> c$.
- I : the maximum uptake constant and K : the Michaelis-Menten constant.

We introduce the Hilbert space :

$$V = \left\{ \psi \in H^1(\Omega),\ \psi_{|\Gamma_2} = 0 \right\}, \qquad \text{with} \quad \|\psi\|_V^2 = \|\psi\|_{L^2(\Omega)}^2 + \|\nabla\psi\|_{L^2(\Omega)}^2.$$

Proposition

Suppose that the vector field q satisfies $|q| \in L^\infty((0,T)\times\Omega)$. *Then there is a unique solution* $c \in V$ *(here* $c \in L^2(0,T;V)$*) such that :*

$$a(t;c,\psi) = L(t;\psi) \qquad \forall\, \psi \in V, \tag{2}$$

where

$$a(t;c,\psi) = \frac{1}{2}\int_\Omega q.\left(\psi\nabla c - c\nabla\psi\right)\,dx + D\int_\Omega \nabla c\;\nabla\psi\,dx \qquad \psi \in V, \tag{3}$$

and

$$L(t;\psi) = \int_{\Gamma_1} h(c)\psi(x).\mathbf{n}\,d\gamma \qquad \psi \in V. \tag{4}$$

Associated cultures (cropping)

- Absorption of nutrients in polluted soils.
  - → Banana needs important amount of :
    - Water, Nitrogen.
    - Use of chemicals : Nitrogen fertilizers !
  - → Solution : Associated plants
- Optimal control
  - → Control of the nutrient concentration, addition of nutrient (associated plant).
  - → Control of systems of incomplete data (pollution).

Associated cultures (cropping)

- Absorption of nutrients in polluted soils.
  - → Banana needs important amount of :
    - Water, Nitrogen.
    - Use of chemicals : Nitrogen fertilizers !
  - → Solution : Associated plants
- Optimal control
  - → Control of the nutrient concentration, addition of nutrient (associated plant).
  - → Control of systems of incomplete data (pollution).

Associated cultures (cropping)

- Absorption of nutrients in polluted soils.
  - → Banana needs important amount of :
    - Water, Nitrogen.
    - Use of chemicals : Nitrogen fertilizers !
  - → Solution : Associated plants
- Optimal control
  - → Control of the nutrient concentration, addition of nutrient (associated plant).
  - → Control of systems of incomplete data (pollution).

Nutrient uptake model with pollution & Optimal control

The nutrient uptake model with pollution is given by the following system :

$$\begin{cases} \alpha \dfrac{\partial c}{\partial t} + q\nabla c - D\Delta c = g & \text{in} \quad Q, \\ (D\nabla c - qc).\overrightarrow{n} = \frac{I\,c}{K} & \text{on} \quad \Sigma_1, \\ (D\nabla c - qc).\overrightarrow{n} = -v & \text{on} \quad \Sigma_2, \\ c(0,x) = 0 & \text{in} \quad \Omega, \end{cases} \tag{5}$$

with $g \in G \subset L^2(Q)$: unknown pollution function, and $v \in L^2(\Sigma_2)$: control function.

$$\text{Minimize :} \quad J(v,g) = \|c(v,g) - \widetilde{c}\|^2_{L^2(\Sigma_1)} + N\|v\|^2_{L^2(\Sigma_2)} \quad \forall\, g \in G. \tag{6}$$

- A natural idea : $\inf_{v\in L^2(\Sigma_2)} \left(\sup_{g\in L^2(Q)} J(v,g) \right)$. But $\sup_{g\in L^2(Q)} J(v,g) = +\infty$!
- Indeed, we have :

$$c(v,g) = c(v,0) + c(0,g), \qquad \text{and} \qquad c(0,g) = \mathcal{A}^* g$$

→ **No-regret control** : $J(v,g) \leq J(0,g),\ \forall g \in G \subset L^2(Q)$.

Nutrient uptake model with pollution & Optimal control

The nutrient uptake model with pollution is given by the following system :

$$\left\{\begin{array}{rcccc} \alpha\dfrac{\partial c}{\partial t} + q\nabla c - D\Delta c & = & g & \text{in} & Q, \\ (D\nabla c - qc).\overrightarrow{n} & = & \frac{I\,c}{K} & \text{on} & \Sigma_1, \\ (D\nabla c - qc).\overrightarrow{n} & = & -v & \text{on} & \Sigma_2, \\ c(0,x) & = & 0 & \text{in} & \Omega, \end{array}\right. \tag{5}$$

with $g \in G \subset L^2(Q)$: unknown pollution function, and $v \in L^2(\Sigma_2)$: control function.

$$\text{Minimize :} \qquad J(v,g) = \|c(v,g) - \tilde{c}\|^2_{L^2(\Sigma_1)} + N\|v\|^2_{L^2(\Sigma_2)} \quad \forall\, g \in G. \tag{6}$$

- A natural idea : $\inf_{v\in L^2(\Sigma_2)}\left(\sup_{g\in L^2(Q)} J(v,g)\right)$. But $\sup_{g\in L^2(Q)} J(v,g) = +\infty$!
- Indeed, we have :

$$c(v,g) = c(v,0) + c(0,g), \qquad \text{and} \qquad c(0,g) = \mathcal{A}^* g$$

→ **No-regret control** : $J(v,g) \leq J(0,g),\ \forall g \in G \subset L^2(Q)$.

Nutrient uptake model with pollution & Optimal control

The nutrient uptake model with pollution is given by the following system :

$$\left\{\begin{array}{rcccl} \alpha\dfrac{\partial c}{\partial t} + q\nabla c - D\Delta c & = & g & \text{in} & Q, \\ (D\nabla c - qc).\overrightarrow{n} & = & \frac{I\,c}{K} & \text{on} & \Sigma_1, \\ (D\nabla c - qc).\overrightarrow{n} & = & -v & \text{on} & \Sigma_2, \\ c(0,x) & = & 0 & \text{in} & \Omega, \end{array}\right. \tag{5}$$

with $g \in G \subset L^2(Q)$: unknown pollution function, and $v \in L^2(\Sigma_2)$: control function.

$$\text{Minimize :} \qquad J(v,g) = \|c(v,g) - \widetilde{c}\|^2_{L^2(\Sigma_1)} + N\|v\|^2_{L^2(\Sigma_2)} \quad \forall\, g \in G. \tag{6}$$

- A natural idea : $\displaystyle\inf_{v\in L^2(\Sigma_2)}\left(\sup_{g\in L^2(Q)} J(v,g)\right)$. But $\displaystyle\sup_{g\in L^2(Q)} J(v,g) = +\infty$!
- Indeed, we have :

$$c(v,g) = c(v,0) + c(0,g), \qquad \text{and} \qquad c(0,g) = \mathcal{A}^* g$$

$\rightarrow$ **No-regret control** : $J(v,g) \leq J(0,g),\ \forall\, g \in G \subset L^2(Q)$.

Definition

We say that the function $u \in L^2(\Sigma_2)$ *is a* **no-regret control**, *if it is a solution of the following new MinMax problem :*

$$\inf_{v \in L^2(\Sigma_2)} \left(\sup_{g \in G} [J(v, g) - J(0, g)] \right). \quad (7)$$

$$J(v, g) - J(0, g) = J(v, 0) - J(0, 0) + 2\langle c(v, 0), c(0, g) \rangle$$

Remark : $\langle c(v, 0), c(0, g) \rangle = \langle \xi(v), g \rangle$, ξ linear sol. to the adjoint problem

Definition

We say that the function $u \in L^2(\Sigma_2)$ is a **no-regret control**, *if it is a solution of the following new MinMax problem :*

$$\inf_{v \in L^2(\Sigma_2)} \left(\sup_{g \in G} [J(v, g) - J(0, g)] \right) . \qquad (7)$$

$$J(v, g) - J(0, g) = J(v, 0) - J(0, 0) + 2\langle\, c(v, 0), c(0, g)\,\rangle$$

Remark : $\langle\, c(v, 0), c(0, g)\,\rangle = \langle\, \xi(v), g\,\rangle$, ξ linear sol. to the adjoint problem

The Low-regret control u_γ is a low-regret control *iff* :

$$\inf_{v \in L^2(\Sigma_2)} \left(\sup_{g \in G} \left(J(v, g) - J(0, g) - \gamma \|g\|_G^2 \right) \right)$$

Definition

We say that the function $u \in L^2(\Sigma_2)$ is a **no-regret control**, *if it is a solution of the following new MinMax problem :*

$$\inf_{v\in L^2(\Sigma_2)} \left(\sup_{g\in G} [J(v,g) - J(0,g)] \right). \tag{7}$$

$$J(v,g) - J(0,g) = J(v,0) - J(0,0) + 2\langle\, c(v,0), c(0,g)\,\rangle$$

Remark : $\langle\, c(v,0), c(0,g)\,\rangle = \langle\, \xi(v), g\,\rangle$, ξ linear sol. to the adjoint problem

The Low-regret control u_γ is a low-regret control *iff* :

$$\inf_{v\in L^2(\Sigma_2)} \left(\sup_{g\in G} \left(J(v,g) - J(0,g) - \gamma\|g\|_G^2\right) \right)$$

$$= \inf_{v\in L^2(\Sigma_2)} \left(J(v,0) - J(0,0) + \sup_{g\in G} \left[2\langle\, \xi(v), g\,\rangle - \gamma\|g\|_G^2\right] \right) \quad \text{(conjugate)}$$

$$= \inf_{v\in L^2(\Sigma_2)} \left(J(v,0) - J(0,0) + \frac{1}{\gamma}\left\|\xi(v)\right\|_{G'} \right) = \inf_{v\in L^2(\Sigma_2)} \mathcal{J}^\gamma(v).$$

Existence of a unique low-regret control

Proposition

The optimal problem $\inf_{v \in L^2(\Sigma_2)} \mathcal{J}^\gamma(v)$ *admits a unique solution* u_γ *called* **low-regret control** *for the NTB system with pollution.*

Existence of a unique low-regret control

Proposition

The optimal problem $\inf_{v\in L^2(\Sigma_2)} \mathcal{J}^\gamma(v)$ *admits a unique solution* u_γ *called* **low-regret control** *for the NTB system with pollution.*

Proof - We recall that $\mathcal{J}^\gamma(v) = J(v,0) - J(0,0) + \dfrac{1}{\gamma}\|\xi(v)\|^2_{L^2(Q)}$ and that $J(v,0) = \|c(v,0) - \tilde{c}\|^2_{L^2(\Sigma_1)} + N\|v\|^2_{L^2(\Sigma_2)}$ and $J(0,0) = \|\tilde{c}\|^2_{L^2(\Sigma_1)}$.

Existence of a unique low-regret control

Proposition

The optimal problem $\inf\limits_{v\in L^2(\Sigma_2)} \mathcal{J}^\gamma(v)$ *admits a unique solution* u_γ *called* **low-regret control** *for the NTB system with pollution.*

Proof - We recall that $\mathcal{J}^\gamma(v) = J(v,0) - J(0,0) + \frac{1}{\gamma}\|\xi(v)\|^2_{L^2(Q)}$ and that

$J(v,0) = \|c(v,0) - \tilde{c}\|^2_{L^2(\Sigma_1)} + N\|v\|^2_{L^2(\Sigma_2)}$ and $J(0,0) = \|\tilde{c}\|^2_{L^2(\Sigma_1)}$.

The cost function $\mathcal{J}^\gamma(v)$ satisfies $\mathcal{J}^\gamma(v) \geq -J(0,0)$, for any $v \in L^2(\Sigma_2)$. Therefore, it exists $k_\gamma = \inf\limits_{v\in L^2(\Sigma_2)} \mathcal{J}^\gamma(v)$. Consider a minimizing sequence $\{v_n(\gamma)\} = \{v_n\}$, then :

$$\|c(v_n,0) - \tilde{c}\|^2_{L^2(\Sigma_1)} + N\|v_n\|^2_{L^2(\Sigma_2)} + \frac{1}{\gamma}\|\xi(v_n)\|^2_{L^2(Q)} \leq k_\gamma + 1 + \|\tilde{c}\|^2_{L^2(\Sigma_1)}.$$

Existence of a unique low-regret control

Proposition

The optimal problem $\inf_{v\in L^2(\Sigma_2)} \mathcal{J}^\gamma(v)$ *admits a unique solution* u_γ *called* **low-regret control** *for the NTB system with pollution.*

Proof - We recall that $\mathcal{J}^\gamma(v) = J(v,0) - J(0,0) + \frac{1}{\gamma}\|\xi(v)\|^2_{L^2(Q)}$ and that

$J(v,0) = \|c(v,0) - \tilde{c}\|^2_{L^2(\Sigma_1)} + N\|v\|^2_{L^2(\Sigma_2)}$ and $J(0,0) = \|\tilde{c}\|^2_{L^2(\Sigma_1)}$.

The cost function $\mathcal{J}^\gamma(v)$ satisfies $\mathcal{J}^\gamma(v) \geq -J(0,0)$, for any $v \in L^2(\Sigma_2)$. Therefore, it exists $k_\gamma = \inf_{v\in L^2(\Sigma_2)} \mathcal{J}^\gamma(v)$. Consider a minimizing sequence $\{v_n(\gamma)\} = \{v_n\}$, then :

$$\|c(v_n,0) - \tilde{c}\|^2_{L^2(\Sigma_1)} + N\|v_n\|^2_{L^2(\Sigma_2)} + \frac{1}{\gamma}\|\xi(v_n)\|^2_{L^2(Q)} \leq k_\gamma + 1 + \|\tilde{c}\|^2_{L^2(\Sigma_1)}.$$

$\rightarrow$ Notations : $\mathcal{A} := \alpha\frac{\partial}{\partial t} + q\nabla - D\Delta$ in Q, and $\mathcal{B} := D\nabla - q$ on Σ.

Proposition

The low-regret control u_γ satisfies to :

$$\langle c(u_\gamma,0)-\tilde{c}, c(w,0)\rangle_{L^2(\Sigma_1)} + N\langle u_\gamma, w\rangle_{L^2(\Sigma_2)} + \langle \frac{1}{\gamma}\xi(u_\gamma), \xi(w)\rangle_{L^2(Q)} = 0, \qquad \forall w \in L^2(\Sigma_2).$$

Proof - We use the Euler-Lagrange formula : $\lim_{\lambda\to 0}\left(\frac{\mathcal{J}^\gamma(u_\gamma+\lambda w)-\mathcal{J}^\gamma(u_\gamma)}{\lambda}\right)=0.$

Proposition

The low-regret control u_γ satisfies to :

$$\langle c(u_\gamma,0)-\tilde{c}, c(w,0)\rangle_{L^2(\Sigma_1)} + N\langle u_\gamma, w\rangle_{L^2(\Sigma_2)} + \langle \frac{1}{\gamma}\xi(u_\gamma), \xi(w)\rangle_{L^2(Q)} = 0, \qquad \forall w \in L^2(\Sigma_2).$$

Proof - We use the Euler-Lagrange formula : $\lim_{\lambda\to 0}\left(\dfrac{\mathcal{J}^\gamma(u_\gamma+\lambda w)-\mathcal{J}^\gamma(u_\gamma)}{\lambda}\right) = 0.$

Theorem

The low-regret control is characterized by the unique quadruplet $\{c_\gamma, \rho_\gamma, \xi_\gamma, p_\gamma\}$ s.t. :

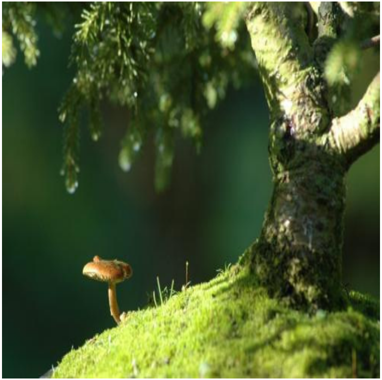
$$\begin{cases} \mathcal{A}c_\gamma = 0, & \mathcal{A}^*\xi_\gamma = 0, & \mathcal{A}\rho_\gamma = \frac{1}{\gamma}\xi_\gamma, & \mathcal{A}^* p_\gamma = 0 & \text{in } Q, \\ (\mathcal{B}c_\gamma).\mathbf{n} = h(c_\gamma), & (\mathcal{B}^*\xi_\gamma).\mathbf{n} = r_\gamma, & (\mathcal{B}\rho_\gamma).\mathbf{n} = h(\rho_\gamma), & -(\mathcal{B}^* p_\gamma).\mathbf{n} = k_\gamma & \text{on } \Sigma_1, \\ (\mathcal{B}c_\gamma).\mathbf{n} = -u_\gamma, & (\mathcal{B}^*\xi_\gamma).\mathbf{n} = 0, & (\mathcal{B}\rho_\gamma).\mathbf{n} = 0, & (\mathcal{B}^* p_\gamma).\mathbf{n} = 0 & \text{on } \Sigma_2, \\ c_\gamma(0) = 0, & \xi_\gamma(T) = 0, & \rho_\gamma(0) = 0, & p_\gamma(T) = 0 & \text{in } \Omega, \end{cases}$$

$$r_\gamma = c(u_\gamma,0) + \tfrac{l}{K}\xi(u_\gamma), \quad m_\gamma = c_\gamma + \tilde{c} + \rho_\gamma, \quad \text{and } k_\gamma = -m_\gamma + \tfrac{l}{K}p_\gamma,$$

with the adjoint equation :

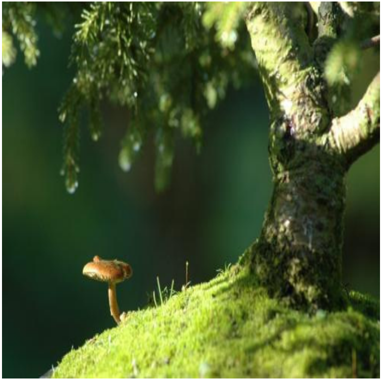
$$p_\gamma + N u_\gamma = 0 \quad \text{in } L^2(Q).$$

Some remarks



- Work in progress
  - → No-regret control
  - → Numerical simulation
  - → Comparison : Numerical curves to the measurements ?
- Nutrient transfert : Plant-fungus association
  - → Mycorhizes ?
  - → Problem of scale ?
  - → Other applications in biology..

Some remarks



- Work in progress
  - → No-regret control
  - → Numerical simulation
  - → Comparison : Numerical curves to the measurements ?
- Nutrient transfert : Plant-fungus association
  - → Mycorhizes ?
  - → Problem of scale ?
  - → Other applications in biology..

References

- **Lions J.-L. (1992)**, *Contrôle à moindres regrets des systèmes distribués*, C. R. Acad. Sciences de Paris, Ser I Maths, 315 pp 1253-1257.
- **Louison L., Omrane A., Ozier-Lafontaine H., Picart D. (2015)**, *Modeling plant nutrient uptake : Mathematical analysis and optimal control*, Evol. Equations and Control Theory, 4 (2) pp 193-203.
- **Louison L., Omrane A.**, *Modeling plant nutrient uptake under pollution : Mathematical analysis and optimal control* (to appear).
- **Ptashnyk M. (2010)**, *Derivation of a macroscopic model for nutrient uptake by hairy-roots*, Nonlinear Analysis : Real World Applications, 11, pp 4586–4596.
- **Roose T. (2001)**, *A mathematical model of plant nutrient uptake*, J. Math. Biol 42, pp 347–360.
- **Tinker P. B., Nye P. H. (2000)**, *Solute movement in the rhizosphere*, Oxford Press University.

Thank you for your attention !