Lindblad master equation in Lie algebra representation

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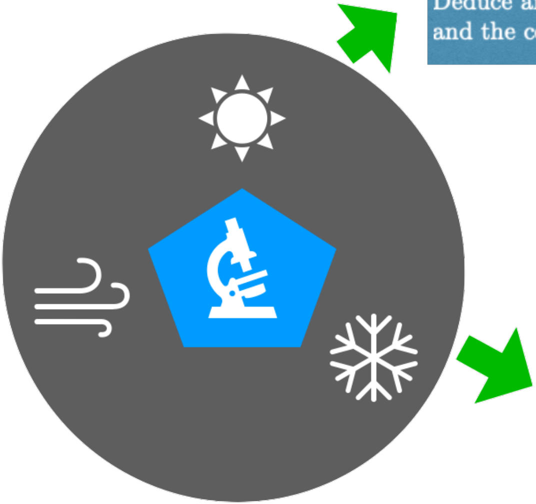


Deduce an interaction Hamiltonian and the corresponding master equation

Use some jumps operators and observe if they are enough to describe the phenomenon

The Lindblad Master equation

The Born-Markov approximation

$$\frac{d}{dt}\rho = -i[H, \rho] + \sum_k \Gamma_k \left(L_k \rho L_k^\dagger - \frac{1}{2} \left\{ L_k L_k^\dagger, \rho \right\} \right) \equiv \Lambda(\rho)$$

1. Valid for a small interaction with the environment (Born's linearity approximation)
2. Markov approximation implies that there is not memory effect from the environment.
3. The Lindblad Master Equation is invariant under unitary transformations of the jump operators L_k.

Standard solution methods

Differential equation system

$$\begin{pmatrix} \dot\rho_{11} & \dot\rho_{12} \\ \dot\rho_{21} & \dot\rho_{22} \end{pmatrix} = \begin{pmatrix} f(\rho) & g(\rho) \\ h(\rho) & m(\rho) \end{pmatrix}$$

Kraus representation theorem

$$\rho(t+\tau) = \phi_\tau(\rho(t))$$

Factorization-type methods

$$\dot\rho = \Lambda(\rho) = A^\dagger A\rho$$

Ket-Bra Entangled states

$$\tfrac{d}{dt}\tilde\rho = \tfrac{1}{i\hbar}[\tilde H, \tilde\rho]$$

Complete Positive Trace Preserving maps

$$\rho(t+\tau) = \phi_\tau(\rho(t))$$

Completely Positive maps

A map $\mathcal{E} : \mathfrak{gl}(\mathcal{H}) \to \mathfrak{gl}(\mathcal{H})$ is said to be CP if it is linear and $\mathcal{E} \otimes \mathbb{I} \in \mathfrak{gl}(\mathcal{H} \otimes \mathcal{H}')$ is positive for every $\mathcal{H}'$

Trace preserving maps

A map $\mathcal{E}$ is TP if $\mathrm{tr}(\mathcal{E}(A)) = \mathrm{tr}(A)$ for all $A \in \mathfrak{gl}(\mathcal{H})$.

The Lindblad master equation implies the existence of a TCP map that describes that evolution

Kraus decomposition theorem

Kraus (1989)

For any CPT map ϕ_t exist a set of operators $M_0, M1, M_2, \dots M_k$ with $k \leq (\dim(\mathcal{H}))^2$ such that

$$\phi_t(\rho) = \sum_\mu M_\mu(t)\rho M_\mu^\dagger(t)$$

with

$$\sum_\mu M_\mu^\dagger M_\mu = \mathbb{I}$$

Find the Krauss representation of an associated linear map is equivalent to solve the associated master equation

$$\dot{\rho}_{\mathrm{I}}(t) = \gamma \sum_{i=0,1} B_i(t)\rho_{\mathrm{I}}(t)B_i^\dagger(t), \quad B_i(t) = e^{-At}L_i e^{At}$$

Lie algebra channels

Definition

Let $\mathfrak{g}$ denote a Lie algebra of dimension k, with basis $\{X_i\}$. Let β be an irreducible $\mathfrak{g}$-representation on $\mathcal{H}$. The LA channel is the one in which a change in the state occurs with probability $p(t)$, caused by the action of $\beta(X_i)$.

The Kraus operators are taken as

$$M_0 = \sqrt{1 - p(t)}\mathbb{I} \quad M_i = \sqrt{\kappa p(t)}\beta(X_i)$$

with

$$\sum_i \beta(X_i)^2 = \kappa^{-1} \cdot \mathbb{I}$$

The Spin-1 Channel

Consider the 3-dimensional representation of $\mathfrak{su}_2$

$$J_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, J_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, J_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

If the density matrix can be written as

$$\boldsymbol{\rho}_v = \frac{1}{3}(\mathbf{1} + v \cdot J), \quad v \in \mathbb{R}^3$$

then

$$\mathcal{E}(\boldsymbol{\rho}_v) = \frac{1}{3}\mathbf{1} + \frac{1-p}{3} v \cdot J + \frac{p}{6}\sum_{a,b} v_b J_a J_b J_a$$

Utility of the LA channels

1. Well defined asymptotic behavior (Iteration Formulas)

$$\lim_{t\to\infty} \rho(t) = \rho_s$$

2. Irreducible Representations Have No Decoherence-Free Subsystems/Subspaces

$$\begin{pmatrix} \dot{\rho}_{11} & \dot{\rho}_{12} \\ \dot{\rho}_{21} & \dot{\rho}_{22} \end{pmatrix} = \begin{pmatrix} f(\rho) & g(\rho) \\ h(\rho) & \rho_{22} \end{pmatrix}$$

3. Irreducible Representations Return to Equilibrium

$$\lim_{t\to\infty} \rho(t) = \rho(0)$$

The G-representation

Consider a master equation given by

$$\dot{\rho}(t) = \Lambda(\rho(t))$$

Let $\{G_a\}$ denote any convenient orthonormal basis set for the space of self-adjoint matrices in $\mathcal{H}$, i.e.

$$G_a^\dagger = G_a, \quad \mathrm{tr}[G_a G_b] = \delta_{ab}$$

Then every $X \in \mathfrak{gl}(H)$ can be written as

$$X = \sum_j x_j G_j$$

with

$$x_j = \mathrm{tr}[G_j X]$$

Then the master equation $\Lambda(\rho)$ can be written as

$$\Lambda(\rho(t)) = \sum_{k,\ell} L_{k,\ell}\, r_\ell(t) G_k \tag{1}$$

with

$$L_{k,\ell} := Tr\left[G_k \Lambda\left(G_\ell\right)\right], \quad r_\ell(t) := Tr\left[G_\ell \rho(t)\right]$$

Also $\phi(\cdot)_t$ can be written in the G-representation as

$$\phi_t[\rho(0)] = \sum_{k,\ell} F_{k,\ell}(t)\, r_\ell(0) G_k$$

with

$$F_{k,\ell}(t) := Tr\left[G_k \phi_t\left(G_\ell\right)\right], \quad r_\ell(0) := Tr\left[G_\ell \rho(0)\right]$$

From the master equation to the CPT map

Taking the time derivative of $\phi_t[\rho(0)]$ we have a natural connection between an master equation and the CPT map

$$\dot{F}_{k,\ell}(t) r_\ell(0) = L_{k,\ell} r_\ell(t)$$

Using

$$r_\ell(t) = \sum_m F_{\ell,m} r_m(0) \tag{2}$$

$$\dot{\mathbf{F}} = \mathbf{L}\mathbf{F}$$

with the natural solution

$$\mathbf{F} = \mathcal{T} \exp\left(\int^t \mathbf{L} d\tau \right)$$

From TCP maps to Kraus-type descomposition

We look for the Kraus-type decomposition

$$\phi_t(\rho(0)) = \sum_{i,j} \mathcal{Z}_{i,j}(t)\beta(X_i)\rho(0)\beta(X_j)$$

with β a representation of the Lie algebra spaned by X_i.
If $\mathcal{H} = \text{span}\{|\alpha_i\rangle\}$, define the symmetric matrix

$$S_{\{i,j\},\{r,s\}} := \langle \alpha_i \left| \phi \left(|\alpha_j\rangle \langle \alpha_s| \right) \right| \alpha_r \rangle$$

We can compute the matrix S from the matrix F

$$S_{\{i,j\},\{r,s\}} = \sum_{k,\ell} F_{k,\ell}(t)\, \text{tr}(G_\ell \,|\alpha_j\rangle \langle \alpha_i|\, G_k \,|\alpha_r\rangle \langle \alpha_s|).$$

Lets consider the following unitary transformation

$$S^{(W)} = W^{\dagger} S W$$

$$\begin{aligned} W_{i,\{r,s\}} &= \mathrm{tr}(\beta(X_i) \left|\alpha_s\right\rangle \left\langle\alpha_r\right|) \\ &= \left\langle\alpha_r\right| \beta(X_i) \left|\alpha_s\right\rangle \end{aligned}$$

This implies that

$$\alpha(X_i) = \sum_{\{r,s\}} W_{i,\{r,s\}} \left|\alpha_r\right\rangle \left\langle\alpha_s\right|$$

Expanding the map $\phi(\cdot)$ in the basis $\{\left|\alpha_i\right\rangle \left\langle\alpha_j\right|\}$

$$\phi(\rho) = \sum_{i,j,r,s} S_{\{i,j\},\{r,s\}} \left|\alpha_i\right\rangle \left\langle\alpha_j\right| \rho \left|\alpha_s\right\rangle \left\langle\alpha_r\right|$$

We get the desired result

$$\phi(\rho) = \sum_{p,t} S^{(W)}_{p,t} \beta(X_p)\, \rho\, \beta^\dagger(X_t)$$

This matrix is unitary iff

$$\sum_i W^*_{i,\{r,s\}} W_{i,\{p,t\}} = \sum_i \langle \alpha_s | \, \beta(X_i)^\dagger \, | \alpha_r \rangle \, \langle \alpha_p | \, \beta(X_i) \, | \alpha_t \rangle = \delta_{r,p} \delta_{s,t}$$

If

$$\beta(X_i) = \sum_{mn} x^n_{m,n} \, | \alpha_m \rangle \, \langle \alpha_n |$$

Then

$$\sum_i \overline{x}^i_{r,s} \overline{x}^i_{p,t} = \delta_{r,p} \delta_{s,t}$$

If the Lie algebra is defined by the structure constants

$$[\,\beta(X_i), \beta(X_j)\,] = \sum_k \epsilon_{i,j,k}\beta(X_k)$$

Then the representation must satisfy

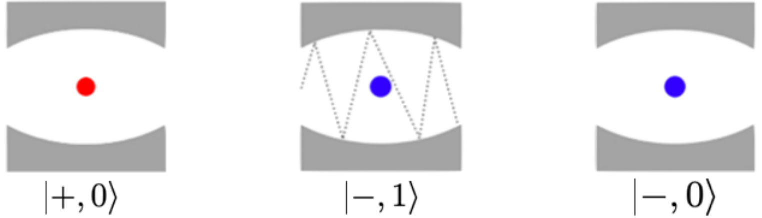
$$x^i_{p,k}x^j_{k,t} = \epsilon_{i,j,k}x^k_{p,t}$$

Objetive

Look for a representation of some appropriate Lie algebra that allows us to obtain the corresponding LA channel

Two-level atom in a diffuse cavity

$$i\hbar\dot{\rho}(t) = [H, \rho] + i\mathcal{L}(\rho)$$



$$H = \hbar\omega a^\dagger a + \hbar\omega\sigma_+\sigma_- + g\left(a^\dagger\sigma_- + a\sigma_+\right) \quad \mathcal{L}(\rho) = \gamma\left(a\rho a^\dagger - \frac{1}{2}\left\{\rho, a^\dagger a\right\}\right)$$

Looking for the G-representation

Consider the following orthogonal matrices

$$G_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad G_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad G_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$G_4 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad G_5 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad G_6 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$G_7 = \begin{pmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad G_8 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & -i & 0 \end{pmatrix} \quad G_9 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}$$

$$L_{k,\ell} = \mathrm{tr}(G_k \Lambda(G_\ell))$$

$$\frac{1}{\hbar}\begin{pmatrix} -\gamma & 0 & 0 & 0 & 0 & 0 & -2g & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2g & 0 & 0 \\ \gamma & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\gamma & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2\hbar\omega & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2g & -2\hbar\omega \\ 2g & -2g & 0 & 0 & 0 & 0 & -\gamma & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\hbar\omega & 2g & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2g & 2\hbar\omega & 0 & 0 & 0 \end{pmatrix}$$

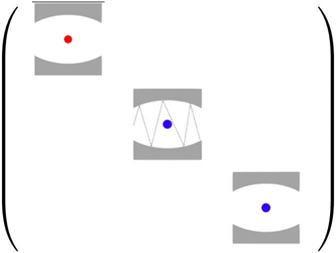
The G-representation of the master equation:

$$\frac{d}{dt}\rho(t) = \sum_{k,\ell} L_{k,\ell} r_\ell(t) G_k$$

$$r(t) = \begin{pmatrix} \rho_{11} \\ \rho_{22} \\ \rho_{33} \\ \rho_{21} + \rho_{12} \\ \rho_{32} + \rho_{23} \\ \rho_{31} + \rho_{13} \\ i\rho_{21} - i\rho_{12} \\ i\rho_{32} - i\rho_{23} \\ i\rho_{31} - i\rho_{13} \end{pmatrix}$$

$$\frac{d}{dt}\rho(t) =$$

$$\frac{1}{i\hbar}\begin{pmatrix} -\rho_{11}i\gamma + g(\rho_{21} - \rho_{12}) & g(\rho_{22} - \rho_{11}) - \rho_{12}\frac{i\gamma}{2} & -\hbar\omega\rho_{13} - g\rho_{23} \\ g(\rho_{11} - \rho_{22}) - \rho_{21}\frac{i\gamma}{2} & g(\rho_{12} - \rho_{21}) & -g\rho_{13} - \hbar\omega\rho_{23} \\ \hbar\omega\rho_{31} + g\rho_{32} & g\rho_{31} + \hbar\omega\rho_{32} & i\gamma\rho_{11} \end{pmatrix}$$

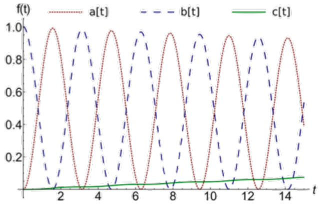


Looking for the CPT map

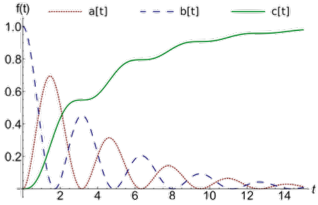
$$\phi_t[\rho(0)] = \sum_{k,\ell} F_{k,\ell}(t)\, r_\ell(0)\, G_k \qquad \mathbf{F}(t) = \exp(\mathbf{L}t)$$

$$\begin{pmatrix}
-3e^{-\frac{2t\gamma}{3\hbar}} & \frac{6}{5}e^{-\frac{2t\gamma}{3\hbar}} & 0 & 0 & 0 & 0 & \frac{7e^{-\frac{2t\gamma}{3\hbar}}\gamma}{10g} & 0 & 0 \\
\frac{6}{5}e^{-\frac{2t\gamma}{3\hbar}} & \frac{3}{5}e^{-\frac{2t\gamma}{3\hbar}} & 0 & 0 & 0 & 0 & \frac{e^{-\frac{2t\gamma}{3\hbar}}\gamma}{10g} & 0 & 0 \\
\frac{9}{5}e^{-\frac{2t\gamma}{3\hbar}} & \frac{9}{5}e^{-\frac{2t\gamma}{3\hbar}} & \frac{2\gamma^2}{27g^2} & 0 & 0 & 0 & -\frac{3e^{-\frac{2t\gamma}{3\hbar}}\gamma}{5g} & 0 & 0 \\
0 & 0 & 0 & e^{-\frac{t\gamma}{\hbar}} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \mathrm{Cos}\left[\frac{2gt}{\hbar}\right] & 0 & -\mathrm{Sin}\left[\frac{2gt}{\hbar}\right] & 0 \\
\frac{7e^{-\frac{2t\gamma}{3\hbar}}\gamma}{10g} & \frac{e^{-\frac{2t\gamma}{3\hbar}}\gamma}{10g} & 0 & 0 & 0 & 0 & -\frac{21}{5}e^{-\frac{2t\gamma}{3\hbar}} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \mathrm{Sin}\left[\frac{2gt}{\hbar}\right] & 0 & \mathrm{Cos}\left[\frac{2gt}{\hbar}\right] & 0 \\
0 & 0 & 0 & 0 & \frac{2gt}{\hbar} & 0 & 0 & 0 & 1
\end{pmatrix}$$

Solution for $\gamma < 4g$

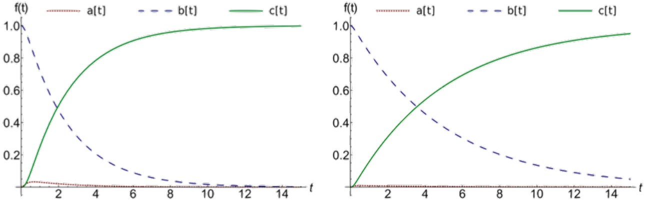


Using $\gamma = 0.01$, $g = 1$; $\gamma = 0.1$, $g = 1$

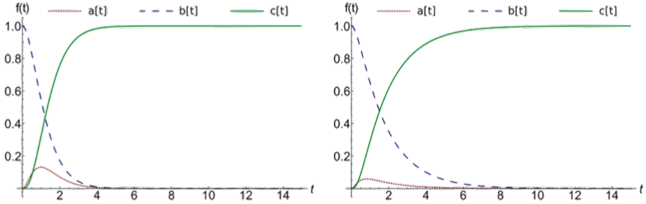


Using $\gamma = 0.05$, $g = 1$; $\gamma = 1$, $g = 1$

Solution for $\gamma > 4g$



Using $\gamma = 4.1,\ g = 1;\ \gamma = 7,\ g = 1$



Using $\gamma = 10,\ g = 1;\ \gamma = 20,\ g = 1$

Looking for a LA channel

We look for a 9-dimensional Lie algebra

Lie algebra	Dimension
$\mathfrak{gl}(n)$	n^2
$\mathfrak{sl}(n)$	$n^2 - 1$
$\mathfrak{so}(n)$	$\frac{n(n-1)}{2}$
$\mathfrak{sp}(2n)$	$\frac{n(2n+1)}{2}$
$\mathfrak{u}(n)$	n^2
$\mathfrak{su}(n)$	$n^2 - 1$

The $\mathfrak{su}(4) \sim \mathfrak{su}(2) \otimes \mathfrak{su}(2)$ generators

$$\lambda_1 = X_4^{1,2} + X_4^{2,1}$$

$$\lambda_2 = -iX_4^{1,2} + iX_4^{2,1}$$

$$\lambda_3 = X_4^{1,1} - X_4^{2,2}$$

$$\lambda_4 = X_4^{1,3} + X_4^{3,1}$$

$$\lambda_5 = -iX_4^{1,3} + iX_4^{3,1}$$

$$\lambda_7 = -iX_4^{2,3} + iX_4^{3,2}$$

$$\lambda_8 = \frac{1}{\sqrt{3}} \left(X_4^{1,1} + X_4^{2,2} - 2X_4^{3,3} \right)$$

$$\lambda_9 = X_4^{1,4} + X_4^{4,1}$$

$$\lambda_{10} = -iX_4^{1,4} + iX_4^{4,1}$$

$$\lambda_{11} = X_4^{4,2} + X_4^{2,4}$$

$$\lambda_{13} = -iX_4^{4,2} + iX_4^{2,4}$$

$$\lambda_{13} = -iX_4^{3,4} + X_4^{4,3}$$

$$\lambda_{15} = \frac{1}{\sqrt{6}}(X_4^{1,1} + X_4^{2,2} + X_4^{3,3} - X_4^{4,4})$$

$$[\lambda_i, \lambda_k] = \sum_l C_{ik} 2i\lambda_l$$

The Dressed representation

We define the following two maps

$$\begin{aligned}\Phi_1 &: \mathcal{H}_a \to \mathbb{R}^2 \\ \Phi_2 &: \mathcal{H}_f \to \mathbb{R}^2\end{aligned}$$

taking the tensor product

$$\begin{aligned}\Phi_1 \otimes \Phi_2 &: \mathcal{H} \to \mathbb{R}^2 \otimes \mathbb{R}^2 \cong \mathbb{R}^4 \\ \Phi_1 \otimes \Phi_2 &: |+,0\rangle \mapsto \left|e_1^2\right\rangle \otimes \left|e_1^2\right\rangle = \left|e_1^4\right\rangle \\ \Phi_1 \otimes \Phi_2 &: |-,1\rangle \mapsto \left|e_2^2\right\rangle \otimes \left|e_2^2\right\rangle = \left|e_4^4\right\rangle \\ \Phi_1 \otimes \Phi_2 &: |-,0\rangle \mapsto \left|e_2^2\right\rangle \otimes \left|e_1^2\right\rangle = \left|e_2^4\right\rangle\end{aligned}$$

and

$$\psi : \left\{\left|e_1^4\right\rangle, \left|e_4^4\right\rangle, \left|e_2^4\right\rangle\right\} \mapsto \left\{\left|e_1^3\right\rangle, \left|e_2^3\right\rangle, \left|e_3^3\right\rangle\right\}$$

We have the dressed representation

$$\phi = \psi \circ (\Phi_1 \otimes \Phi_2) : \{|+,0\rangle, |-,1\rangle, |-,0\rangle\} \mapsto \left\{\left|e_1^3\right\rangle, \left|e_2^3\right\rangle, \left|e_3^3\right\rangle\right\}$$

$\mathfrak{su}(4) generators in Dressed representation$

$$\lambda_1 = \frac{1}{\sqrt{2}}\left(X_3^{1,3} + X_3^{3,1}\right) \qquad \lambda_9 = \frac{1}{\sqrt{2}}\left(X_3^{1,2} + X_3^{2,1}\right)$$

$$\lambda_2 = \frac{1}{\sqrt{2}}\left(-iX_3^{1,3} + iX_3^{3,1}\right) \qquad \lambda_{10} = \frac{1}{\sqrt{2}}\left(-iX_3^{1,2} + iX_3^{2,1}\right)$$

$$\lambda_3 = X_3^{1,1} \qquad \lambda_{11} = \frac{1}{\sqrt{2}}\left(X_3^{3,2} + X_3^{2,3}\right)$$

$$\lambda_4 = 0 \qquad \lambda_{12} = \frac{1}{\sqrt{2}}\left(-iX_3^{3,2} + iX_3^{2,3}\right)$$

$$\lambda_5 = 0 \qquad \lambda_{13} = 0$$

$$\lambda_7 = 0 \qquad \lambda_{15} = X_3^{3,3}$$

$$\lambda_8 = X_3^{2,2}$$

The $\mathfrak{su}(4)$ channel

$$W = \begin{pmatrix} 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{-1}{\sqrt{2}} & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{-i}{\sqrt{2}} & 0 & \frac{i}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{i}{\sqrt{2}} & 0 & \frac{-i}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

We get the desired result

$$\phi(\rho) = \sum_{p,t} S^{(W)}_{p,t} \lambda_p \, \rho \, \lambda_t m^\dagger$$

Thank you for the attention