Selfsimilar solutions to coagulation and fragmentation equations

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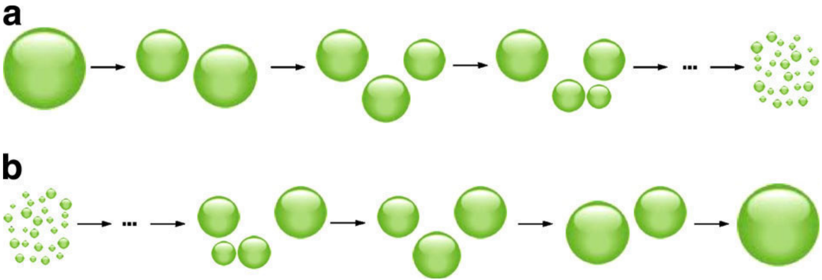
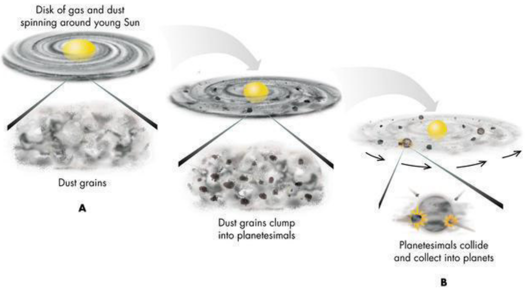
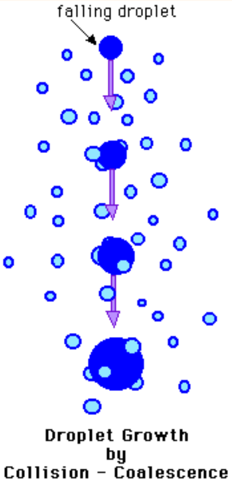
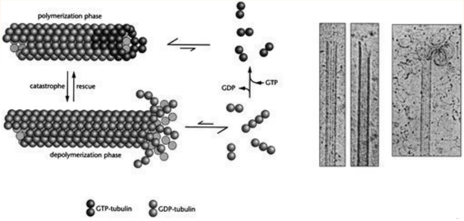


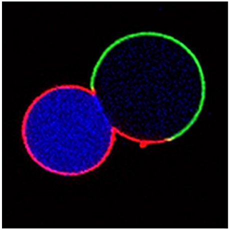
Applications to Coagulation-fragmentation

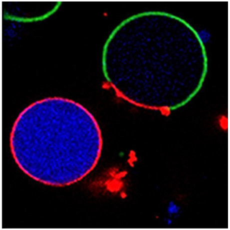


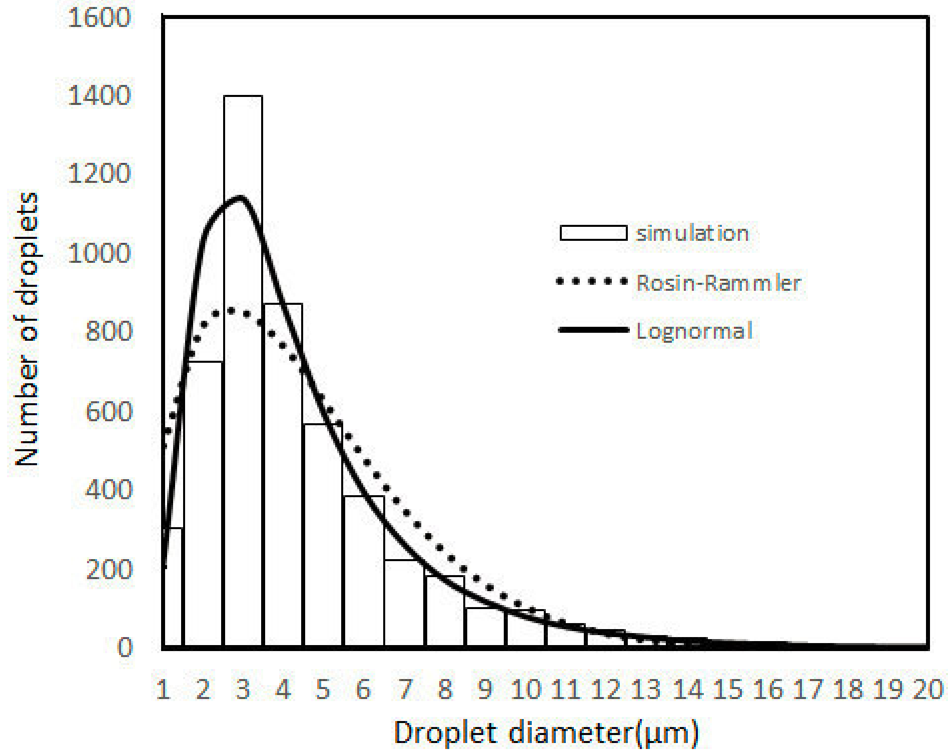


More complex applications

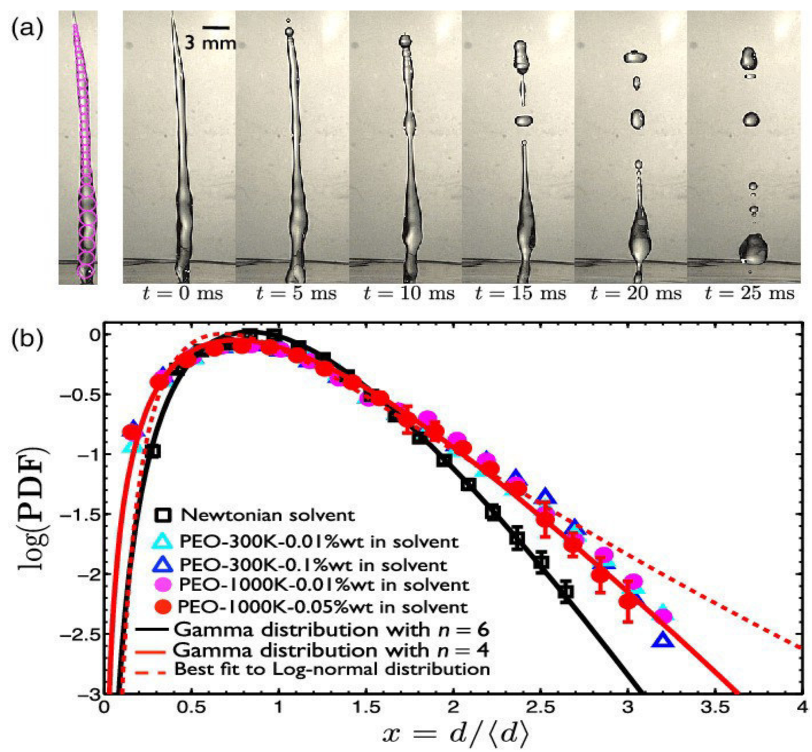








Reddy, Banerjee, 2017



McKinley et al, 2016

The coagulation-fragmentation equation

One can write down a coagulation-fragmentation equation of balance of mass:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{c}(x,t) = \boldsymbol{Q}_{\mathsf{C}}(\mathrm{c}) - \boldsymbol{Q}_{\mathsf{F}}(\mathrm{c}),$$

$$\boldsymbol{Q}_{\mathsf{C}}(\mathrm{c}) := \frac{1}{2}\int_0^x K(x-y,y)\,\mathrm{c}(x-y,t)\,\mathrm{c}(y,t)\;\mathrm{d}y - \mathrm{c}(x,t)\int_0^\infty K(x,y)\,\mathrm{c}(y,t)\;\mathrm{d}y$$

$$\boldsymbol{Q}_{\mathsf{F}}(\mathrm{c}) := \beta(x)\,\mathrm{c}(x,t) - \int_x^\infty \mathrm{c}(y,t)\,\frac{\beta(y)}{y}\mathrm{B}\left(\frac{x}{y}\right)\;\mathrm{d}y$$

where $\beta(x)$ is the fragmentation rate of clusters of size x and $\mathrm{B}(u)$ is the daughter fragments distribution kernel.
The mass depending and symmetric operator $K(x,y)$ is known as the coagulation kernel and describes the rate at which particles of the given size coagulate.

Unfortunately, most of the physically interesting models correspond to equations that are not exactly solvable.

Pure coagulation: the Smoluchowski equation

- It has been possible to discover analytical solutions exclusively for a limited number of kernels; the principal cases include: $K_c(x,y) = 1$, $K_+(x,y) = x + y$, $K_\times(x,y) = xy$.
- For more general kernels, studies have essentially relied on numerical methods.
- Existence, positivity and uniqueness result for all time for kernels growing at most **linearly**.

INTERESTS:

- **singularities** in finite time occur for some Kernels: very strong coagulation creates infinitely dense clusters.
- **self-similar solutions**.

Laplace transform: $$\varphi(\eta,t) = \int_0^\infty e^{-\eta x} c(x,t)dx$$

Regularized Laplace transform: $$\Phi(\eta,t) = -\int_0^\infty (e^{-\eta x}-1)xc(x,t)dx$$

$$\boxed{\begin{aligned} \partial_t\phi(\eta,t) &= -(\phi(\eta,t))^2, & \lambda&=0;\\ \partial_t\phi(\eta,t)-\phi(\eta,t)\,\partial_\eta\phi(\eta,t) &= -\phi(\eta,t), & \lambda&=1;\\ \partial_t\Phi(\eta,t)-\Phi(\eta,t)\,\partial_\eta\Phi(\eta,t) &= 0, & \lambda&=2. \end{aligned}}$$

$$K(x,y) = (xy)^{1-\varepsilon}$$

Fractional Burgers equation: $$\boxed{\partial_t\Phi(\eta,t) - D^{-\varepsilon}\Phi(\eta,t)\partial_\eta(D^{-\varepsilon}\Phi(\eta,t)) = 0}$$

Nonlocal transport equation: $\theta_t = D^{(\gamma)}(\theta)\theta_x,$ MAF, Eggers, to appear in Nonlinearity 2019

(Toy model for Euler eqn.) $D^{(0)}(\theta) = H(\theta).$ Existence of singularities: A. Córdoba, D. Córdoba, MAF, Ann. of Math. 2005

$$\theta = (t_0 - t)^{\alpha}\Theta\left(\frac{x}{(t_0-t)^{\beta}}\right)$$

$$\alpha - 1 = 2\alpha - (1+\gamma)\beta$$

$$\beta = \frac{1+\alpha}{1+\gamma}$$

$$\theta \sim |x|^{\frac{\alpha(1+\gamma)}{1+\alpha}}, \text{ as } |x| \to 0$$

$$\gamma = 0 \to |x|^{0.541...}$$
$$\gamma = 1 \to |x|^{\frac{4}{3}}$$

Selfsimilarity of the second Kind (Barenblatt)

$$\alpha \approx 1.181\ldots \text{ for } \gamma = 0,$$

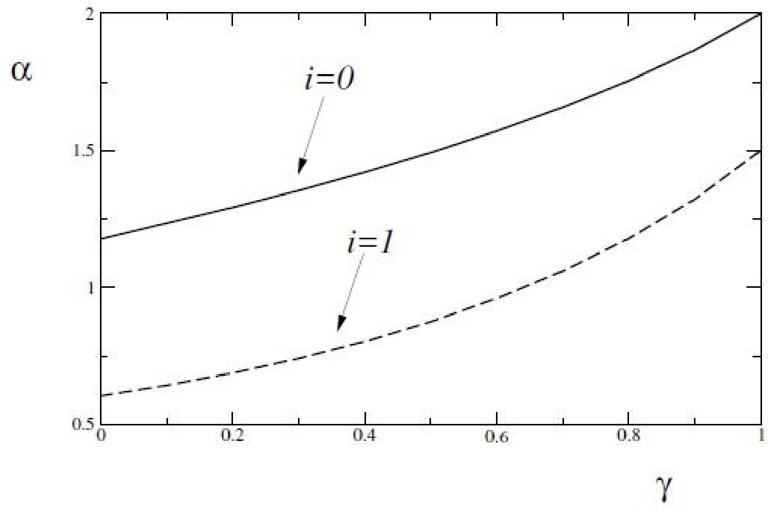


Figure 2. The exponent α as a function of γ. The ground state ($i = 0$) is shown as the solid line, the first unstable branch ($i = 1$) is the dashed line. For $\gamma = 1$, $\alpha = 2$ in the ground state, and $\alpha = 3/2$ in the first unstable state.

Singularity = gelation

GELATION:

Starting from an initial particle size distribution $c_0 = c(x, 0)$ such that all its moments $M_i(0) = \int_0^\infty x^i c_0(x)\, dx$ are bounded, there is a certain time t^* such that all moments $M_i(t)$ for $i \geq i_0$ diverge.

The distribution $c(x, t)$ develops an **heavy algebraic tail in finite time**.

Up to $t < t^*$ total conservation of mass is guaranteed, but after t^* the first moment M_1 is no longer conserved and starts to decrease.

This phenomenon is called finite time gelation and indicates the aggregation of the particles in a cluster of infinite density that drains mass from the coagulating system with finite x-mass.

An interesting problem is the continuation of **post-gelling solutions**.

Some mathematical well-posedness results I

	case:	hypothesis and setting:	results:	references:
pure coagulation	asymptotically sub-linear vs asymptotically sub-quadratic:	$K(x,y) \le \kappa_0(1+x+y)$, or $\kappa_0(1+x+y) < K(x,y) \le \kappa_1 xy$	$\exists!$, $T^*=\infty$ $\exists!$, $T^*<\infty$ (estimation)	Classic reference: e.g. Norris99
	homogeneous non-gelling coagulation:	$K(x,y) \le \kappa_0\left(x^\lambda + y^\lambda\right)$, $\lambda \in (-\infty,1]$	$\exists!$ weak sense, $T^*=\infty$	Fournier-Laurençot 2006
	homogeneous gelling coagulation:	$\kappa_1(xy)^{\frac{\lambda}{2}} \le K(x,y) \le \kappa_2\left(xy^{\lambda-1} + x^{\lambda-1}y\right)$, with λ the degree of homogeneity of K and $\lambda \in (1,2]$	$\exists!$ weak sense, $T^*<\infty$, lower bound	Fournier-Laurençot 2006
	faster-growing, singular kernel:	various formulations, for example: $K(x,y) \ge A\left((1+x)^\lambda + (1+y)^\lambda\right)$, $\lambda > 1$	instantaneous gelling solutions	Carr and da Costa 1992

Some mathematical well-posedness results II

	case:	hypothesis and setting:	results:	references:
pure fragmentation	singular rate of fragmentation:	$\beta(x)=x^{\lambda}$, $\lambda<0$	"shattering" critic time is explicitly known	McGrady–Ziff 87 Cheng–Redner 90
	non-singular case:	mass conserving distribution $\int_0^x yb(y,x)\,\mathrm{d}y=x$	well-posed in $\mathrm{L}^1_{x\mathrm{d}x}$ mass-conservation	Lamb 2004
	homogeneous rate with infinite fragments:	$\nu\in(-2,-1]$ and $B(u)=(\nu+2)u^{\nu}$	well-posedness	Cepeda 2014
	finite positive moments in the non-singular case:	$M_k(c_0)<\infty$ for all $k\in\mathbb{N}$	$M_k<\infty$	Equicontinuous semigroups in Fréchet spaces: Banasiak-Lamb 2014

Some mathematical well-posedness results III

	case:	hypothesis and setting:	results:	references:
coagulation-fragmentation	fragmentation dominates over coagulation:	$K(x,y) \leq \kappa_1 (xy)^\lambda, \beta(x) \geq \kappa_2 x^\gamma$, $\lambda \in [0,1], \gamma \in (-1,\infty), 2+\gamma > 2\lambda$	a mass conserving solution exists: $T^* = \infty$	Many strategies, Escobedo Laurençot and Perthame 2003
	strong fragmentation and weak coagulation:	$\beta(x) = x^\lambda, B(u) = (\nu+2)u^\nu$, $\nu \in (-1,0], 0 \leq \sigma \leq 1, \sigma < \lambda$, and $K(x,y) \leq \kappa_2((1+x)^\sigma + (1+y)^\sigma)$	global-in-time and unicity	Banasiak, Lamb and Langer 2013
	strong fragmentation and strong coagulation:	the same as above and $0 \leq \sigma \leq \tau < \lambda$ and $K(x,y) \leq \kappa_1 (x^\sigma y^\tau + x^\tau y^\sigma)$	local solution and uniqueness	as above
	detailed balance condition:	let $M \in L^1_1$ be a non-zero function $K(x,y)M(x)M(y) = b(x,y)M(x+y)$.	solutions converge to an equilibrium: $c_{[M_1(0)]}(x)$ depending only on the initial mass	Carr-daCosta 94, Dubowski-Stewart 96
	other equilibria:	strong fragmentation, small initial mass (the latter is a technical condition)	solutions converge to an equilibrium $c_{[M_1(0)]}(x)$ L^2 exponential trend	Fournier–Mischler 2004

Smoluchowski coagulation equation and self-similarity

A relevant feature is the **role of the self-similar solutions**.

The long term regime in known models approaches to a self-similar profile.

Self-similar approach: gelling case

- **Homogeneus multiplicative kernel** $K(x,y) = (xy)^{1-\varepsilon}$; for $0 \leq \varepsilon < \frac{1}{2}$ we are in gelation regime. Possible solutions can behave asymptotically as $t \to t^*$:
$$c(x,t) = (t^* - t)^{\alpha}\, \psi\left((t^* - t)^{\beta} x\right)$$
- In this case, the self-similar solutions have not been determined explicitly and, more remarkably, from numerical experiments, the similarity exponents cannot be determined from simple dimensional considerations (`cf. Lee, 2001`).
- So they seem to belong to the class of the so called **self-similarity of the second kind**.
- Remember: when $\varepsilon < 0$, the strong coagulation rate yields *instantaneous gelation*.

Substituting:

$$-(\beta(3-2\varepsilon)-1)\psi(\xi)-\beta\xi\psi'(\xi)=$$

$$=\frac{1}{2}\int_{[0,\xi]}(\xi-y)^{1-\varepsilon}\psi(\xi-y)\,y^{1-\varepsilon}\psi(y)\,dy-\xi^{1-\varepsilon}\psi(\xi)\int_{\mathbb{R}+}y^{1-\varepsilon}\psi(y)\,dy.$$

where, again, $\alpha=(3-2\varepsilon)\,\beta-1$, from dimensional considerations.

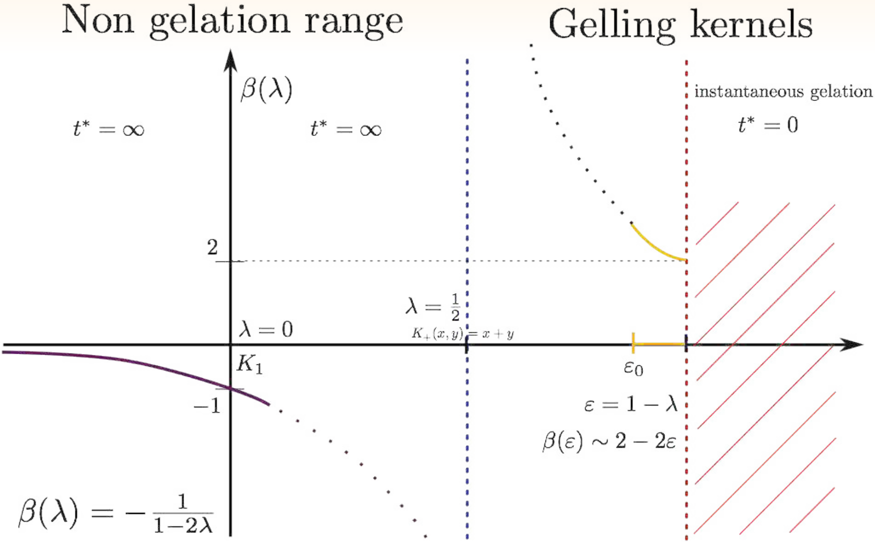
We have $\beta<0$ in non-gelling cases and $\beta>0$ otherwise. Therefore, we study the self-similar problem for all $\lambda-$multiplicative kernels:

$$|\beta|\,\mathbf{T}\,[\psi]+\mathbf{Q}_{\mathbf{C},\lambda}\,[\psi]=0$$

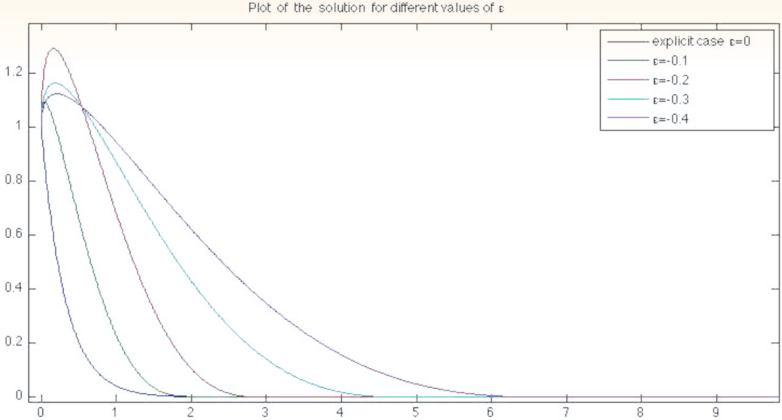
The results on coagulation

A visual overview

Picture: self-similarity exponents



Picture: gelling solutions profiles



The solutions are multiplied times $\xi^{3-2\varepsilon-\frac{1}{\beta}}$. Two behaviors: (I) polynomial expansion at the origin, (II) quasi-exponential decay at infinity.

The gelling case result: main ingredients

Recall:

$$
\begin{aligned}
&- \left(\beta \left(3 - 2\varepsilon\right) - 1\right) \psi(\xi) - \beta \xi \psi'(\xi) = \\
&= \frac{1}{2} \int_{[0,\xi]} \left(\xi - y\right)^{1-\varepsilon} \psi\left(\xi - y\right)\, y^{1-\varepsilon} \psi\left(y\right)\, dy - \xi^{1-\varepsilon} \psi\left(\xi\right) \int_{\mathbb{R}^+} y^{1-\varepsilon} \psi\left(y\right)\, dy.
\end{aligned}
$$

Laplace transform

We studied the s.-s. solutions to Smoluchowski equation close to the product kernel. Our strategy involves a *regularized Laplace transform*:

- Transform: $\Phi(\eta) = -\int_{\mathbb{R}^+} (e^{-\eta\xi} - 1)\xi\psi(\xi)\, d\xi.$
- Inverse transform: $\psi(\xi) = \frac{1}{2\pi i}\frac{1}{\xi^2}\int_{-i\infty}^{i\infty} e^{\eta\xi}\Phi'(\eta)\, d\eta.$

ONE FORMALLY DERIVES A NEW PROBLEM:

$$-\left(\left(1-2\varepsilon\right)\beta - 1\right)\Phi(\eta) + \beta\eta\Phi'(\eta) = \frac{1}{2}\frac{d}{d\eta}\left[D_\eta^{-\varepsilon}\Phi(\eta)\right]^2,$$

with:

$$D_\eta^{-\varepsilon}\Phi(\eta) = -\int_{\mathbb{R}^+} (e^{-\eta\xi} - 1)\xi^{1-\varepsilon}\psi(\xi)\, d\xi.$$

The explicit case

If $\varepsilon = 0$, we can deduce previously known results:
Laplace variable equation reduces to a first order ordinary differential equation, and we can easily obtain the implicit expression:

$$\eta(\Phi) = k\Phi^{\frac{\beta}{\beta-1}} + \Phi$$

With the **specific election of $\beta = 2$**, this equation is a second order polynomial; we can find its zeros and obtain two possible solutions, but we only consider the one giving a positive ψ:

$$\Phi_0\left(\eta\right) = 2\pi\left(-1 + \sqrt{1 + \frac{\eta}{\pi}}\right), \qquad \psi\left(\xi\right) = \frac{1}{\xi^{\frac{5}{2}}} e^{-\pi\xi}.$$

We can also study the equation for different values of β, but the inverse transform ψ presents algebraic tails.

Small perturbation of the product Kernel

Starting from an explicit solution, we look for a branch of solutions perturbatively.

$$\beta(\varepsilon) = 2 + \varepsilon\lambda(\varepsilon)$$

$$\Phi(\eta) = \Phi_0(\eta) + \varepsilon\Phi_1(\eta)$$

where Φ_1 should be controlled in a suitable norm and the decay properties of Φ permit studing the decay properties of ψ

The first fixed point theorem

Theorem.

There exists an $\varepsilon_0 > 0$ *and a function*

$$\lambda(\varepsilon) = 2 + O(\varepsilon)$$

such that for any $0 < \varepsilon < \varepsilon_0$ *and with* $\beta(\varepsilon) = 2 + \varepsilon\lambda(\varepsilon)$ *there exists a unique solution to Smoluchowski's self-similar equation (up to rescaling) satisfying:*

$$\int_0^\infty \xi^{\frac{7}{2}-\frac{\delta}{2}} \psi(\xi)\, d\xi < \infty$$

for any $0 < \delta \ll 1$.

Both the **behaviour of Φ for $\eta \sim 0$ and at infinity must be controlled**. Then, decaying properties of Φ give estimates on moments like the $\frac{7}{2}$-th above.

Setting of the nonlinear problem

Let $\Phi\left(\eta\right)=\Phi_0\left(\eta\right)+\varepsilon\Phi_1\left(\eta\right).$ We get the Laplace self-similar equation for Φ_1:

$$\Phi_1-2\eta\frac{d}{d\eta}\Phi_1+\frac{d}{d\eta}\left(\Phi_0\Phi_1\right)=F_0(\eta)+\varepsilon L\Phi_1+\varepsilon Q(\Phi_1,\Phi_1)$$

Linear and quadratic terms in Φ_1 are $O\left(\varepsilon\right)$, so, calling $N\left[\Phi_1\right]$ the right hand side, we define a mapping that assigns to a given $\overline{\Phi}_1$ the solution to

$$\Phi_1-2\eta\frac{d}{d\eta}\Phi_1+\frac{d}{d\eta}\left(\Phi_0\Phi_1\right)=N\left[\overline{\Phi}_1\right]$$

this is done in the following.

Steps

- Establish a Hardy-like inequality: $\mathcal{M}_\gamma \leq \int_{-\infty}^{\infty} \left(W\left(|k|\right)\right)^2 \left|\Phi'\left(ik\right)\right|^2 \, \mathrm{d}k$;
- this gives a functional space Y to look for a solution $\Phi\left(\eta\right)$.
- The solution must be close (a sphere of small radius) to a suitable function $h\left(\eta\right)$;
- and its behavior *close to zero* determines the decay of $\psi\left(\xi\right)$ *at infinity*. We want ψ to decay faster than some power.
- An analyticity condition on $h\left(\eta\right)$ -that is: eliminating an $\eta^2 \log \eta$ term- fixes the value of β and makes ψ decay faster than such power.
- Now use a fixed-point theorem (the tricky part is to deal with h at the same time).
- Finally, use the moment-equation to obtain the strong result: **all higher moments are also bounded**.

Analysis of the expansion:

1. Given $\overline{\Phi}_1(\eta) = \overline{a_1}\eta - \frac{1}{\pi}\overline{a_1}\eta^2 + o(|\eta|^2)$ one can compute $\left(N\overline{\Phi}_1\right)(\eta) = \overline{b_1}\eta + \overline{b_2}\eta^2 + O(|\eta|^3)$ as $\eta \to 0$;
2. Consider $\rho(\eta) = \frac{\overline{b_1}\eta + \overline{b_2}\eta^2}{1-\frac{\eta^2}{2}}$ and
$h(\eta) \equiv \frac{\left(\sqrt{1+\frac{\eta}{\pi}}-1\right)^2}{\sqrt{1+\frac{\eta}{\pi}}} \int_{\eta_0}^{\eta} \frac{\sqrt{1+\frac{w}{\pi}}}{\left(\sqrt{1+\frac{w}{\pi}}-1\right)^2} \frac{\rho(w)}{\Phi_0(w)-2w} dw;$
3. Then, integrating the differential equation for Φ_1,

$$\Phi_1(\eta) - h(\eta) = \frac{\left(\sqrt{1+\frac{\eta}{\pi}}-1\right)^2}{\sqrt{1+\frac{\eta}{\pi}}} \int_0^{\eta} \frac{\sqrt{1+\frac{w}{\pi}}}{\left(\sqrt{1+\frac{w}{\pi}}-1\right)^2} \frac{\left(N\overline{\Phi}_1\right)(w) - \rho(w)}{\Phi_0(w) - 2w} dw;$$

4. Then $\Phi_1 - h$ is a $o\left(\eta^2\right)$ and we can compute explicitly

$$h(\eta) = \overline{b_1}\eta - \frac{1}{\pi}\overline{b_1}\eta^2 - \left(\overline{b_2} + \frac{3}{4\pi}\overline{b_1}\right)\eta^2 \ln \eta + O(|\eta|^3 \log|\eta|).$$

The analyticity condition

The presence of the $\eta^2 \ln \eta$ logarithmic term in the Laplace solution implies that the corresponding inverse transform presents an algebraic decay:

$$\psi(\xi) \simeq_{\xi \to \infty} \frac{\sin(|\varepsilon| \pi)}{\xi^{4+O(\varepsilon)}}.$$

For a **faster decay**,

$$\overline{b_2} + \frac{3}{4\pi} \overline{b_1} = 0$$

which fixes $\beta(\varepsilon) = 2 + 2\varepsilon + O(\varepsilon^2)$. Higher orders can be also **explicitly computed** imposing more terms in the expansion of Φ_1. This **analyticity condition** characterizes the self-similar problem as one of the **second kind**.

The norm

We obtain estimates of $(\Phi_1(\eta) - h(\eta))$ in terms of $J(\eta) \equiv \left(N\left[\overline{\Phi}_1\right](\eta) - \rho(\eta)\right)$ in an appropriate functional space:

Definition

Let X be the space of functions f such that:

$$\|f(k)\|_X \equiv \int_{-\infty}^{\infty} \left(\frac{1}{|k|^3} + \frac{1}{|k|^{\frac{3}{2}}}\right)^2 |f(k)|^2 \, dk < \infty.$$

Let Y be the subspace of X whose functions are such that:

$$\|f(k)\|_Y \equiv \|f(k)\|_X + \left\|k \frac{d}{dk} f(k)\right\|_X < \infty.$$

How to obtain the $\frac{7}{2} - th$ moment

By Banach's fixed-point theorem, we find the solution to the nonlinear problem such that $\|\Phi_1(ik) - h(ik)\|_Y < \infty$ and hence:

$$\int_{-\infty}^{\infty} \left(\frac{1}{|k|^2} + \frac{1}{|k|^{\frac{1}{2}}} \right)^2 \left| \Phi_1'(ik) - h'(ik) \right|^2 dk < \infty.$$

To conclude the theorem, we employ

$$\int_{\mathbb{R}^+} \xi^{(\gamma+2)} |\psi(\xi)| \, d\xi < C \int_{\mathbb{R}} \left| \Phi'(k) \right|^2 \left(|k|^{1-2\gamma-\delta} + |k|^{1-2\gamma+\delta} \right) dk$$

with $\gamma = \frac{3}{2} - \frac{\delta}{2}$ and $0 < \delta \ll 1$.

Faster decay for the self-similar solution

Theorem. [illegible]

Under the previous hipotesis, we can conclude that for any $|\varepsilon| < \varepsilon_0$ and with the same $\beta(\varepsilon) = 2 + \varepsilon\lambda(\varepsilon)$ found before, there exists a unique solution (up to rescaling) with all its moments M_α $(\alpha \geq 2)$ bounded.

Remarkably, *no further restriction* is placed upon ε_0 or β.

We multiply the s.s. Smoluchowski equation at both sides by ξ^{α} with $\alpha > \frac{5}{2}$ and integrate to obtain:

$$\left(\alpha - 2 + 2\varepsilon + \frac{1}{\beta}\right)\mathcal{M}_{\alpha} = \frac{1}{2\beta}\int_{\mathbb{R}+}\int_{[0,\xi]} \left((\xi - y) + y\right)^{\alpha}(\xi - y)^{1-\varepsilon} y^{1-\varepsilon}\psi(\xi - y)\,\psi(y)\,dyd\xi$$

$$-\frac{1}{\beta}\mathcal{M}_{\alpha+1-\varepsilon}\mathcal{M}_{1-\varepsilon}$$

and we use:

$$(\xi - y)^{\alpha} + y^{\alpha} + C_1\left((\xi - y)^{\alpha-1} y + y^{\alpha-1}(\xi - y)\right) \leq \left((\xi - y) + y\right)^{\alpha} \leq$$

$$\leq (\xi - y)^{\alpha} + y^{\alpha} + C_2\left((\xi - y)^{\alpha-1} y + y^{\alpha-1}(\xi - y)\right).$$

Using the preceding inequalities, we can bound:

$$\frac{C_1}{\beta}\mathcal{M}_{\alpha-\varepsilon}\mathcal{M}_{2-\varepsilon} \leq \left(\alpha - (-2\varepsilon + 2) + \frac{1}{\beta}\right)\mathcal{M}_{\alpha} \leq \frac{C_2}{\beta}\mathcal{M}_{\alpha-\varepsilon}\mathcal{M}_{2-\varepsilon}$$

We consider now the **pure fragmentation equation**.

Recall...

$\hookrightarrow$

The original equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{c}\left(x,t\right)=-\beta\left(x\right)\mathrm{c}\left(x,t\right)+\int\limits_{x}^{\infty}\mathrm{c}\left(y,t\right)\frac{\beta\left(y\right)}{y}\mathrm{B}\left(\frac{x}{y}\right)\,\mathrm{d}y$$

with $\beta\left(x\right)=\gamma x^{\gamma}$, and $\gamma\geq 0$ (non-shattering case).
$\mathrm{B}:[0,1]\to\mathbb{R}^{+}$ is the *relative fragmentation rate* and verifies the normalization property:

$$\int\limits_{0}^{1}u\mathrm{B}\left(u\right)\mathrm{d}u=1.$$

Getting the self-similar fragmentation equation

We seek a function like $(t')^{\alpha}\varphi\left(\frac{x}{(t')^{\beta}}\right)$ such that the mass moment M_1, *and also all the other moments are conserved.*

Such symmetry requirement permits determining the self-similar formulation so that the similarity problem does actually belong to the *first kind self-similarity*.

A SELF-SIMILAR PROBLEM OF THE FIRST KIND:

$$\mathrm{c}(x,t) = t^{\frac{2}{\gamma}}\varphi(\xi),$$

with $\xi = xt^{\frac{1}{\gamma}}$. Imposing in the evolution equation, we get:

$$2\varphi(\xi) + \xi\frac{\mathrm{d}}{\mathrm{d}\xi}\varphi(\xi) = -\gamma\xi^{\gamma}\varphi(\xi) + \int_{\xi}^{\infty}\varphi(\eta)\,\gamma\eta^{\gamma-1}\,\mathrm{B}\left(\frac{\xi}{\eta}\right)\mathrm{d}\eta$$

that can be also written:

$$\mathbf{T}[\varphi](x) = \mathbf{Q_F}[\varphi](x)$$

A general well-posedness result

Both existence, uniqueness and convergence to the self-similar profiles can be obtained under general conditions as it has been done in the work of Escobedo, Mischler and Rodriguez Ricard (2005).

The Mellin transform approach

- Consider the transform $\Phi(z) = \int_0^\infty x^{z-1}\varphi(x)\,\mathrm{d}x$.
- We also define $\Theta(z) = \int_0^1 u^{z-1}\mathrm{B}(u)\,\mathrm{d}u$.

The new problem

$$(2-z)\,\Phi(z) = (\Theta(z)-1)\,\gamma\Phi(z+\gamma)\,,$$

We are interested in evaluating Φ at $z = is$ in order to apply the inverse Mellin transform:

$$\varphi(x) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} x^{-z}\Phi(z)\,dz.$$

The Carleman problem

Some references to Wiener-Hopf methods and how to solve this class of problems can be found in Escobedo Velázquez (2010).

First of all, we introduce the new variable $\zeta = e^{z \cdot 2\pi i \cdot \frac{1}{\gamma}}$ and define $f(\zeta) = \text{Log}(\Phi(z))$, $\mathcal{T}(\zeta) = \Theta(z)$.
The Carleman problem now reads:

$$f(\zeta + 0 \cdot i) - f(\zeta - 0 \cdot i) = \text{Log}\left(\gamma \frac{\mathcal{T}(\zeta) - 1}{2 - \frac{\gamma}{2\pi i}\text{Log}\zeta}\right)$$
$$= \text{Log}\gamma + \text{Log}(\mathcal{T}(\zeta) - 1) - \text{Log}\left(2 - \frac{\gamma}{2\pi i}\text{Log}\zeta\right),$$

where f is analytic in $\mathbb{C} \setminus \mathbb{R}^+$.

The typical strategy and a variation

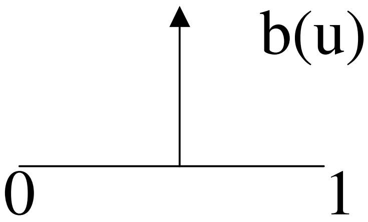
This problem belongs to the general class of problems of the form:

$$f\left(\zeta + 0\cdot i\right) - f\left(\zeta - 0\cdot i\right) = G(\zeta)$$

with solution

$$f\left(\zeta\right) = \frac{1}{2\pi i}\int_0^\infty G(s)\frac{ds}{s-\zeta},$$

provided $G(s)$ decays sufficiently fast at infinity so that the integral is well defined.

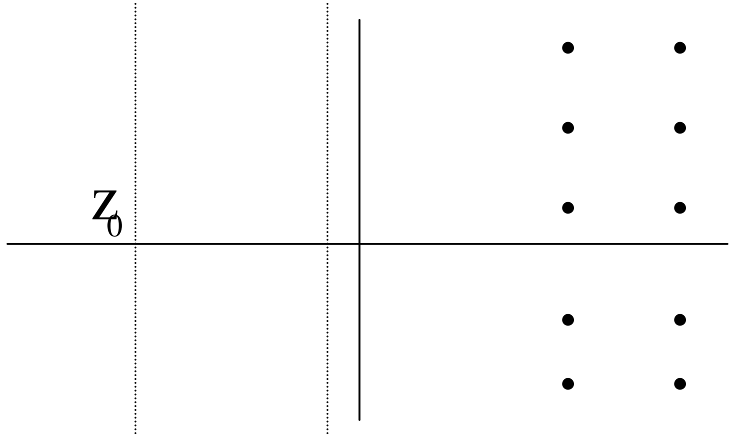


$$\Phi(z) = (\gamma/4)^{\frac{z}{\gamma}} \frac{1}{\Gamma\left(\frac{2-z}{\gamma}+1\right)} e^{\frac{\ln 2}{2\gamma}z^2 - \frac{\ln 2}{2}z} \prod_{n=1}^{\infty} \frac{1}{1-\frac{2^{z-2}}{2^{n\gamma}}}$$

Mitosis

$$\varphi(\xi) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \xi^{-z}\Phi(z)\,dz,$$

$$\frac{df}{dz} = 0$$



where

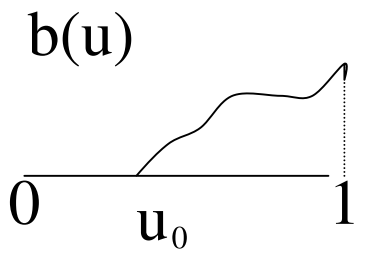
$$\begin{aligned} f(z,\xi) &= -z\ln\xi + z\frac{1}{\gamma}\ln\frac{\gamma}{4} - \left(\frac{2-z}{\gamma}\right)\ln\left(\frac{2-z}{\gamma}\right) + \frac{2-z}{\gamma} - \frac{1}{2}\log(-z) \\ &\quad + \frac{\ln 2}{2\gamma}z^2 - \frac{\ln 2}{2}z - \sum \ln\left(1-\frac{2^{z-2}}{2^{n\gamma}}\right) + O(1) \end{aligned}$$

$$\varphi(\xi) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{f(z,\xi)}dz = \frac{1}{2\pi i}\int_{z_0-i\infty}^{z_0+i\infty} e^{f(z,\xi)}dz = \frac{1}{2\pi i}\int_{z_0-i\infty}^{z_0+i\infty} e^{f(z_0,\xi)+\frac{1}{2}\frac{d^2f}{dz^2}(z_0,\xi)(z-z_0)^2+\cdots}dz$$

$$\sim \quad C\xi^{-\frac{1}{2\ln 2}(4\ln 2+2\ln(\ln 2)-2\ln\gamma+\gamma\ln 2+2)}$$

$$\times e^{-\frac{1}{2}\frac{\gamma}{\ln 2}(\ln\xi)^2+\frac{1}{\ln 2}\ln\xi\ln|\ln\xi|+\frac{3}{2}\frac{1}{\gamma\ln 2}(\ln|\ln\xi|)^2-\frac{1}{2\gamma\ln 2}\left(6\ln(\ln 2)-2\ln\gamma-\gamma\ln 2+\gamma^2\ln 2+4\right)\ln|\ln\xi|+o(1)}$$

as $\xi \to 0$.



$$b\left(u\right)=\begin{cases}0, & u\in\left[0,u_0\right]\\ c_1\left(u-u_0\right)^{\mu}+o\left(\left(u-u_0\right)^{\mu}\right), & u\in\left(u_0,u_0+\delta\right)\\ b\left(1\right)+c_2\left(1-u\right)^{\nu}+o\left(\left(1-u\right)^{\nu}\right), & u\in\left(1-\delta,1\right)\end{cases}.$$

Then

$$\Theta\left(z\right)=\Gamma\left(1+\mu\right)\frac{u_0^{z+\mu}}{\left(-z-\mu\right)^{1+\mu}}+O(1/z)$$

$$\boxed{\begin{aligned}\varphi\left(\xi\right)\quad\sim\quad & C\xi^{\frac{1}{\ln u_0}\left(\left(\mu-\ln u_0+1\right)-\left(\mu+1\right)\ln\left(-\frac{\gamma}{\ln u_0}\right)\right)}\\ & \times e^{\frac{1}{2}\frac{\gamma}{\ln u_0}\left(\ln\xi\right)^2-\frac{\left(\mu+1\right)}{\ln u_0}\ln\left(\left|\ln\xi\right|\right)\ln\xi+\frac{\left(\mu+1\right)^2}{2\gamma\ln u_0}\ln^2\left|\ln\xi\right|+\frac{2\left(\mu+1\right)}{2\gamma\ln u_0}\left(\ln u_0+\ln\left(-\frac{\gamma}{\ln u_0}\right)+\mu\ln\left(-\frac{\gamma}{\ln u_0}\right)\right)\ln\left|\ln\xi\right|}\end{aligned}}$$

Properties of the self-similar solution φ

The self-similar solution $\varphi(\xi)$ given as the inverse Mellin transform of Φ verifies:

$$\int_0^\infty e^{\xi^\gamma} \frac{\varphi(\xi)\, d\xi}{\xi^{1+\mu-\tau} + \xi^{1+\mu+\tau}} < \infty$$

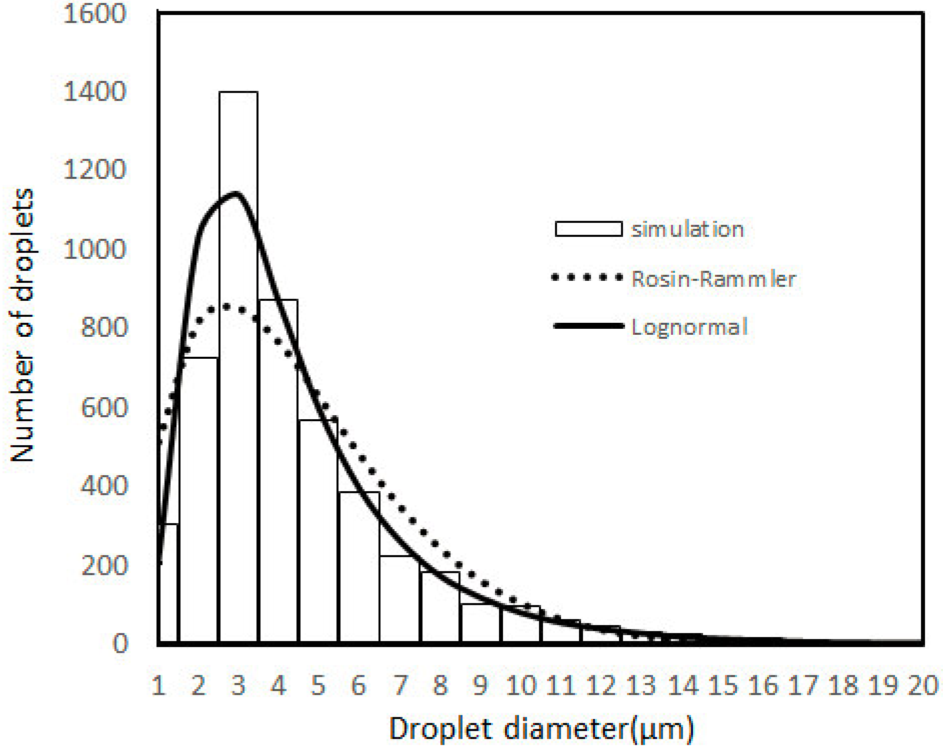
for any $0 < \tau < \min\{\mu, \gamma\}$.

Moreover, $\mathbf{M}_\alpha = \int_0^\infty \xi^\alpha \varphi(\xi)\, d\xi < \infty$ **for each** $\alpha > -\mu - 1$. The solution has the following **regularity property**: for all $l > -\mu + n$,

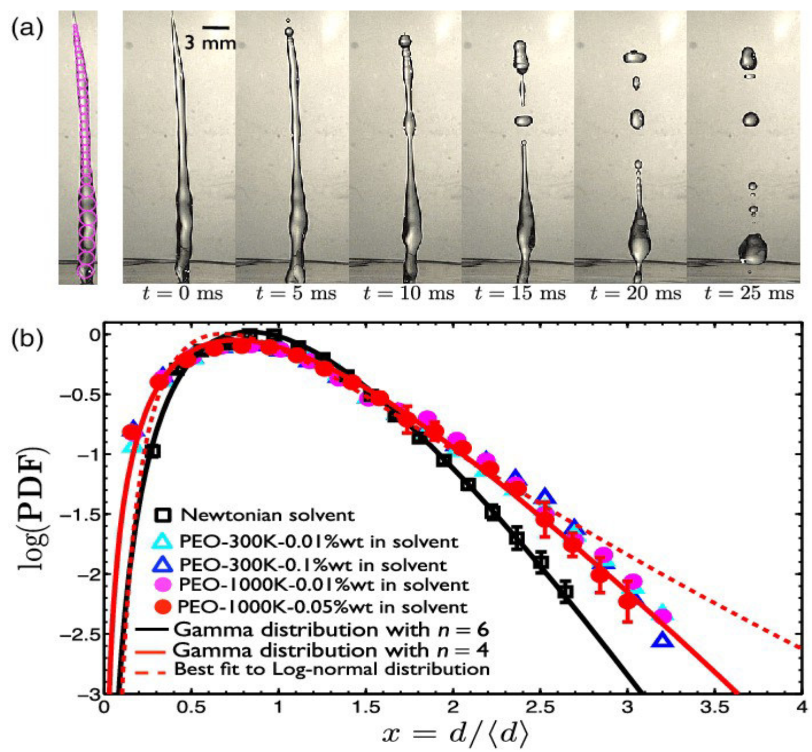
$$\frac{d^n}{d\xi^n}\left(\xi^l \varphi(\xi)\right) \leq \frac{1}{2\pi} \int_{-\infty}^{\infty} |(n-is)(n-1-is)\cdots(1-is)||\Phi(is+l-n)|ds < \infty,$$

or also

$$\left|\frac{d^n}{d\xi^n}\varphi(\xi)\right| \leq C_n \xi^{\mu-n-\delta}, \text{ with } \delta > 0,\ \delta \ll 1.$$



Reddy, Banerjee, 2017



McKinley et al, 2016

Conclusions: Smoluchowski equation

We propose a **general scaling hypothesis** for coagulation:

- A first existence and uniqueness theorem for rapidly decaying self-similar solutions in the gelling range.
- Asymptotic and analytical results in wide ranges of homogeneity λ degree.
- Different techniques to compute β.
- Numerical evidences.

Our works on this subject:

- Breschi-Fontelos, Nonlinearity 2014. *Self-similar solutions of the second kind representing gelation in finite time for the Smoluchowski equation.*
- Breschi-Fontelos. *On global in time self-similar solutions of Smoluchowski equation with multiplicative kernel.* (in PhD theses, to submit).

Conclusions: fragmentation equation

We employed Wiener-Hopf's techniques in the complex plane to achieve:

- explicit formulas;
- control on asymptotic behaviors of the self-similar solution;
- regularity properties (under the hypothesis of continuous fragmentation kernel);
- new unexpected asymptotics for the compact support case (as well as a different explicit formula for the Mellin transform).

Still many interesting problems to study! An important one: **what about non-continuous kernels?**