Turnpike property for the two and three dimensional Navier–Stokes equations[1]

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VIII Partial Differential Equations, optimal design and numerics
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Plan of the Talk

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Motivation

We consider the dynamics

$$(1) \qquad \begin{cases} \dot{x}(t) &= f(x(t), u(t)), \\ x(0) &= x_0, \end{cases}$$

and a corresponding optimal control problem

$$\min_u J^T(u) := \int_0^T f^0(x(t), u(t))dt, \qquad x \text{ solution of } (1),$$

and the stationary analogue problem

$$\min_u J_s(u) := f^0(x, u), \qquad \text{with the constraint } f(x, u) = 0.$$

We assume that both J^T and J_s admit minimal control (and state).

Motivation

The problem is the following: to analyze the convergence of the trajectories and controls which are optimal in $[0, T]$ toward the stationary state and control which are optimal for the corresponding stationary regime.

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Motivation

The problem is the following: to analyze the convergence of the trajectories and controls which are optimal in $[0, T]$ toward the stationary state and control which are optimal for the corresponding stationary regime.

- How does this fact depend of the model under consideration?
- Does depend on the type of control problem?
- In aeronautics, the optimal shape design problems are addressed in a steady context. Is this model reduction justified?

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Mathematical background

- Let $\Omega \subset \mathbb{R}^i$, with $i = 2, 3$ be a bounded and simply connected domain, with boundary $\partial\Omega$ of class C^2.
- We denote the Sobolev spaces $\mathbf{H}^1(\Omega) = H^1(\Omega; \mathbb{R}^i)$, $\mathbf{H}_0^1(\Omega) = H_0^1(\Omega; \mathbb{R}^i)$, $\mathbf{H}^{-1}(\Omega) = H^{-1}(\Omega; \mathbb{R}^i)$, for $i = 2, 3$, and we consider the following spaces

$$\begin{aligned} \mathbf{V} &= \{\mathbf{v} \in \mathbf{H}_0^1(\Omega) : \ \text{div } \ \mathbf{v} = 0\}, \\ \mathbf{H} &= \{\mathbf{v} \in \mathbf{L}^2(\Omega) : \ \text{div } \ \mathbf{v} = 0, \ \gamma_{\mathbf{n}}\mathbf{v} = 0\}, \end{aligned}$$

  where $\gamma_{\mathbf{n}}$ denotes the normal component of the trace operator.
- The spaces $\mathbf{V}, \mathbf{H}$, and $\mathbf{V}'$ satisfies

$$\mathbf{V} \subset \mathbf{H} = \mathbf{H}' \subset \mathbf{V}'$$

  with dense and continuous imbedding.

- The trilinear form $b : \mathbf{V} \times \mathbf{V} \times \mathbf{V} \to \mathbb{R}$ is the variational formulation of the nonlinearity term $(\mathbf{y} \cdot \nabla)\mathbf{v}$ given by

$$b(\mathbf{y}, \mathbf{v}, \mathbf{w}) = \int_\Omega ((\mathbf{y} \cdot \nabla \mathbf{v})) \cdot \mathbf{w} dx.$$

- We know that the trilinear form b satisfies the following properties, which are fundamental for the study of the Navier–Stokes equations.

Lemma (Temam Book)

1. $b(\mathbf{y}, \mathbf{v}, \mathbf{w}) + b(\mathbf{y}, \mathbf{w}, \mathbf{v}) = 0$, $\forall \mathbf{y} \in \mathbf{V}$, $\forall \mathbf{v}, \mathbf{w} \in \mathbf{H}^1(\Omega)$.
2. $b(\mathbf{y}, \mathbf{v}, \mathbf{v}) = 0$, $\forall \mathbf{y} \in \mathbf{V}$, $\forall \mathbf{v} \in \mathbf{H}^1(\Omega))$.
3. $b(\mathbf{y}, \mathbf{v}, \mathbf{w}) = ((\nabla \mathbf{v})^T \mathbf{w}, \mathbf{y})$, $\forall \mathbf{y}, \mathbf{v}, \mathbf{w} \in \mathbf{H}^1(\Omega)$.
4. For all $\mathbf{y} \in \mathbf{V}$ and all $\mathbf{v}, \mathbf{w} \in \mathbf{H}^1(\Omega)$ we have

$$|b(\mathbf{y}, \mathbf{v}, \mathbf{w})| \leq C \|\mathbf{y}\|^{1/2}_{\mathbf{L}^2(\Omega)} \|\mathbf{y}\|^{1/2}_{\mathbf{H}^1(\Omega)} \|\mathbf{v}\|^{1/2}_{\mathbf{L}^2(\Omega)} \|\mathbf{v}\|^{1/2}_{\mathbf{H}^1(\Omega)} \|\mathbf{w}\|_{\mathbf{H}^1(\Omega)}. \tag{2}$$

- $A\mathbf{y} = -P(\Delta \mathbf{y})$, where Δ is the vector Laplacian, and P is the orthogonal projector from $\mathbf{L}^2(\Omega))$ onto $\mathbf{H}$, called the Leray projector.
- Let B be the nonlinear operator $B: \ W(0,T) \to \mathbf{L}^2(0,T;\mathbf{V}')$, where $W(0,T) := \{\mathbf{y} \in \mathbf{L}^2(0,T;\mathbf{V}) : \ \mathbf{y}_t \in L^2(0,T;\mathbf{V}')\}$, for $\mathbf{y} \in W(0,T)$, $\mathbf{w} \in L^2(0,T;\mathbf{V}')$ defined by

  (3)
  $$\langle B(\mathbf{y}), \mathbf{w}\rangle_{L^2(\mathbf{V}'),L^2(\mathbf{V})} = \int_0^T \langle (B(\mathbf{y}))(t), \mathbf{w}(t)\rangle_{\mathbf{V}',\mathbf{V}} dt = \int_0^T b(\mathbf{y}(t),\mathbf{y}(t),\mathbf{w}(t))dt.$$

Proposition (Wachsmuth '06)

1. $\mathbf{y} \to B(\mathbf{y})$ is differentiable from $\mathbf{V}$ into $\mathbf{V}'$, and we have

$$\langle B'(\bar{\mathbf{y}})\mathbf{w}, \mathbf{v}\rangle_{L^2(\mathbf{V}'),L^2(\mathbf{V})} = \int_0^T [b(\bar{\mathbf{y}}, \mathbf{w}(t), \mathbf{v}(t)) + b(\mathbf{w}(t), \bar{\mathbf{y}}, \mathbf{v}(t))]dt.$$

2. Let $B'(\mathbf{y})^*$ denote the adjoint of $B'(\mathbf{y})$ for the duality between $\mathbf{V}$ and $\mathbf{V}'$, then we have

$$\langle B'(\bar{\mathbf{y}})^*\mathbf{v}, \mathbf{w}\rangle = \int_0^T [b(\mathbf{w}(t), \bar{\mathbf{y}}, \mathbf{v}(t)) - b(\bar{\mathbf{y}}, \mathbf{v}(t), \mathbf{w}(t))]dt.$$

3. As for quadratic functions, the second derivative is independent of $\bar{\mathbf{y}}$:

$$\langle B''(\bar{\mathbf{y}})[\mathbf{w}_1, \mathbf{w}_2], \mathbf{v}\rangle_{L^2(\mathbf{V}'),L^2(\mathbf{V})} = \int_0^T [b(w_1(t), w_2(t), v(t)) + b(w_2(t), w_1(t), v(t)$$

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2D

Given $T > 0$, we denote $\Omega_T = \Omega \times (0, T)$ and $\Gamma_T = \partial\Omega \times (0, T)$. We consider the incompressible Navier–Stokes problem in the two–dimensional case

$$
(4) \qquad \left\{ \begin{array}{rcll}
\mathbf{y}_t - \mu\Delta\mathbf{y} + (\mathbf{y}\cdot\nabla)\mathbf{y} + \nabla p & = & \mathbf{u} & , \quad \text{in } Q_T, \\
\operatorname{div} \mathbf{y} & = & 0 & , \quad \text{in } Q_T, \\
\mathbf{y} & = & 0 & , \quad \text{on } \Gamma_T, \\
\mathbf{y}(x,0) & = & \mathbf{y}_0(x) & , \quad x \in \Omega,
\end{array} \right.
$$

where the forcing term $\mathbf{u}$ is in $L^2(0, T; \mathbf{L}^2(\Omega))$, the initial data $\mathbf{y}_0$ is in $\mathbf{V}$, and the kinematic viscosity $\mu > 0$ (constant).

Theorem

There exists a unique weak solution of (4) satisfying for all $T > 0$

$$
(\mathbf{y}, p) \in (C([0, T]; \mathbf{V}) \cap L^2(0, T; \mathbf{H}^2(\Omega)) \cap \mathbf{V})) \times L^2(0, T; H^1(\Omega) \cap L^2_0(\Omega)),
$$

and

$$
\mathbf{y}_t \in L^2(0, T; \mathbf{H}).
$$

We consider the stationary Navier–Stokes equations

$$(5) \qquad \begin{cases} -\mu\Delta \mathbf{y} + (\mathbf{y}\cdot\nabla)\mathbf{y} + \nabla p &= \mathbf{u} \quad , \quad \text{in } \Omega, \\ \operatorname{div} \mathbf{y} &= 0 \quad , \quad \text{in } \Omega, \\ \mathbf{y} &= 0 \quad , \quad \text{on } \partial\Omega, \end{cases}$$

where $\mathbf{u} \in \mathbf{L}^2(\Omega)$.

Theomem

If $\|\mathbf{u}\|_{\mathbf{L}^2(\Omega)} \leq C(\Omega)\mu^2$, then the problem (5) has a unique weak solution

$$\mathbf{y} \in \mathbf{H}^2(\Omega) \cap \mathbf{V} \ , \quad p \in H^1(\Omega).$$

Optimal control problem

Find $\mathbf{u}^T \in L^2(0, T; \mathbf{H})$, $\mathbf{y}^T$ *is the solution of* (4) *associated to* $\mathbf{u}^T$, *minimizing the functional*

$$(6) \qquad J^T(\mathbf{u}) = \frac{1}{2}\int_0^T \|\mathbf{y}(t) - \mathbf{x}^d\|^2_{\mathbf{L}^2(\Omega)} dt + \frac{k}{2}\int_0^T \|\mathbf{u}(t)\|^2_{\mathbf{L}^2(\Omega)} dt + \mathbf{q}_0 \cdot \mathbf{y}(T),$$

where $\mathbf{x}^d \in \mathbf{L}^2(\Omega)$ *is desired state,* $\mathbf{q}_0 \in \mathbf{L}^2(\Omega)$ *and* $k > 0$ *is a constant*.

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where $\mathbf{x}^d \in \mathbf{L}^2(\Omega)$ *is desired state,* $\mathbf{q}_0 \in \mathbf{L}^2(\Omega)$ *and* $k > 0$ *is a constant.*

Theorem

Let $\mathbf{y}_0 \in \mathbf{V}$. There exists at least an element $\mathbf{u}^T \in \mathbf{L}^2(0,T;\mathbf{H})$, and $\mathbf{y}^T \in C([0,T];\mathbf{V}) \cap L^2(0,T;\mathbf{H}^2(\Omega))$ such that the functional $J^T(\mathbf{u})$ attains its minimum at $\mathbf{u}^T$.

Theorem (Abergel and Temam 1990)

Let $(\mathbf{y}^T, \mathbf{u}^T)$ be an optimal pair for problem (8). The following equality holds

$$\mathbf{u}^T + \mathbf{q}^T = 0,$$

where $\mathbf{q}^T$ is the adjoint state that of the linearized adjoint problem

$$\begin{cases} -\mathbf{q}_t - \mu\Delta\mathbf{q}^T + (\nabla\mathbf{y}^T)^T\mathbf{q}^T - (\mathbf{y}^T\cdot\nabla)\mathbf{q}^T + \nabla\tilde{p} & = & \mathbf{y}^T - \mathbf{x}^d & , & \text{in } Q_T, \\ \text{div } \mathbf{q}^T & = & 0 & , & \text{in } Q_T, \\ \mathbf{q}^T & = & 0 & , & \text{on } \Gamma_T, \\ \mathbf{q}^T(x,T) & = & \mathbf{q}_0 & , & x\in\Omega. \end{cases}$$

Optimal control problem

Our optimal control problem is find $\overline{\mathbf{u}}$, $\overline{\mathbf{y}}$ being the solution of (5) associated to $\overline{\mathbf{u}}$, minimizing the functional

$$J(\mathbf{u}) = \frac{1}{2}\|\mathbf{y} - \mathbf{x}^d\|^2_{\mathbf{L}^2(\Omega)} + \frac{\alpha}{2}\|\mathbf{u}\|^2_{\mathbf{L}^2(\Omega)},$$

where $\mathbf{x}^d \in \mathbf{L}^2(\Omega)$ is a target and $\alpha > 0$ is a constant.

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where $\mathbf{x}^d \in \mathbf{L}^2(\Omega)$ is a target and $\alpha > 0$ is a constant.

Theorem

There exists at least an element $\overline{\mathbf{u}} \in \mathbf{L}^2(\Omega)$ such that the functional $J(\mathbf{u})$ attains its minimum at $\overline{\mathbf{u}}$.

Theorem (J. De los Reyes 2004)

Let $(\overline{\mathbf{u}}, \overline{\mathbf{y}})$ be an optimal solution such that $\mu > \mathcal{M}(\overline{\mathbf{y}})$, where $\mathcal{M}(\mathbf{y}) = \sup_{\mathbf{v} \in \mathbf{V}} \frac{|b(\mathbf{v}, \mathbf{v}, \mathbf{y})|}{\|\mathbf{v}\|_{\mathbf{V}}^2}$. Then there exists $\overline{\mathbf{q}} \in \mathbf{V}$ such that satisfies the following optimality system in variational sense

$$\begin{cases} -\mu\Delta\overline{\mathbf{y}} + (\overline{\mathbf{y}} \cdot \nabla)\overline{\mathbf{y}} + \nabla\overline{p} = -\overline{\mathbf{q}} & , \text{ in } \Omega, \\ \operatorname{div} \overline{\mathbf{y}} = 0 & , \text{ in } \Omega, \\ \overline{\mathbf{y}} = 0 & , \text{ on } \partial\Omega, \\ -\mu\Delta\overline{\mathbf{q}} - (\overline{\mathbf{y}} \cdot \nabla)\overline{\mathbf{q}} + (\nabla\overline{\mathbf{y}})^T\overline{\mathbf{q}} + \nabla\pi = \overline{\mathbf{y}} - \mathbf{x}^d & , \text{ in } \Omega \\ \operatorname{div} \overline{\mathbf{q}} = 0 & , \text{ in } \Omega, \\ \overline{\mathbf{q}} = 0 & , \text{ on } \partial\Omega. \end{cases}$$

Main Theorem

Theorem (S.Z. 2018)

We assume that the tracking term $\|\overline{\mathbf{y}} - \mathbf{x}^d\|_{\mathbf{V}}$ is sufficiently small, $\mu > \mathcal{M}(\overline{\mathbf{y}})$, and $\mathbf{z}_0 = \mathbf{y}_0 - \overline{\mathbf{y}} \in X_\sigma$. Then, there exists some $\epsilon > 0$ such that for every $\mathbf{y}_0, \mathbf{q}_0$ with

$$\|\mathbf{y}_0 - \overline{\mathbf{y}}\|_{\mathbf{L}^2(\Omega)} + \|\mathbf{q}_0 - \overline{\mathbf{q}}\|_{\mathbf{L}^2(\Omega)} \leq \epsilon,$$

there exists a solution of the evolutionary optimality system such that

$$\|\mathbf{y}^T(t) - \overline{\mathbf{y}}\|_{\mathbf{L}^2(\Omega)} + \|\mathbf{q}^T(t) - \overline{\mathbf{q}}\|_{\mathbf{L}^2(\Omega)} \leq C(e^{-\gamma t} + e^{-\gamma(T-t)}), \quad \forall t < T,$$

where $\gamma > 0$ is the stabilizing rate of the linearized optimality system.

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We consider the evolutionary Navier–Stokes problem in 3D

$$
(7) \qquad \begin{cases} \mathbf{y}_t - \mu\Delta\mathbf{y} + (\mathbf{y}\cdot\nabla)\mathbf{y} + \nabla p &= \mathbf{f} + \mathbf{u} \quad , \quad \text{in } Q_T, \\ \text{div } \mathbf{y} &= 0 \quad , \quad \text{in } Q_T, \\ \mathbf{y} &= 0 \quad , \quad \text{on } \Gamma_T, \\ \mathbf{y}(x,0) &= \mathbf{y}_0(x) \quad , \quad x \in \Omega. \end{cases}
$$

We consider the space

$$
W(0,T) := \{\mathbf{y} \in L^2(0,T;\mathbf{V}) : \ \mathbf{y}_t \in L^2(0,T;\mathbf{V}')\}.
$$

We will assume that $\mathbf{f}, \mathbf{u} \in L^2(0,T;\mathbf{L}^2(\Omega))$ and $\mathbf{y}_0 \in \mathbf{V}$. We shall say that $\mathbf{y} \in W(0,T)$ is a weak solution of (7) if

$$
\begin{cases} \mathbf{y}_t + \mu A\mathbf{y} + B\mathbf{y} &= \mathbf{f} + \mathbf{u} \quad , \quad \text{on } (0,T), \\ \mathbf{y}(0) &= \mathbf{y}_0 \quad , \end{cases}
$$

where A is the Stokes operator

- It is a well–known result that there exists at least one weak solution $\mathbf{y} \in W(0, T)$ of (7). However, is still an open problem the uniqueness of such solution.

[1]H. Sohr and W. von Wahl. Generic solvability of the equations of Navier-Stokes. *Hiroshima Mathematical Journal*, 17(3):613–625, 1987.

- It is a well–known result that there exists at least one weak solution $\mathbf{y} \in W(0,T)$ of (7). However, is still an open problem the uniqueness of such solution.
- Alternatively, we consider strong solutions of (7). These are solutions with $\mathbf{y} \in L^p(0,T;\mathbf{H}^2(\Omega)\cap\mathbf{H}^1_0(\Omega))$ and $\mathbf{y}_t \in L^p(0,T;\mathbf{L}^p(\Omega))$, for some $2 \leq p < \infty$.

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- From the work of Sohr and von Wahl (1987)[1], we have the following: a weak solution $\mathbf{y}$ of (7) is strong if $\mathbf{y} \in L^s(0,T;\mathbf{L}^q(\Omega))$ holds for some $s, q \in (0,\infty)$ with $\frac{2}{s} + \frac{3}{q} \leq 1$. Therefore, we consider the following class of regular solutions.

Definition

Let $\mathbf{f}, \mathbf{u} \in L^2(0,T;\mathbf{L}^2(\Omega))$ and $\mathbf{y}_0 \in \mathbf{V}$. We shall say that $\mathbf{y}$ is a strong solution of (7) if it is a weak solution and

$$\mathbf{y} \in L^8(0,T;\mathbf{L}^4(\Omega)).$$

[1] H. Sohr and W. von Wahl. Generic solvability of the equations of Navier-Stokes. *Hiroshima Mathematical Journal*, 17(3):613–625, 1987.

- It is well–known that (7) does not have more than one strong solution. However, is still an open problem the existence of strong solutions of (7) for arbitrary $T > 0$.

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- It is well–known that (7) does not have more than one strong solution. However, is still an open problem the existence of strong solutions of (7) for arbitrary $T > 0$.

Theorem (Casas 1998)

[a]Let us assume that $(\mathbf{y}, p)$ is a strong solution of (7). Then $\mathbf{y} \in \mathbf{H}^{2,1}(\Omega) \cap C([0,T];\mathbf{V})$ and $p \in L^2(0,T;H^1(\Omega))$. Moreover, there exists an increasing function $\eta : [0,+\infty) \to [0,+\infty)$ depending only on Ω and μ such that

$$\|\mathbf{y}\|_{\mathbf{H}^{2,1}(\Omega)} \leq \eta \left(\|\mathbf{y}_0\|_{\mathbf{V}} + \|\mathbf{f} + \mathbf{u}\|_{L^2(0,T;\mathbf{L}^2(\Omega))} + \|\mathbf{y}\|_{L^8(0,T;\mathbf{L}^4(\Omega))} \right).$$

[a]E. Casas. An optimal control problem governed by the evolution Navier-Stokes equations. *Optimal control of viscous flow*, 59:79–95, 1998

Theorem

Let us assume that system (7) has a strong solution for some $\overline{\mathbf{u}} \in L^2(0, T; \mathbf{L}^2(\Omega))$. Then there exists an open neighborhood $\mathcal{A}_0$ of $\overline{\mathbf{u}}$ in $L^2(0, T; \mathbf{L}^2(\Omega))$ such that (7) has a strong solution for every $\mathbf{u} \in \mathcal{A}_0$. Moreover, the mapping $G : \mathcal{A}_0 \to \mathbf{H}^{2,1}(\Omega) \cap C([0, T]; \mathbf{V})$, defined by $G(\mathbf{u}) = \mathbf{y}_{\mathbf{u}}$, is of class C^∞.[a]

[a] E. Casas and K. Chrysafinos. Analysis of the velocity tracking control problem for the 3d evolutionary navier–stokes equations. *SIAM Journal on Control and Optimization*, 54(1):99–128, 2016.

- As a consequence of the previous Theorem, we deduce that the set of controls $\mathbf{u} \in L^2(0, T; \mathbf{L}^2(\Omega))$ for which there exists a strong solution $\mathbf{y_u}$ is open. Hereafter, this set will be denoted by $\mathcal{A}$.

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- This set $\mathcal{A}$

$$\mathcal{A} := \{\mathbf{u} \in L^2(0, T; \mathbf{L}^2(\Omega)) : \text{ (7) has a strong solution } \mathbf{y_u}\}$$

is an open subset of $L^2(0, T; \mathbf{L}^2(\Omega))$ and is dense in the norm $L^s(0, T; \mathbf{L}^q(\Omega))$ for all $s, q \in (0, \infty)$ with $4 < \frac{2}{s} + \frac{3}{q}$ (Sohr and von Wahl (1987)).

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- In particular we have that for any $\mathbf{u} \in L^2(0, T; \mathbf{L}^2(\Omega))$ and any $\varepsilon > 0$, there exists $\mathbf{v}_\varepsilon \in L^2(0, T; \mathbf{L}^2(\Omega))$ with $\|\mathbf{v}_\varepsilon\|_{L^1(0,T;\mathbf{L}^1(\Omega))}$ such that $\mathbf{u} + \mathbf{v}_\varepsilon \in \mathcal{A}$.

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Optimal control problem

- We consider the functional $J : A_\varepsilon \to \mathbb{R}$ defined by

$$(8) \quad J^T(\mathbf{u}) = \frac{1}{8}\int_0^T \|\mathbf{y}(t) - \mathbf{x}^d\|^8_{\mathbf{L}^4(\Omega)} dt + \frac{\alpha}{2}\int_0^T \|\mathbf{u}(t)\|^2_{\mathbf{L}^2(\Omega)} dt + \mathbf{q}_0 \cdot \mathbf{y}(T),$$

  where $\mathbf{x}^d \in \mathbf{L}^2(\Omega)$ is desired state, $\mathbf{q}_0 \in \mathbf{L}^2(\Omega)$, $\alpha > 0$ is a constant, and A_ε is the set of controls $\mathbf{u} \in L^2(0, T; \mathbf{L}^2(\Omega))$ for which there exists a strong solution $\mathbf{y}_u$ of (7).
- We assume that the set of admissible controls $\mathcal{U}_{ad} = A_\varepsilon \cap \mathcal{U}_{a,b}$ satisfies

$$(9) \qquad \mathcal{U}_{ad} \neq \emptyset,$$

  where

$$\mathcal{U}_{a,b} := \{\mathbf{u} \in L^2(0, T; \mathbf{L}^2(\Omega)) \ : \ a_j \leq \mathbf{u}_j(x,t) \leq b_j \text{ for a.e. } (x,t) \in Q_T, \ 1 \leq j \leq 3\},$$

  where $-\infty \leq a_j \leq b_j \leq +\infty$ for $1 \leq j \leq 3$.

Theorem (Casas and Crysafinos (2016))

The optimal control problem has at least one solution. Moreover, for any local solution $\mathbf{u}^T$, there exists $\mathbf{y}^T, \mathbf{q}^T \in \mathbf{H}_{2,1}(Q_T) \cap C([0,T];\mathbf{V})$ and $p^T, \pi^T \in L^2(0,T;H^1(\Omega))$ such that

$$\left\{\begin{array}{rcll} \mathbf{y}_t^T - \mu\Delta\mathbf{y}^T + (\mathbf{y}^T\cdot\nabla)\mathbf{y}^T + \nabla p^T & = & \mathbf{f} + \mathbf{u}^T & , \\ \text{div } \mathbf{y}^T & = & 0 & , \\ \mathbf{y}^T & = & 0 & , \\ \mathbf{y}^T(x,0) & = & y_0(x) & , \\ -\mathbf{q}_t^T - \mu\Delta\mathbf{q}^T - (\mathbf{y}^T\cdot\nabla)\mathbf{q}^T + (\nabla\mathbf{y}^T)^T\mathbf{q}^T + \nabla\pi^T & = & \|\mathbf{y}^T - \mathbf{x}^d\|_{\mathbf{L}^4(\Omega)}^4 |\mathbf{y}^T - \mathbf{x}^d|^2(\mathbf{y}^T - \mathbf{x}^d) & , \\ \text{div } \mathbf{q}^T & = & 0 & , \\ \mathbf{q}^T & = & 0 & , \\ \mathbf{q}^T(x,T) & = & \mathbf{q}_0 & , \end{array}\right.$$

Optimal control problem

Consider the functional

$$J(\mathbf{u}) = \frac{1}{8}\|\mathbf{y} - \mathbf{x}^d\|^8_{\mathbf{L}^4(\Omega)} + \frac{\alpha}{2}\|\mathbf{u}\|^2_{\mathbf{L}^2(\Omega)}, \tag{10}$$

where $\mathbf{x}^d \in \mathbf{L}^2(\Omega)$ is a target and $\alpha > 0$ is a constant.

Theorem

Let $(\overline{u}, \overline{y})$ be an optimal solution. Then there exists $q \in V$ such that satisfies the following optimality system in a variational sense

(11)

$$\left\{\begin{array}{rcll} -\mu\Delta\overline{y} + (\overline{y}\cdot\nabla)\overline{y} + \nabla\overline{p} & = & -\dfrac{q}{\alpha} & , \text{ in } \Omega, \\ \operatorname{div}\ \overline{y} & = & 0 & , \text{ in } \Omega, \\ \overline{y} & = & 0 & , \text{ on } \partial\Omega, \\ -\mu\Delta q - (\overline{y}\cdot\nabla)q + (\nabla\overline{y})^T q + \nabla\pi & = & \|\overline{y} - x^d\|^4_{L^4(\Omega)}|\overline{y} - x^d|^2(\overline{y} - x^d) & , \text{ in } \Omega \\ \operatorname{div}\ q & = & 0 & , \text{ in } \Omega, \\ q & = & 0 & , \text{ on } \partial\Omega. \end{array}\right.$$

Main Theorem

Theorem

We assume that the tracking term $\|\overline{\mathbf{y}} - \mathbf{x}^d\|_{\mathbf{V}}$ is sufficiently small, $\mu > \mathcal{M}(\overline{\mathbf{y}})$, and $\mathbf{z}_0 = \mathbf{y}_0 - \overline{\mathbf{y}} \in X_\sigma$. Then, there exists some $\epsilon > 0$ such that for every $\mathbf{y}_0, \mathbf{q}_0$ with

$$\|\mathbf{y}_0 - \overline{\mathbf{y}}\|_{\mathbf{L}^2(\Omega)} + \|\mathbf{q}_0 - \overline{\mathbf{q}}\|_{\mathbf{L}^2(\Omega)} \leq \epsilon,$$

there exists a solution of the evolutionary optimality system such that

$$\|\mathbf{y}^T(t) - \overline{\mathbf{y}}\|_{\mathbf{L}^2(\Omega)} + \|\mathbf{q}^T(t) - \overline{\mathbf{q}}\|_{\mathbf{L}^2(\Omega)} \leq C(e^{-\gamma t} + e^{-\gamma(T-t)}), \quad \forall t < T,$$

where $\gamma > 0$ is the stabilizing rate of the linearized optimality system.

Future work and open problems

- Moore realistic boundary data: Nonlinear Navier–slip boundary condition

$$(\sigma(\mathbf{y}, p) \cdot n)_{tg} + (A(x,t)\mathbf{y})_{tg} = g,$$

  where $\sigma(\mathbf{y}, p) = \nabla \mathbf{y} + \nabla \mathbf{y}^T - pI$.

- Optimal shape design for $2D$ Navier–Stokes equations in large time.

References

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