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Quantum observers from Poisson-Lie geometry

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7TH INT. WORKSHOP ON NEW CHALLENGES IN QUANTUM MECHANICS: INTEGRABILITY AND SUPERSYMMETRY

Overview

1. **Quantum observers**
2. **Observers as oriented time-like geodesics**
3. **Noncommutative observers from Poisson geometry**
4. **Important example: κ-Poincaré**
5. **Final remarks**

1. Quantum observers

Quantum observers

What is an **observer**? In general, it could be defined as **any system with the ability to perform measurements**.

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- **Special relativity**: Observer $\sim$ **inertial reference frame**.
- **Quantum mechanics**: Observer $\sim$ any system with the ability to perform quantum measurements. In particular, **any system is a quantum system**.

Figure: Observer in special relativity - Inertial observer (Apollo 11, 16 July 1969)

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Figure: Quantum mechanical observer - Quantum system (Schrödinger's cat)

Figure: Quantum observer - Inertial observer described by quantum mechanics

2. Observers as oriented time-like geodesics

However, we need something more concrete to work with. We take the idea of observer from special relativity (SR), so for us

Definition (Classical observer)

Given a predetermined spacetime M (smooth manifold endowed with a Lorentzian metric and the associated Levi-Civita connection), **a classical observer in M is just an oriented time-like geodesic**.

It is clear that oriented time-like geodesics in M are exactly worldlines of free particles, so this definition agrees with the one from SR. We denote the space of oriented time-like geodesics in M by $L(M)$.

In particular, in (1+1)D Minkowski (flat) spacetime, time-like geodesics are just straight lines inside the light cone.

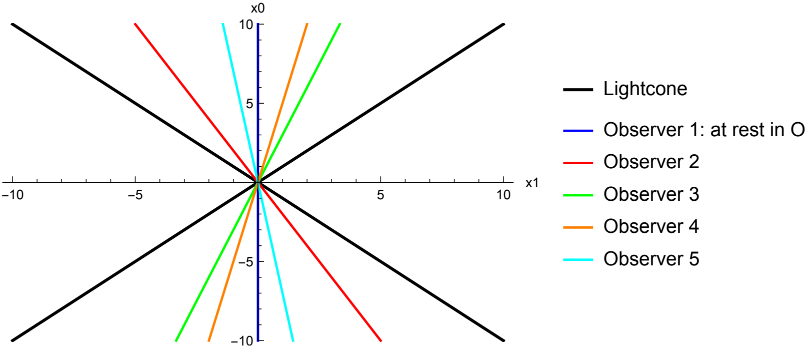

Figure: Lightcone and time-like geodesics

Any two observers inside the lightcone are **causally connected**.

Unfortunately, the space of oriented time-like geodesics $L(M)$ for a general spacetime M is a very complicated object [1]. For example:

- $L(M)$ is a topological space, but not necessarily Hausdorff.
- Even if $L(M)$ is Hausdorff, the topological manifold could not admit an smooth atlas (and thus not be a smooth manifold).

However, if M is a **simply connected Lorentzian space of constant curvature**, then the space $L(M)$ is a smooth manifold [2]. Moreover, $L(M)$ is a **homogeneous space**.

In particular, we are interested here in (3+1)D Minkowski spacetime $\mathbf{M}^{3+1}$, which satisfies this condition. So both $\mathbf{M}^{3+1}$ and $L(\mathbf{M}^{3+1})$ are homogeneous spaces.

[1] Beem, 1991.

[2] Alekseevsky, Guilfoyle, Klingenberg, 2011.

Let us briefly recall the notion of action of a Lie group on a manifold and of a homogeneous space.

Definition

An **action of a Lie group** G on a smooth manifold M is a homomorphism $\alpha : G \to \mathrm{Diff}(M)$ such that the map $\alpha(g) : M \to M, m \to \alpha(g)\, m$ is smooth for all $g \in G$. We say that an action α is **transitive** if for all $m, n \in M$, there exists at least one element $g \in G$ such that $\alpha(g)m = n$. A manifold endowed with a transitive Lie group action is called a **homogeneous space**.

Homogeneous spaces are identified with certain coset spaces in the following sense:

Theorem

Let α be a transitive action of a Lie group G on a smooth manifold M. Then for any $m \in M$ the map

$$\begin{aligned} \beta_m : G/H_m &\to M \\ gH_m &\to \alpha_g(m) \end{aligned}$$

is a diffeomorphism which commutes with the action of G. (It is assumed that the group G acts on G/H_m by left translations.)

3. M^{3+1} and $L(\mathsf{M}^{3+1})$ as homogeneous spaces

Let $\mathfrak{g} = \mathrm{Lie}(G)$ the Lie algebra of the Poincaré Lie group, with commutators

$$
\begin{array}{lll}
[J_a, J_b] = \epsilon_{abc} J_c, & [J_a, P_b] = \epsilon_{abc} P_c, & [J_a, K_b] = \epsilon_{abc} K_c, \\
[K_a, P_0] = P_a, & [K_a, P_b] = \delta_{ab} P_0, & [K_a, K_b] = -\epsilon_{abc} J_c, \\
[P_0, P_a] = 0, & [P_a, P_b] = 0, & [P_0, J_a] = 0.
\end{array}
$$

where P_α are the generators of translations, J_a of rotations and K_a of boosts.

G acts transitively in both $\mathbf{M}^{3+1}$ and $L(\mathbf{M}^{3+1})$, with the **stabilizer of a point (Lorentz subalgebra $\mathfrak{l} \simeq \mathfrak{so}(3,1)$)**, given by

$$
[J_a, J_b] = \epsilon_{abc} J_c, \qquad [J_a, K_b] = \epsilon_{abc} K_c, \quad [K_a, K_b] = -\epsilon_{abc} J_c.
$$

Also, the **stabilizer of a worldline $\mathfrak{h} \simeq \mathfrak{so}(3) \times \mathbb{R}$** is

$$
[P_0, J_a] = 0, \qquad [J_a, J_b] = \epsilon_{abc} J_c.
$$

Minkowski spacetime $\mathbf{M}^{3+1}$

We define **local coordinates**

$$(x^\alpha, \xi^a, \theta^a) : U \subset G \to \mathbb{R}^{10}$$

on the **Poincaré group** G by the inverse map of

$$G_M = \exp x^0 P_0 \exp x^1 P_1 \exp x^2 P_2 \exp x^3 P_3 \times \\ \times \exp \xi^1 K_1 \exp \xi^2 K_2 \exp \xi^3 K_3 \times \exp \theta^1 J_1 \exp \theta^2 J_2 \exp \theta^3 J_3 \,,$$

and so the Lorentz subgroup L is parametrized by

$$L = \exp \xi^1 K_1 \exp \xi^2 K_2 \exp \xi^3 K_3 \exp \theta^1 J_1 \exp \theta^2 J_2 \exp \theta^3 J_3 .$$

So, **Minkowski spacetime is $\mathbf{M}^{3+1} = G_M/L$** with coordinates

$$x^\alpha : U' \subset \mathbf{M}^{3+1} = G/L \to \mathbb{R}^4$$

Space of worldlines $\mathrm{L}(\mathbf{M}^{3+1})$

We define **local coordinates**

$$(\eta^a, y^\alpha, \phi^a) : U \subset G_0 \to \mathbb{R}^{10}$$

on the **Poincaré group** G by the inverse map of

$$G_{\mathcal{W}} = \exp \eta^1 K_1 \exp y^1 P_1 \exp \eta^2 K_2 \exp y^2 P_2 \exp \eta^3 K_3 \exp y^3 P_3 \times \\ \times \exp \phi^1 J_1 \exp \phi^2 J_2 \exp \phi^3 J_3 \exp y^0 P_0.$$

and so the stabilizer of a worldline H is parametrized by

$$H = \exp \phi^1 J_1 \exp \phi^2 J_2 \exp \phi^3 J_3 \exp y^0 P_0.$$

The space of worldlines is $\mathcal{W} = L(\mathbf{M}^{3+1}) = G_W/H$ with coordinates

$$(y^\alpha, \eta^a) : U' \subset \mathcal{W} = G/H \to \mathbb{R}^6$$

Change of coordinates between $\mathbf{M}^{3+1}$ and $\mathrm{L}(\mathbf{M}^{3+1})$

The explicit diffeomorphism on the **Poincaré Lie group** G in both parametrizations is

$$x^\alpha = f^\alpha(y^\beta, \eta^a), \qquad \xi^a = \eta^a, \qquad \theta^a = \phi^a,$$

where

$$\begin{aligned}
f^0(y^\alpha, \eta^a) &= y^1 \sinh \eta^1 + \\
&\quad \cosh \eta^1 \left(y^2 \sinh \eta^2 + \cosh \eta^2 (y^0 \cosh \eta^3 + y^3 \sinh \eta^3) \right), \\
f^1(y^\alpha, \eta^a) &= y^1 \cosh \eta^1 + \\
&\quad \sinh \eta^1 \left(y^2 \sinh \eta^2 + \cosh \eta^2 (y^0 \cosh \eta^3 + y^3 \sinh \eta^3) \right), \\
f^2(y^\alpha, \eta^a) &= y^2 \cosh \eta^2 + \sinh \eta^2 (y^0 \cosh \eta^3 + y^3 \sinh \eta^3), \\
f^3(y^\alpha, \eta^a) &= y^0 \sinh \eta^3 + y^3 \cosh \eta^3 .
\end{aligned}$$

Metric structure in $\mathbf{M}^{3+1}$ and $\mathrm{L}(\mathbf{M}^{3+1})$

Both spaces $\mathbf{M}^{3+1}$ and $\mathrm{L}(\mathbf{M}^{3+1})$ are naturally endowed with a metric structure. However, while the (pseudo) Riemannian metric **(Minkowski metric)** in M

$$\mathrm{d}s^2 = (\mathrm{d}x^0)^2 - (\mathrm{d}x^1)^2 - (\mathrm{d}x^2)^2 - (\mathrm{d}x^3)^2,$$

is well-known, the space **$\mathrm{L}(\mathbf{M}^{3+1})$ presents an invariant foliation**, with a **'main' metric**

$$\mathrm{d}s^2_{(1)} = (\cosh\eta^2)^2(\cosh\eta^3)^2(\mathrm{d}\eta^1)^2 + (\cosh\eta^3)^2(\mathrm{d}\eta^2)^2 + (\mathrm{d}\eta^3)^2,$$

and a **'subsidiary' in each leaf**

$$\mathrm{d}s^2_{(2)} = (\mathrm{d}y^1)^2 + (\mathrm{d}y^2)^2 + (\mathrm{d}y^3)^2, \qquad \boldsymbol{\eta} = \boldsymbol{\eta}_0,$$

so each leaf is isometric to three-dimensional Euclidean space.

Metric structure in $\mathbf{M}^{3+1}$ and $\mathrm{L}(\mathbf{M}^{3+1})$

The **geodesic distance** [3] is given by

$$\cosh\chi = \cosh\eta^1 \cosh\eta^2 \cosh\eta^3.$$

This metric structure shows how the **three-velocity space of special relativity is hyperbolic**.

In the low rapidity regime (i.e. take $c \to \infty$) we recover the well-known expressions of classical mechanics

$$\mathrm{d}s^2_{(1)} = (\mathrm{d}\eta^1)^2 + (\mathrm{d}\eta^2)^2 + (\mathrm{d}\eta^3)^2,$$
$$\chi^2 = (\eta^1)^2 + (\eta^2)^2 + (\eta^3)^2.$$

[3] Herranz, Santander, 1998.

3. Noncommutative observers from Poisson geometry

Until now, we have only treated the space of classical observers. In order to introduce **quantum observers in noncommutative spacetimes**, we need the following definition.

Definition

A **Poisson-Lie group** is a Poisson manifold (G, π_G) such that the Lie group multiplication $\mu : (G \times G, \pi_G \oplus \pi_G) \to (G, \pi_G)$ that is a Poisson map.

The Poisson-Lie group [4] condition can be restated in terms of the Poisson bivector on G alone, and it is just

$$\pi_G(\mu(g_1, g_2)) = (L_{g_1})_* \pi_G(g_2) + (R_{g_2})_* \pi_G(g_1).$$

[4] Drinfel'd, 1987.
Chari, Pressley, 1994.

The tangent counterpart of a Poisson-Lie group is a **Lie bialgebra** $(\mathfrak{g}, \delta)$ where the **cocommutator** $\delta : \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}$ satisfies

i) **(Co-Jacobi condition)**
$\sum_{cycl} (\delta \otimes \mathrm{id}) \circ \delta(X) = 0 \quad \forall X \in \mathfrak{g}$

ii) **(1-cocycle conditon)**
$\delta([X, Y]) = \mathrm{ad}_X \delta(Y) - \mathrm{ad}_Y \delta(X), \quad \forall X, Y \in \mathfrak{g}$

Particular cases of 1-cocycles are **1-coboundaries**

$$\delta(X) = \mathrm{ad}_X r \quad \forall X \in \mathfrak{g}$$

with $r \in \mathfrak{g} \otimes \mathfrak{g}$ a skew-symmetric solution of the **modified classical Yang-Baxter equation (mCYBE)**.

Theorem

Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. Consider a skew-symmetric solution of the mCYBE $r \in \bigwedge^2 \mathfrak{g}$ defining a coboundary Lie bialgebra $(\mathfrak{g}, \delta)$ by $\delta(X) = \mathrm{ad}_X r$ for all $X \in \mathfrak{g}$. Then the **unique Poisson-Lie structure on G whose tangent space is $(\mathfrak{g}, \delta)$** *is defined by the Poisson bivector*

$$\pi_G = \sum_{i,j} r^{ij} \left(X_i^L \otimes X_j^L - X_i^R \otimes X_j^R \right).$$

where X_i^L and X_i^R are left- and right-invariant vector fields on G.

The Poisson version of a homogeneous space is

Definition

A **Poisson homogeneous space** is a Poisson manifold (M, π_M) endowed with a transitive Lie group action $\alpha : (G \times M, \pi_G \oplus \pi_M) \to (M, \pi_M)$ that is a Poisson map.

This can be expressed in terms of the Poisson bivectors on M and G, by

$$\pi_M(\alpha(g, m)) = (\alpha_g)_* \pi_M(m) + (\alpha_m)_* \pi_G(g).$$

Using the diffeomorphism $M \simeq G/H_m$ for some $m \in M$ and an **appropriate parametrization of the Lie group G** (as previously shown) we have that

$$\pi_M = p_*(\pi_G)$$

where $p : G \to G/H_m$ is the canonical projection.

Theorem

Let $(G/H, \pi)$ be a Poisson homogeneous space and $\mathfrak{g} = \mathrm{Lie}(G)$ and $\mathfrak{h} = \mathrm{Lie}(H)$. A sufficient condition to have a well-defined Poisson homogeneous space is the **coisotropy condition**

$$\delta(\mathfrak{h}) \subset \mathfrak{h} \wedge \mathfrak{g}.$$

If moreover H is a Poisson subgroup of G, so

$$\delta(\mathfrak{h}) \subset \mathfrak{h} \wedge \mathfrak{h},$$

then we say that the Poisson homogeneous space is of **Poisson subgroup type**.

4. Important example: κ-Poincaré

Consider the **skew-symmetric solution of the mCYBE**

$$r = \frac{1}{\kappa}(K_1 \wedge P_1 + K_2 \wedge P_2 + K_3 \wedge P_3) \in \mathfrak{g} \otimes \mathfrak{g}$$

which defines the **quantum universal enveloping algebra** [5] **(QUEA)** $\mathcal{U}_{1/\kappa}(\mathfrak{g})$ with **deformed commutation relations**

$$\begin{array}{lll} [J_a, J_b] = \epsilon_{abc} J_c, & [J_a, P_b] = \epsilon_{abc} P_c, & [J_a, K_b] = \epsilon_{abc} K_c, \\ [K_a, P_0] = P_a, & [K_a, P_b] = \delta_{ab} P_0, & \\ [P_0, P_a] = 0, & [P_a, P_b] = 0, & [P_0, J_a] = 0, \end{array}$$

$$[K_a, P_b] = \delta_{ab} \left(\frac{\kappa}{2} \left(1 - \mathrm{e}^{-2P_0/\kappa} \right) + \frac{1}{2\kappa} \mathbf{P}^2 \right) - \frac{1}{\kappa} P_a P_b \,,$$

and **coproduct**

$$\Delta(P_0) = P_0 \otimes 1 + 1 \otimes P_0, \qquad \Delta(J_a) = J_a \otimes 1 + 1 \otimes J_a,$$

$$\Delta(P_a) = P_a \otimes 1 + \mathrm{e}^{-P_0/\kappa} \otimes P_a,$$

$$\Delta(K_a) = K_a \otimes 1 + \mathrm{e}^{-P_0/\kappa} \otimes K_a + \frac{1}{\kappa} \epsilon_{abc} P_b \otimes J_c \,.$$

[5] Lukierski, Nowicki, Tolstoi, 1991.

The **Lie bialgebra $(\mathfrak{g}, \delta)$** is just the first order of the previous QUEA, with **cocommutator**

$$\delta = \Delta_0 - \sigma \circ \Delta_0 : \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}$$

taking the explicit form

$$\begin{aligned}
\delta(P_0) &= \delta(J_a) = 0, \qquad \delta(P_a) = \frac{1}{\kappa} P_a \wedge P_0, \\
\delta(K_1) &= \frac{1}{\kappa}(K_1 \wedge P_0 + J_2 \wedge P_3 - J_3 \wedge P_2), \\
\delta(K_2) &= \frac{1}{\kappa}(K_2 \wedge P_0 + J_3 \wedge P_1 - J_1 \wedge P_3), \\
\delta(K_3) &= \frac{1}{\kappa}(K_3 \wedge P_0 + J_1 \wedge P_2 - J_2 \wedge P_1),
\end{aligned}$$

which completely defines the **κ-Poincaré Poisson-Lie group**, which is just the semiclassical counterpart of the κ-Poincaré quantum group.

κ-Minkowski spacetime

The **coisotropy condition** is satisfied for the Lorentz subalgebra

$$\delta(\mathfrak{l}) \subset \mathfrak{l} \wedge \mathfrak{g},$$

so $\mathbf{M}^{3+1} = G/L$ is a well-defined Poisson homogeneous space, defined by

$$\{x^0, x^a\} = -\frac{1}{\kappa} x^a, \qquad \{x^a, x^b\} = 0.$$

It can be straightforwardly quantized

$$[\hat{x}^0, \hat{x}^a] = -\frac{\hbar}{\kappa} \hat{x}^a, \qquad [\hat{x}^a, \hat{x}^b] = 0.$$

This is the **famous κ-Minkowski spacetime** [6], extensively used in quantum gravity models, where the quantum parameter κ is related to the Planck length.

[6] Lukierski, Nowicki, Tolstoi, 1991.

κ-space of worldlines

In this case H is a **Poisson-Lie subgroup**:

$$\delta(\mathfrak{h}) = 0 \subset \mathfrak{h} \wedge \mathfrak{g},$$

so $L(\mathbf{M}^{3+1}) = G/H$ is well-defined and

$$\{y^1, y^2\} = \frac{1}{\kappa}\left(y^2 \sinh\eta^1 - \frac{y^1 \tanh\eta^2}{\cosh\eta^3}\right),$$

$$\{y^1, y^3\} = \frac{1}{\kappa}\left(y^3 \sinh\eta^1 - y^1 \tanh\eta^3\right),$$

$$\{y^2, y^3\} = \frac{1}{\kappa}\left(y^3 \cosh\eta^1 \sinh\eta^2 - y^2 \tanh\eta^3\right),$$

$$\{y^1, \eta^1\} = \frac{1}{\kappa}\frac{\left(\cosh\eta^1 \cosh\eta^2 \cosh\eta^3 - 1\right)}{\cosh\eta^2 \cosh\eta^3},$$

$$\{y^2, \eta^2\} = \frac{1}{\kappa}\frac{\left(\cosh\eta^1 \cosh\eta^2 \cosh\eta^3 - 1\right)}{\cosh\eta^3},$$

$$\{y^3, \eta^3\} = \frac{1}{\kappa}\left(\cosh\eta^1 \cosh\eta^2 \cosh\eta^3 - 1\right),$$

$$\{y^a, \eta^b\} = 0, \quad a \neq b, \qquad \{\eta^a, \eta^b\} = 0\,.$$

κ-space of worldlines

The diffeomorphism

$$q^1 = \frac{\cosh \eta^2 \cosh \eta^3}{\cosh \eta^1 \cosh \eta^2 \cosh \eta^3 - 1}\, y^1, \qquad p^1 = \eta^1,$$
$$q^2 = \frac{\cosh \eta^3}{\cosh \eta^1 \cosh \eta^2 \cosh \eta^3 - 1}\, y^2, \qquad p^2 = \eta^2,$$
$$q^3 = \frac{1}{\cosh \eta^1 \cosh \eta^2 \cosh \eta^3 - 1}\, y^3, \qquad p^3 = \eta^3,$$

shows that **the space of worldlines is symplectic** (outside $(\eta^1, \eta^2, \eta^3) \neq (0,0,0)$)

$$\{q^a, q^b\} = \{p^a, p^b\} = 0, \qquad \{q^a, p^b\} = \frac{1}{\kappa}\, \delta_{ab}.$$

and so, similarly to the spacetime, it can be straightforwardly **quantized**

$$[\hat{q}^a, \hat{q}^b] = [\hat{p}^a, \hat{p}^b] = 0, \qquad [\hat{q}^a, \hat{p}^b] = \frac{\hbar}{\kappa}\, \delta_{ab} \mathbb{I}.$$

5. Final remarks

- We have showed a **completely new idea for introducing noncommutativity in the space of worldlines**.
- This can be interpreted as **noncommutative quantum observers arising from quantum group symmetries**.
- Hopefully, this can produce **testable results from a quantum gravity perspective**.
- The construction presented is **completely general** and can be applied to **any quantum deformation**.
- Details in: Ballesteros, G-S, Herranz, *Phys Lett B.* **792**, 175-181 (2019).
- Work is in progress for the case of the κ-(A)dS noncommutative spacetime (non-vanishing cosmological constant) recently constructed (see Ballesteros, G-S, Herranz, *Phys Lett B.* **796**, 93-101 (2019)).

- Physical consequences from a quantum gravity perspective in: Ballesteros, Gubitosi, G-S, Herranz, Mercati (In preparation), giving rise to 'fuzzy' worldlines:

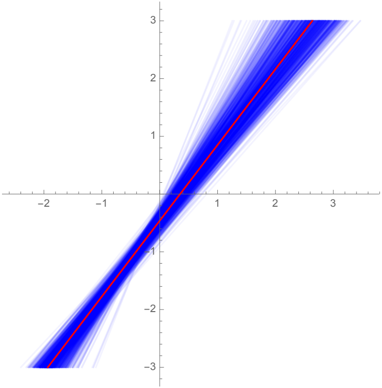

Figure: Fuzzy worldlines arising from the model.

Thanks for your attention!

Figure: Photo by Aydin Büyüktas.