Groups, special functions and rigged Hilbert spaces

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Contents

MOTIVATION

- In the last years we have been involved in a program of revision of the connection between special functions, in particular, the COP, with Lie groups, differential eq. and Hilbert spaces. So, in some sense we are revisiting the connection between symmetry and QM.
- This program has been motivated for previous works on superintegrable Hamiltonian systems
  - The energy levels are degenerate in the quantum case. The origin of this degeneration is the existence of a symmetry group.
  - Usually special functions (COP) appear as solutions of the Schrödinger eq. for this kind of systems.

- We have considered the ladder algebraic structure for different COP, like Hermite, Legendre, Laguerre, ALP, SH, . . .
- In any case we have obtained a symmetry group (dynamical group or spectrum generating group).
- The corresponding COP has associated a particular representation of its symmetry Lie group.
    - For instance, for the ALP, SH, $SO(3,2)$ is the symmetry group and in both cases they support a UIR with quadratic Casimir $-5/4$.
    - Both are bases of square integrable functions defined on $(-1,1)\times\mathbb{Z}$ and on the sphere S^2, respectively.
- In any case we get discrete bases and continuous bases.

- Rigged Hilbert Spaces are suitable frameworks when a description of quantum states requires of discrete bases, or complete orthonormal sets, and generalized continuous bases like those used in the Dirac formalism.
- This is a common situation arisen when we deal with special functions, which hold discrete labels and depend on continuous variables.
- We present a framework where special functions, Lie algebras and rigged Hilbert spaces are fully integrated.
- To illustrate our statement we consider the particular case of the Spherical Harmonics. They are of wide use in Physics and recently in Signal Processing when 3D signals are considered.

Introduction: mathematical overview

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Too wide class: all functions are (or could be in future) special

Possible sub-classes:

- Hypergeometric functions (Askey scheme)
- Functions associated to representations of Lie groups (Wigner (Talman), Miller, Vilenkin)
- Functions with additional properties (Truesdell)

Berry

What are they [special functions], other than just names of mathematical objects that are useful only in situations of contrived simplicity? Why are we so pleased when a complicated calculation 'comes out' as a Bessel function, or Laguerre polynomial? What determines which functions are 'special'? (Berry [2001], p. 12)

One reason for the continuing popularity of special functions could be that they enshrine sets of recognizable and communicable patterns and so constitute a common currency. [. . .] [P]erhaps special functions provide an economical and shared culture analogous to books: places to keep our knowledge in, so that we can use our heads for better things. (Berry [2001], p. 12)

M. Berry, "Why are special functions special?", *Phys. Today* **54** (2001) 11.

2.- Representations of Lie groups

Wigner (Talman), Miller, Vilenkin

Talman's conclusion is that

> [t]he group theoretic treatment shows that the special functions are special only in that they are related to specific groups. The usefulness of group representation theory for the solution of a variety of physical problems makes it natural that representation matrix elements are important special functions for many problems in mathematical physics. It may further be true that the properties that can be derived group theoretically are their most important ones, since they originate from the 'geometric' properties of the functions. ([1968], p. 2)

Talman's work was inspired by lectures by Eugene P. Wigner who states in the introduction to Talman's book that

> [. . .] the common point of view from which the special functions are here considered, and also the natural classification of their properties, destroys some of the mystique which has surrounded, and still surrounds, these functions. Whether this is a loss or a gain remains for the reader to decide ([1968], p. xii).

J D Talman, *Special functions: a group theoretic approach* (New York: Benjamin, 1968).

3.- Functions with additional properties

Truesdell:

Special functions are defined by a set of formal properties

physics [...]' ([1948], p. 8). This class of *familiar* special functions include Bessel, Legendre, Laguerre, and Hermite functions which have the following 'major formal properties' in common:

> (1) They satisfy ordinary linear differential equations of second order; (2) they satisfy ordinary linear difference equations of second order; (3) with suitable weight functions they form complete sets of orthogonal functions over suitable intervals; (4) they satisfy linear differential-difference [recurrence] relations. The first three of these properties after very long and thorough investigations by numerous excellent mathematicians have yielded but slight clews [sic] to the discovery of such relations as [those 35

C Truesdell, *Annals in Math. Studies* **18** (Princeton: Princeton Univ. Press 1948).

Algebraic special functions

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Starting from 2) Vilenkin and 3) Truesdell we define

Algebraic special functions (ASF). Assumptions:

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We get:

1. defining 2nd order differential equation are obtained from:
   - factorization method
   - Casimir of all possible subalgebras of $\mathcal{G}$
   - Casimir of the full algebra $\mathcal{G}$ (in a particular representation)
2. the set of operators acting on the Hilbert space is homomorphic to the Universal Enveloping Algebra (UEA) built on $\mathcal{G}$.

ALGEBRAIC SPECIAL FUNCTIONS

RANK 1 : Hermite ($h(1), \mathcal{C} = 0$); Laguerre, Legendre ($su(1,1), \mathcal{C} = -1/4$)

RANK 2 : Assoc. Legendre, Spher. Harm. ($so(3,2), \mathcal{C} = -5/4$);
Gegenbauer, Jacobi $P_n^{(\alpha,\alpha)}(x)$,
Assoc. Laguerre ($h(1) \oplus h(1) \xrightarrow[\text{procedure}]{\text{Schwinger}} so(3,2), \mathcal{C} = -5/4$);
Zernike ($su(1,1) \oplus su(1,1)$);

RANK 3 : Jacobi $P_n^{(\alpha,\beta)}(x)$ ($su(2,2), \mathcal{C} = -1/4$)

⋮ ⋮

RANK n : . . . . (?)

Rigged Hilbert Spaces (Gelf'and triplet)

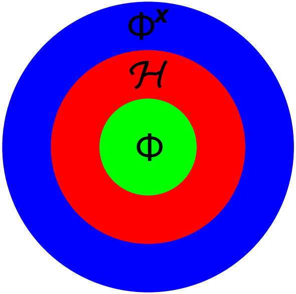
- RHS were introduced by Gel'fand and collaborators in connection with the spectral theory of self-adjoint oper. and proved, together with Maurin, the nuclear spectral theorem (1955-1959).
- The RHS formulation of QM was introduced by Bohm and Roberts (1965)
- In the last years Feichtinger has applied RHS in time-frequency analysis and in Gabor analysis with applications in signal processing.
- There are several reasons to assert that Hilbert spaces are not sufficient for a thoroughly formulation of QM even within the non-relativistic context:
  - The Dirac formulation, where eigenvectors of oper. with eigenvalues in the continuous spectrum play a crucial role.
  - Furthermore, these eigenvectors, not in the Hilbert space of square integrable wave funct., are widely used in QM.
  - A proper definition of Gamow vectors, which are widely used in calculations including unstable quantum systems, are also non-normalizable.
  - Formulations of time asymmetry in QM may require the use of tools different from, but including to, Hilbert spaces.
- The proper tool that includes naturally the Hilbert space containing pure states on which observables act and all other characteristics is the RHS.

Rigged Hilbert Space (RHS)

is a triplet of spaces

$$\Phi \subset \mathcal{H} \subset \Phi^{\times}$$

$\mathcal{H}$ is an infinite dim. (separable) Hilbert space.
Φ (*space of test vectors*) is a dense subspace of $\mathcal{H}$ endowed with its own topology.
This topology on Φ is finer (contains more open sets) than the topology that Φ has as subspace of $\mathcal{H}$.
$\Phi^{\times}$ is the dual /antidual space of Φ with a topology compatible with $(\Phi, \Phi^{\times})$



$$\Phi \subset \mathcal{H} \subset \Phi^{x}$$

One straightforward consequence of this fact is that all sequences which converge on Φ, also converge on $\mathcal{H}$, the converse being not true.

The difference between topologies gives rise that the dual space of Φ is bigger than $\mathcal{H}$, which is self-dual.

Definition

Let A be a densely defined oper. on $\mathcal{H}$ such that Φ be a subspace of its domain and that

$$\forall\, \varphi \in \Phi \qquad \Longrightarrow \qquad A\varphi \in \Phi$$

we say that Φ reduces A or that Φ is invariant under the action of A, $A\Phi \subset \Phi$.

Property

In this case, A may be extended unambiguously to the anti-dual $\Phi^\times$ ($A^\times$) by making use of the duality formula:

$$\langle A\varphi | F\rangle = \langle \varphi | A^\times\, F\rangle\,, \qquad \forall\, \varphi \in \Phi\,, \qquad \forall\, F \in \Phi^\times$$

If A is continuous on Φ, then $A^\times \equiv A$ is continuous on $\Phi^\times$.

The topology on Φ is given by an infinite countable set of norms

$$\{||-||_{n=1}^{\infty}\}$$

A linear operator A on Φ is continuous

if and only if for each norm $||-||_n$ there is a $K_n > 0$ and a finite sequence of norms $||-||_{p_1}, ||-||_{p_2}, \dots, ||-||_{p_r}$ such that for any $\varphi \in \Phi$, one has

$$\boxed{||A\varphi||_n \leq K_n \, (||\varphi||_{p_1} + ||\varphi||_{p_2} + \cdots + ||\varphi||_{p_r})}$$

The same result applies to check the continuity of any linear or antilinear mapping

$$F : \Phi \longmapsto \mathbb{C}$$

In this case, the norm $||A\varphi||_n$ should be replaced by the modulus $|F(\varphi)|$.

Spherical harmonics (SH)

Let us consider the hollow unit sphere S^2 in $\mathbb{R}^3$. Any point of S^2 is characterized by two angular variables (θ, ϕ) s.t. : $0 \leq \theta \leq \pi$, $0 \leq \phi < 2\pi$

Spherical harmonics have the following explicit form:

$$Y_l^m(\theta, \phi) = \frac{1}{\sqrt{2\pi}} \sqrt{\frac{(l-m)!}{(l+m)!}}\, e^{im\phi}\, P_l^m(\cos\theta)$$

where $l \in \mathbb{N}$, $m \in \mathbb{Z}$ with $|m| \leq l$, P_l^m are the Associated Legendre Polynomials

As is well know, SH verify the following diff. eq.

$$\left(\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \sin\theta\, \partial\theta + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^2} + l(l+1) \right) Y_l^m(\theta, \phi) = 0$$

SH are an orthonormal basis for the Hilbert space $L^2(S^2, d\Omega)$, with the measure $d\Omega := d(\cos\theta)\, d\phi$, of Lebesgue square integrable func. on S^2

Any function $f(\theta,\phi) \in L^2(S^2, d\Omega)$ admits the span

$$f(\theta,\phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} f_{l,m} \sqrt{l+1/2}\, Y_l^m(\theta,\phi)$$

where the series converges in the sense of the norm in $L^2(S^2, d\Omega)$.

Necessary and sufficient cond. for convergence : $\displaystyle \sum_{l=0}^{\infty} \sum_{m=-l}^{l} |f_{l,m}|^2 < \infty$

The coefficients $f_{l,m}$ are given by the expression

$$f_{l,m} = \int_0^{2\pi} d\phi \int_0^{\pi} d(\cos\theta)\, [Y_l^m]^*(\theta,\phi)\, f(\theta,\phi)\,,$$

SH satisfy the relations of orthonormality and completeness

$$\int_{S^2} d\Omega\ [Y_l^m]^*\ (\theta,\phi)\ (l+1/2)\ Y_{l'}^{m'}(\theta,\phi) = \delta_{l,l'}\,\delta_{m,m'}\ ,$$

$$\sum_{l=|m|}^{+\infty} \sum_{m=-\infty}^{+\infty} [Y_l^m]^*\ (\theta,\phi)\ (l+1/2)\ Y_l^m(\theta',\phi') = \delta\,(\cos\theta - \cos\theta')\ \delta(\phi-\phi')\,,$$

with $\delta\,(\cos\theta - \cos\theta') = \delta(\theta-\theta')/\sin\theta$.

Symmetries of the spherical harmonics

The SH span a Hilbert space supporting a UIR of $SO(3)$, although is less acknowledged that they also span a space supporting a UIR with quadratic Casimir $C_2 = -5/4$ of the de anti-de Sitter group $SO(3,2)$

Symmetries of the Associated Legendre Polynomials

The ALPs are solutions of the gen. Legendre eq.

$$(1-x^2)y'' - 2xy' + \left(l(l+1) - \frac{m^2}{1-x^2} \right) y = 0\,.$$

Many recurrence formulae are supported by the ALP suggesting the presence of a hidden large symmetry. We consider as starting point:

$$\begin{aligned}
(x^2-1){P'}_l^m(x) &= \sqrt{1-x^2}P_l^{m+1}(x) + m\,x\,P_l^m(x) \\
(x^2-1){P'}_l^m(x) &= -(l+m)(l-m+1)\sqrt{1-x^2}P_l^{m-1}(x) - m\,x\,P_l^m(x) \\
(x^2-1){P'}_l^m(x) &= -(l+1)\,x\,P_l^m(x) + (l-m+1)P_{l+1}^m(x) \\
(x^2-1){P'}_l^m(x) &= l\,x\,P_l^m(x) - (l+m)P_{l-1}^m(x)
\end{aligned}$$

Definition eq. depends from m^2 only. But $\{P_l^m(x)\}$ have a different normalization for $m > 0$ and $m < 0$: $P_l^{-m}(x) = (-1)^m \frac{(l-m)!}{(l+m)!} P_l^m(x)$

Using one of the principles of the XX-century Physics: save the symmetries we make the following rescaling

$$\boxed{T_l^m(x) := \sqrt{\frac{(l-m)!}{(l+m)!}} P_l^m(x)} \quad \Longrightarrow \quad T_l^{-m}(x) = (-1)^m T_l^m(x)$$

This rescaling of $P_l^m(x)$ makes easier the exhibition of the underlying symmetry algebra.

But, as eigenvectors of different eigenvalues of the hermitian operator M, they are different vectors in the vector space $\{T_l^m(x)\}$

Orthogonality and completeness of the $\{T_l^m(x)\}$ for fix value of m are :

$$\begin{aligned} \int_{-1}^{1} T_l^m(x)\,(l+1/2)\,T_{l'}^m(x)dx &= \delta_{ll'} \qquad (l, l' \geq |m|) \\ \sum_{l=|m|}^{\infty} T_l^m(x)\,(l+1/2)\,T_l^m(y) &= \delta(x-y) \end{aligned}$$

Introducing the operators X and D_X of the configuration space

$$Xf(x) = x\,f(x) \qquad D_x f(x) = f'(x) \qquad [X, D_x] = -1;$$

and the 2 other operators L and M

$$LT_l^m(x) = l\,T_l^m(x), \qquad MT_l^m(x) = mT_l^m(x)$$

From the recurrence formulae, we obtain the ladder operators $J_\pm$ and $K_\pm$:

$$\begin{aligned} J_\pm &:= \mp\sqrt{1-X^2}D_x - \frac{X}{\sqrt{1-X^2}}M \\ K_\pm &:= -(1-X^2)D_x + X(L+1/2 \pm 1/2) \end{aligned}$$

They act on the $T_l^m(x)$ as:

$$\boxed{\begin{aligned} J_\pm T_l^m(x) &= \sqrt{(l\mp m)(l\pm m+1)}\,T_l^{m\pm 1}(x) \\ K_\pm T_l{}^m(x) &= \sqrt{(l-m+1/2\pm 1/2)(l+m+1/2\pm 1/2)}\,T_{l\pm 1}{}^m(x) \end{aligned}}$$

Although L and M are diagonal on the $T_l^m(x)$ they do not commute neither with the configuration space, nor with $J_\pm$ and $K_\pm$:

$$[M, L] = 0 \quad [M, J_\pm] = \pm J_\pm \quad [M, K_\pm] = 0 \quad [L, J_\pm] = 0 \quad [L, K_\pm] = \pm K_\pm$$

$so_K(2,1)$

We see that

$$[K_+, K_-] = -2K_3\,, \qquad [K_3, K_\pm] = \pm K_\pm\,; \qquad K_3 := L + 1/2\,.$$

Hence $\{K_\pm, K_3\}$ generate so(2,1)

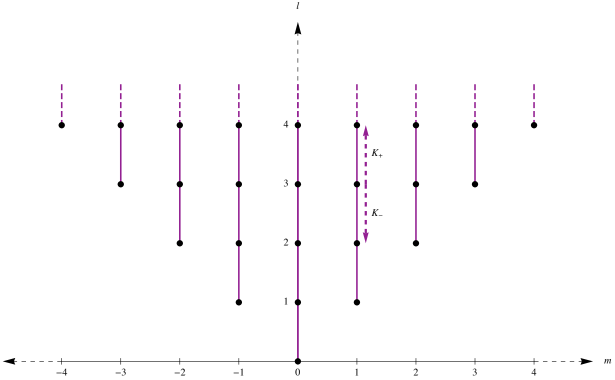
so(2, 1)–Casimir is related to m

$$\mathcal{C}_2\, T_l^m(x) = \left(K_3^2 - \frac{1}{2}\{K_+, K_-\}\right) T_l^m(x) = (m^2 - 1/4)\, T_l^m(x)$$

Using the operatorial form of $K_\pm$, gives again, up to the irrelevant global factor $(1 - X^2)$, the generalized Legendre equation:

$$\mathcal{C}_2 - (M^2 - 1/4) = (1 - X^2)\left((1 - X^2)D_x^2 - 2XD_x + L(L+1) - \frac{1}{1 - X^2}M^2\right) \equiv 0$$

$\{T_l^m(x)\}(m \text{ fixed})$ UIR– $so(2,1) \equiv \langle K_\pm, K_3 \rangle$



$T_l^m(x)$ (black points) in $so(2,1) \equiv \langle K_\pm, K_3 \rangle$ UIR (vertical lines)

$so_J(3)$

In a similar way we find a $so(3)$ algebra

$$[J_+, J_-] = 2J_3\,, \qquad [J_3, J_\pm = \pm J_\pm[\,; \qquad J_3 := M$$

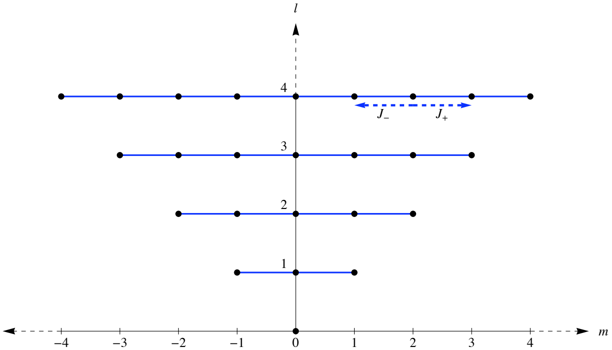
with Casimir

$$\mathcal{C}_2\, T_l^m(x) = \left(J_3^2 + \frac{1}{2}\{J_+, J_-\}\right) T_l^m(x) = l(l+1)\, T_l^m(x)$$

in the differential representation of the operators $J_\pm$, again

$$\mathcal{C}_2 - L(L+1) = -\left((1-X^2)D_x^2 - 2XD_x + L(L+1) - \frac{1}{1-X^2}M^2\right) \equiv 0$$

$T_l^m(x) so(3) \equiv \langle J_\pm, J_3 \rangle$ UIR



$T_l^m(x)$ (black points) in $so(3)$ UIR (horizontal lines)

R–operators

$$\boxed{R_\pm := [K_\pm, J_\pm]}$$

we have

$$R_\pm T_l{}^m(x) \quad = \quad \sqrt{(l+m+1\pm 1)(l+m\pm 1)}\, T_{l\pm 1}{}^{m\pm 1}(x),$$

and

$$R_\pm \quad = \quad -X\sqrt{1-X^2}D_x - \frac{1}{\sqrt{1-X^2}}M - \sqrt{1-X^2}(L+1/2\pm 1/2)$$

$R_\pm$ and $R_3 := L + M + 1/2$ span an other $so(2,1)$ algebra

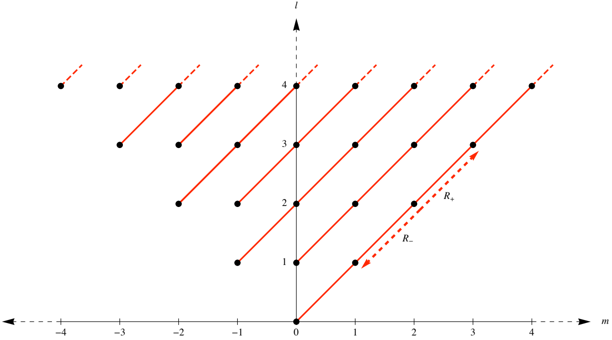
$$[R_+, R_-] = -4R_3, \qquad [R_3, R_\pm] = \pm 2R_\pm.$$

with Casimir equation

$$R_3^2 - \frac{1}{2}\{R_+, R_-\} + \frac{3}{4} \equiv 0$$

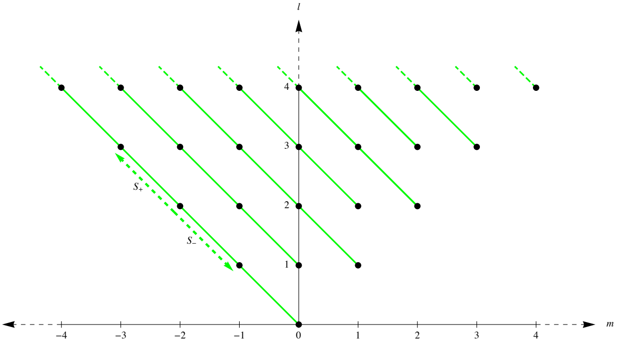
From differential form of $R_\pm$ again we recover the gen. Legendre eq.

$T_l^m(x) so(2,1) \equiv \langle R_\pm, R_3 \rangle$ UIR



$T_l^m(x)$ (black points) in $so(2,1) \equiv \langle R_\pm, R_3 \rangle$ UIR (inclined lines)

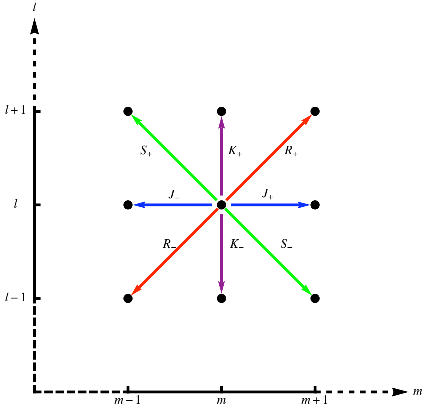
$T_l^m(x) so(2,1) \equiv \langle S_\pm, S_3 \rangle$ UIR



$T_l^m(x)$ (black points) in $so(2,1) \equiv \langle S_\pm, S_3 \rangle$ UIR (inclined lines)

$$\boxed{S_\pm := [K_\pm, J_\mp]} \qquad S_\pm T_l^m(x) = \sqrt{(l-m+1\mp 1)(l-m\pm 1)}\, T_{l\pm 1}^{m\mp 1}(x),$$

$T_l^m(x)$ B_2 Root system



Root system of B_2 (2-dim Cartan subalgebra at the origin ($\langle L, M \rangle$))

so(3,2)

From commutation relations of $K_\pm$, $L_\pm$, $R_\pm$, $S_\pm$, L and M the Lie algebra

$$\boxed{so(3,2)}$$

(real form of B_2), is obtained.

Associated Legendre equation is related also to the Casimir of $so(3,2)$

$$C_2^{so(3,2)} + 5/4 \equiv X^2 \left((1 - X^2) D_x^2 - 2XD_x + L(L+1) - \frac{1}{1 - X^2} M^2 \right) = 0$$

SPHERICAL HARMONICS

(well known representations of $SO(3)$ for l fixed)

$$Y_l^m(\theta, \phi) = \frac{1}{\sqrt{2\pi}} \sqrt{\frac{(l-m)!}{(l+m)!}} e^{im\phi} P_l^m(x) = \frac{1}{\sqrt{2\pi}} e^{im\phi} T_l^m(x)\,, \quad x = \cos\theta$$

(ϕ, m) pair of conjugate variables

$$\boxed{(K_\pm, J_\pm, R_\pm, S_\pm) \longrightarrow (K'_\pm = K_\pm, J'_\pm = e^{\pm i\phi} J_\pm, R'_\pm = e^{\pm i\phi} R_\pm, S'_\pm = e^{\mp i\phi} S_\pm)}$$

Bases related to the spherical harmonics

The action of the operators of the Cartan subalgebra of the Lie algebra $so(3,2)$, (rank 2, a noncompact real form of B_2), L and M, on SH is

$$\begin{aligned} L\,Y_l^m(\theta,\phi) &= l\,Y_l^m(\theta,\phi)\,, \\ M\,Y_l^m(\theta,\phi) &= m\,Y_l^m(\theta,\phi)\,. \end{aligned}$$

Abstract Hilbert space $\mathcal{H}$

We also consider an abstract Hilbert space $\mathcal{H}$ support of an equivalent UIR of $SO(3,2)$ of Casimir $\mathcal{C}_{so(3,2)} = -5/4$

Let $\widetilde{L}, \widetilde{M}$ be the elements of the Cartan subalgebra of $so(3,2)$ in this repr.

$$\widetilde{L}\,|l,m\rangle = l\,|l,m\rangle\,, \quad \widetilde{M}\,|l,m\rangle = m\,|l,m\rangle\,, \qquad \begin{cases} l = 0,1,2,\ldots \\ -l \leq m \leq l \in \mathbb{Z} \end{cases}$$

The set of eigenvectors $\{|l,m\rangle\}$ is a discrete basis for $\mathcal{H}$

Discrete basis $\{|l, m\rangle\}$

The eigenvectors $|l, m\rangle$ fulfill the properties of orthogonality and completeness

$$\langle l, m|l', m'\rangle = \delta_{l,l'}\,\delta_{m,m'}\,, \qquad \boxed{\sum_{l=0}^{\infty}\sum_{m=-l}^{l} |l, m\rangle\langle l, m| = \mathcal{I}} \ \text{identity on } \mathcal{H}$$

Any $|f\rangle \in \mathcal{H}$ may be written as

$$|f\rangle = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} f_{l,m}\,|l, m\rangle \qquad \text{if and only if} \qquad \sum_{l=0}^{\infty}\sum_{m=-l}^{l} |f_{l,m}|^2 < \infty\,.$$

With

$$\boxed{f_{l,m} = \langle l, m|f\rangle}$$

So that

$$|f\rangle = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} |l, m\rangle\langle l, m|f\rangle$$

$\mathcal{H} \to L^2(S^2, d\Omega)$

We may easily establish a canonical injection S

$$\begin{array}{ccc} \mathcal{H} & \stackrel{S}{\longmapsto} & L^2(S^2, d\Omega) \\ |l, m\rangle & \longmapsto & S\,|l, m\rangle = \sqrt{l + 1/2}\; Y_l^m(\theta, \phi) \end{array}$$

and extended by linearity and continuity to the whole $\mathcal{H}$.

S is a unitary mapping

$$S\,|l, m\rangle = \sqrt{l + 1/2}\;\; Y_l^m(\theta, \phi) \quad \Rightarrow \quad S^{-1}\,\left(Y_l^m(\theta, \phi)\right) = |l, m\rangle / \sqrt{l + 1/2}$$

and we relate the operators L, M on $L^2(S^2, d\Omega)$ and $\widetilde{L}, \widetilde{M}$ on $\mathcal{H}$ by

$$\boxed{\widetilde{L} = S^{-1}\, L\, S\,, \qquad \widetilde{M} = S^{-1}\, M\, S}$$

For any $|f\rangle = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} f_{l,m}\,|l, m\rangle \in \mathcal{H}$

$$S|f\rangle = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} f_{l,m}\, S\,|l, m\rangle = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} f_{l,m}\, \sqrt{l + 1/2}\; Y_l^m(\theta, \phi)$$

Continuous basis, $\{|\theta,\phi\rangle\}$

We introduce a continuous basis, $\{|\theta,\phi\rangle\}$, depending on the values of the angles θ and ϕ with the help of the discrete basis $\{|l,m\rangle\}$ by

$$\boxed{\langle\theta,\phi|l,m\rangle := \sqrt{l+1/2}\, Y_l^m(\theta,\phi)} \qquad 0\le\theta\le\pi\,, 0\le\phi<0$$

This def. may be extended to any $|f\rangle\in\mathfrak{G}$, where $\mathfrak{G}$ is a dense subspace in $\mathcal{H}$

$$\boxed{\langle\theta,\phi|f\rangle = f(\theta,\phi)} := S\,|f\rangle = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} f_{l,m}\,\sqrt{l+1/2}\,Y_l^m(\theta,\phi)\,,$$

recovering the expansion of $f(\theta,\phi)$

$\langle\theta,\phi|$ cannot be defined as a continuous mapping on $\mathcal{H}$, although QM textbooks give the first identity, which for any $|f\rangle\in\mathcal{H}$ is indeed merely formal.

We shall define in a proper way $\langle\theta,\phi|$ as a linear continuous functional over a space $\mathfrak{G}$, with the property that the triplet $\boxed{\mathfrak{G}\subset\mathcal{H}\subset\mathfrak{G}^\times}$ be a RHS.

Here, $|\theta,\phi\rangle\in\mathfrak{G}^\times$ and $\langle f|\theta,\phi\rangle := \langle\theta,\phi|f\rangle^*$

Some standard formulas

that will have proper meaning when we define the space $\mathfrak{G} \subset \mathcal{H}$

For any $|f\rangle, |g\rangle \in \mathfrak{G}$ and since $S\,|f\rangle, S\,|g\rangle \in L^2(S^2, d\Omega)$ (S unitary) we have

$$\langle f|g\rangle = \langle Sf|Sg\rangle = \int_{S^2} d\Omega\, f^*(\theta,\phi)\, g(\theta,\phi) = \int_{S^2} d\Omega\, \langle f|\theta,\phi\rangle\, \langle\theta,\phi|g\rangle\,.$$

We may derive the completeness relation (now $\mathbb{I}$ is a formal identity) given by

$$\boxed{\mathbb{I} = \int_{S^2} d\Omega\, |\theta,\phi\rangle\langle\theta,\phi| = \int_0^{2\pi} d\phi \int_0^{\pi} d(\cos\theta)\, |\theta,\phi\rangle\langle\theta,\phi|}$$

Applying $\mathbb{I}$ on $|f\rangle \in \mathfrak{G}$ we find a well known formal formula for $\mathbb{I}$:

$$\mathbb{I}\,|f\rangle = |f\rangle = \int_{S^2} d\Omega\, \langle\theta,\phi|f\rangle\, |\theta,\phi\rangle \;= \boxed{\int_{S^2} d\Omega\, f(\theta,\phi)\, |\theta,\phi\rangle}$$

For given θ and ϕ almost elsewhere with respect to the Lebesgue measure, $f(\theta,\phi)$ is the coef. of $|\theta,\phi\rangle$ in the span of $|f\rangle \in \mathfrak{G}$ in terms of $\{|\theta,\phi\rangle\}$.

From $f(\theta,\phi) = \langle\theta,\phi|f\rangle$ we may derive the well known orthogonality relation

$$\boxed{\langle\theta,\phi\,|\,\theta',\phi'\rangle = \delta\,(\cos\theta - \cos\theta')\;\delta(\phi-\phi')}$$

Also we get that

$$|l,m\rangle = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta\, \sqrt{l+1/2}\, Y_l^m(\theta,\phi)\, |\theta,\phi\rangle$$

The *discrete* basis $\{|l,m\rangle\}$ in terms of the *continuous* basis $\{|\theta,\phi\rangle\}$

From the identity $\mathcal{I}$ on $\mathcal{H}$ and $\langle\theta,\phi|f\rangle = \langle\theta,\phi|\mathcal{I}|f\rangle$, we have

$$f(\theta,\phi) = \langle\theta,\phi|f\rangle = \langle\theta,\phi|\mathcal{I}|f\rangle = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \langle\theta,\phi|l,m\rangle\langle l,m|f\rangle$$

For any pair of vectors $|f\rangle, |g\rangle \in \mathfrak{G}$ and using the previous relations we get

$$\langle f|g\rangle = \int_{S^2} d\Omega\, f^*(\theta,\phi)\, g(\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} f^*_{l,m}\, g_{l,m}$$

$$||f||^2 = \int_0^{2\pi} d\phi \int_0^{\pi} d(\cos\theta)\, |f(\theta,\phi)|^2 = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} |f_{l,m}|^2 .$$

Presentation of the continuous/discrete bases as in the standard QM .

Meaningful operators for spherical harmonics

As discrete basis, we have used a complete orthonormal set of eigenvectors of the Cartan operators $\widetilde{L}$ and $\widetilde{M}$: $\{|l,m\rangle\}$.

This opens the door for a discussion on other relevant operators acting on $\mathcal{H}$ or, equivalently through the unitary mapping S, on $L^2(S^2, d\Omega)$

Let $\widetilde{O}$ be an oper. densely defined on $\mathcal{H}$, which may be bounded or not

$$\boxed{O := S\widetilde{O}S^{-1}}, \qquad S : \mathcal{H} \longmapsto L^2(S^2, d\Omega)$$

acts on $L^2(S^2, d\Omega)$ and shares the properties with $\widetilde{O}$ Since $Sf = \langle\theta,\phi|f\rangle$

$$S\widetilde{O}|l,m\rangle = \langle\theta,\phi|\widetilde{O}|l,m\rangle\,.$$

Then, provided that $|l,m\rangle$ lies in the domain of $\widetilde{O}$

$$S\widetilde{O}|l,m\rangle = S\widetilde{O}S^{-1}|l,m\rangle = \sqrt{l+1/2}\,O\,Y_l^m(\theta,\phi)\,.$$

Hence

$$\boxed{O\,Y_l^m(\theta,\phi) = \frac{1}{\sqrt{l+1/2}}\,\langle\theta,\phi|\widetilde{O}|l,m\rangle}$$

Operators on $L^2(S^2, d\Omega)$: $\Theta, \Phi, L, M, D_\Theta, D_\Phi$

determine operators on $\mathcal{H}$ via S (i.e., $\widetilde{O} = \mathcal{S}^{-1} O \mathcal{S}$):

- Θ and Φ: multiplication operators by θ and ϕ, respectively.

  As θ and ϕ are angular variables, both operators are bounded and, therefore, defined on the whole $L^2(S^2, d\Omega)$.

- L and M (the multiplication oper. by l and m, respect.) are unbounded
- $D_\Theta := d/d\theta$ and $D_\Phi := d/d\phi$ are also unbounded
- We denote the corresponding oper. on $\mathcal{H}$ as $\widetilde{\Theta}$, $\widetilde{\Phi}$, $\widetilde{L}$, $\widetilde{M}$, $\widetilde{D}_\Theta$ and $\widetilde{D}_\Phi$,

They are not all independent. In fact after the definition of the SH

$$-i\, D_\Phi \equiv M \quad \Longrightarrow \quad -i\, \widetilde{D}_\Phi \equiv \widetilde{M}$$

The eq. defining the SH can be formally expressed in terms of the above oper.

$$\left(D_\Theta^2 + \cot\Theta\ D_\Theta + \frac{1}{\sin^2\Theta}\, D_\Phi^2 + L(L+1) \right)\ Y_l^m(\theta, \phi) = 0\,,$$

- Θ and Φ have purely absolutely continuous spectrum, which is $[0,\pi]$ in the first case and $[0,2\pi]$ in the second one.
  - They commute, so that they must have a basis of simultaneous eigenvectors: $\{|\theta,\phi\rangle\}$.
  - These eigenvectors do not belong to the Hilbert space $L^2(S^2, d\Omega)$, although they are properly defined on a larger structure: a RHS
- The spectrum of both L and M is discrete and degenerate.
  - L, M commute and their simultaneous eigenvector basis is $Y_{l,m}(\theta,\phi)$.
  - In this case, $Y_{l,m}(\theta,\phi)$ form an orthonormal basis (complete orthonormal set) on $L^2(S^2, d\Omega)$, the image by S of the orthonormal basis $\{|l,m\rangle\}$ in $\mathcal{H}$.
- Thus, we are playing with discrete and continuous basis, a game that leads us out of the Hilbert space.
- The situation is saved using RHS

RHS formulation

Now, let us consider the space $\mathfrak{D}$ of all funct. $f(\theta,\phi)$ in $L^2(S^2,d\Omega)$ that fulfill

$$\left|\left|f(\theta,\phi)\right|\right|_n^2 := \sum_{l=0}^{\infty}\sum_{m=-l}^{l} (l+|m|+1)^{2n}\cdot\left|f_{l,m}\right|^2 < \infty\,, \qquad n = 0,1,2,\ldots$$

All the finite linear combinations of SH are in $\mathfrak{D}$, so that this space is dense in $L^2(S^2,d\Omega)$.

Thus, we have defined a family of norms $||-||_n$ on $\mathfrak{D}$, which gives a topology s.t. $\mathfrak{D}$ is a Frèchet space (metrizable and complete).

Since for $n=0$ we have the Hilbert space norm, the canonical injection from $\mathfrak{D}$ into $L^2(S^2,d\Omega)$ is continuous.

Denoting by $\mathfrak{D}^\times$ the dual space of $\mathfrak{D}$, we have constructed the following RHS

$$\mathfrak{D} \subset L^2(S^2,d\Omega) \subset \mathfrak{D}^\times$$

Continuity of generators of $so(3,2)$ on $\mathfrak{D} \subset L^2(S^2, d\Omega) \subset \mathfrak{D}^\times$

$$
\begin{aligned}
J_\pm \, Y_l^m(\theta,\phi) \quad &: \quad = \sqrt{(l \mp m)(l \pm m + 1)} \, Y_l^{m\pm 1}(\theta,\phi)\,, \\
K_+ \, Y_l^m(\theta,\phi) \quad &:= \quad \sqrt{(l+1)^2 - m^2} \, Y_{l+1}^m(\theta,\phi)\,, \\
K_- \, Y_l^m(\theta,\phi) \quad &:= \quad \sqrt{l^2 - m^2} \, Y_{l-1}^m(\theta,\phi)\,, \\
R_+ \, Y_l^m(\theta,\phi) \quad &:= \quad \sqrt{(l+m+2)(l+m+1)} \, Y_{l+1}^{m+1}(\theta,\phi)\,, \\
R_- \, Y_l^m(\theta,\phi) \quad &:= \quad \sqrt{(l+m)(l+m-1)} \, Y_{l-1}^{m-1}(\theta,\phi)\,, \\
S_+ \, Y_l^m(\theta,\phi) \quad &:= \quad \sqrt{(l-m+2)(l-m+1)} \, Y_{l+1}^{m-1}(\theta,\phi)\,, \\
S_- \, Y_l^m(\theta,\phi) \quad &:= \quad \sqrt{(l-m)(l-m-1)} \, Y_{l-1}^{m+1}(\theta,\phi)\,, \\
L\,(M) \, Y_l^m(\theta,\phi) \quad &:= \quad l\,(m) \, Y_l^m(\theta,\phi)\,.
\end{aligned}
$$

Since l goes from 0 to ∞, these oper. are all unbounded. So their respective domains are densely defined on $L^2(S^2, d\Omega)$, but not on the whole space.

These oper. are defined on the whole $\mathfrak{D}$ and are continuous with the topology on $\mathfrak{D}$

The proof is simple and it is essentially the same with all these operators

Proof for K_+ : (1) $\mathfrak{D}$ is invariant under K_+

$\forall\, f = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} f_{l,m}\,\sqrt{l+1/2}\;Y_l^m(\theta,\phi) \in \mathfrak{D}$ we have that

$$K_+\,f(\theta,\phi) = \sum_{l,m} f_{l,m}\,\sqrt{l+1/2}\,\sqrt{(l+1)^2-m^2}\;Y_{l+1}^m(\theta,\phi)$$

To show that this vector is in $\mathfrak{D}$, we have to prove that for any $n = 0,1,2,\ldots,$

$$||K_+\,f(\theta,\phi)||_n^2 = \sum |f_{l,m}|^2\,((l+1)^2-m^2)\,(l+1+|m|+1)^{2n} < \infty$$

Since

$$\begin{aligned} (l+1+|m|+1)^{2n} &\leq 2^{2n}\,(l+|m|+1)^{2n} \\ (l+1)^2-m^2 &\leq (l+|m|+1)^2 \end{aligned}$$

$$||K_+\,f(\theta,\phi)||_n^2 \leq 2^{2n}\sum |f_{l,m}|^2\,(l+|m|+1)^{2n+2} = 2^{2n}\,||f(\theta,\phi)||_{n+1}^2$$

which obviously converges. Therefore $\boxed{K_+\mathfrak{D} \subset \mathfrak{D}}$

(2) Continuity of K_+ on $\mathfrak{D}$ $\quad ||K_+ f(\theta,\phi)||_n \leq 2^n\,||f(\theta,\phi)||_{n+1}$,
which satisfies the continuity condition for all $n = 0,1,2,\ldots$.

Using the duality formula ($\langle K_+ f|F\rangle = \langle f|K_+^\times\,F\rangle$, $\forall\, f \in \mathfrak{D}$, $\forall\, F \in \mathfrak{D}^\times$)
we extend K_+ to a weakly continuous oper. on $\mathfrak{D}^\times$

Same properties can be proven for the other generators of $so(3,2)$, included the generators of the Cartan subalgebra, L, M that are also unbounded.

All of them can be extended to weakly continuous oper. on $\mathfrak{D}^\times$ by means of the duality formula

It is straightforward to show that all elements of the UEA are also continuous oper. on $\mathfrak{D}$ as well as on the antidual $\mathfrak{D}^\times$

In addition, the same RHS serves as a support for a repr. of Lie algebra as continuous oper. on it and the same for its UEA

RHS: $\mathfrak{G} \subset \mathcal{H} \subset \mathfrak{G}^\times$

The main tool to define the RHS $\mathfrak{G} \subset \mathcal{H} \subset \mathfrak{G}^\times$ is the unitary mapping S

$$\boxed{\mathfrak{G} := S^{-1}\mathfrak{D}}$$

hence the topology on $\mathfrak{G}$ is the transported topology from $\mathfrak{D}$ by S.
So that if $|f\rangle \in \mathfrak{G}$, the semi-norms are

$$\left|\left||f\rangle\right|\right|_n^2 = \sum_{l,m} (l + |m| + 1)^{2n} \left|f_{l,m}\right|^2 < \infty\,, \qquad n = 0, 1, 2, \ldots\,.$$

The topology on $\mathfrak{G}$ uniquely defines $\mathfrak{G}^\times$.

There exists a one-to-one continuous mapping $S^\times \equiv S$ from $\mathfrak{G}^\times$ onto $\mathfrak{D}^\times$ with continuous inverse, which is given by an extension of S defined via duality

$$\langle Sf|S^\times F\rangle = \langle f|F\rangle\,, \qquad |f\rangle \in \mathfrak{G},\ F \in \mathfrak{G}^\times\,,$$

$$\begin{array}{ccccc} \mathfrak{D} & \subset & L^2(S^2, d\Omega) & \subset & \mathfrak{D}^\times \\ S^{-1}\downarrow & & S^{-1}\downarrow & & S^{-1}\downarrow \\ \mathfrak{G} & \subset & \mathcal{H} & \subset & \mathfrak{G}^\times \end{array}\,.$$

If O satisfies $O\mathfrak{D} \subset \mathfrak{D}$ with continuity, same property works for $\widetilde{O} = S^{-1}\,O\,S$

Continuity of other relevant operators

The relevant oper. for the discussion on SH are:

- Φ: is a bounded oper. on $L^2(S, d\Omega)$.
- Θ: is again a bounded oper. on $L^2(S, d\Omega)$.
  - It is relevant only through its funct. $\sin\Theta$ and $\cos\Theta$, which are also well defined bounded oper. on $L^2(S, d\Omega)$.
  - $\sin\Theta$ and $\cos\Theta$ are continuous oper. on $\mathfrak{D}$
  - $\sin^{-1}\Theta$ is also continuous oper. on $\mathfrak{D}$
    This may be a little surprising since $\sin^{-1}\theta$ is not bounded in $[0, 2\pi]$.
    But we are discussing the continuity of oper. on a topology different albeit stronger than the Hilbert space topology, on which $\sin^{-1}\Theta$ is not continuous.
- $-iD_\Phi \equiv M$ and $-iD_\Theta$ are not bounded on $L^2(S, d\Omega)$.
  But they are continuous on $\mathfrak{D}$

The functional $\langle\theta,\phi|$ on $\mathfrak{G}$: Properties

It is defined for almost all θ and ϕ with $0 \le \theta < \pi$ and $0 \le \phi < 2\pi$ as a functional on all $|f\rangle \in \mathfrak{G}$ as

$$\boxed{\langle\theta,\phi|f\rangle := f(\theta,\phi) = S|f\rangle}$$

Let us fix arbitrary $|f\rangle \in \mathfrak{G}$, θ and ϕ and we can write

$$\langle\theta,\phi|f\rangle = f(\theta,\phi) = \begin{cases} \displaystyle\sum_{l=0}^{\infty}\sum_{m=-l}^{l} \langle\theta,\phi|l,m\rangle\langle l,m|f\rangle \\ \displaystyle\sum_{l=0}^{\infty}\sum_{m=-l}^{l} f_{l,m}\sqrt{l+1/2}\,Y_l^m(\theta,\phi) \end{cases}$$

$f_{l,m} = \langle l,m|f\rangle$: coef. of the span of $|f\rangle$ in terms of the discrete basis $\{|l,m\rangle\}$.

It defines a linear continuous mapping

$$\begin{array}{cccc} \langle\theta,\phi| : & \mathfrak{G} & \longmapsto & \mathbb{C} \\ & |f\rangle & \longmapsto & \langle\theta,\phi|f\rangle = f(\theta,\phi) \end{array}$$

Since $\langle f|\theta,\phi\rangle = \langle\theta,\phi|f\rangle^*$, it follows that
$|\theta,\phi\rangle$ is a continuous anti-linear functional on $\mathfrak{G}$: $|\theta,\phi\rangle \in \mathfrak{G}^\times$.

Proof.- The linearity is obvious. The proof for the continuity relies on a version of$||Af||_n \leq K_n \sum_{i=1}^{r} ||f||_{p_i}$ valid for linear/antilinear mappings from $\mathfrak{G}$ into $\mathbb{C}$

$$\left|F(f)\right| \leq C\left\{\left|\left|f\right|\right|_{p_1} + \cdots + \left|\left|f\right|\right|_{p_k}\right\}, \quad C > 0, \ \forall |f\rangle \in \mathfrak{G}$$

Let us go back to $\langle\theta, \phi|f\rangle$ and let us write it as

$$\langle\theta, \phi|f\rangle = \sum f_{l,m} \, (l + |m| + 1)^p \left(\frac{\sqrt{l + 1/2}}{(l + |m| + 1)^p} \, Y_l^m(\theta, \phi) \right),$$

where $p \in \mathbb{N}$ s.t. $p \geq 1$. Using the Schwarz inequality

$$|\langle\theta, \phi|f\rangle| \leq \sqrt{\sum \left|f_{l,m}\right|^2 (l + |m| + 1)^{2p}} \times \sqrt{\sum \frac{l + 1/2}{(l + |m| + 1)^{2p}} \left|Y_l^m(\theta, \phi)\right|^2}$$

The first series converges because $|f\rangle \in \mathfrak{G}$ and it is not more than $\left|\left||f\rangle\right|\right|_p$. The 2nd series also converges for $p \geq 2$: it is enough to take into account

$$\left|Y_l^m(\theta, \phi)\right| \leq \frac{1}{\sqrt{2\pi}}, \quad \forall \theta, \phi; \qquad l + |m| + 1 \geq l + 1$$

We conclude that for any $|f\rangle \in \mathfrak{G}$ and any fixed $p = 2, 3, 4, \ldots$, one has

$$\left|\langle\theta, \phi|f\rangle\right| \leq C_p \left|\left|f\right|\right|_p,$$

This proves the continuity of the linear functional $\langle\theta, \phi|$ on $\mathfrak{G}$ for fixed θ and ϕ.

Properties of $|\theta, \phi\rangle \in \mathfrak{G}^\times$

Let us remember that :

- $S : \mathcal{H} \to L^2(S^2, d\Omega)$ is a unitary mapping with the property that $S : \mathfrak{G} \to \mathfrak{D}$ continuously and with continuous inverse,
- the duality formula $\langle F|f\rangle = \langle SF|Sf\rangle$ defines, for all $|f\rangle \in \mathfrak{G}$ and $F \in \mathfrak{G}^\times$, an extension of S, $S : \mathfrak{G}^\times \to \mathfrak{D}^\times$, continuous with continuous inverse.
- Therefore, $S|\theta, \phi\rangle$ exists and is in $\mathfrak{D}^\times$.

Then, we define

$$\boxed{\widehat{\cos\Theta} := S^{-1} \cos\Theta\, S}$$

which by construction is a continuous oper. from $\mathfrak{G}$ into itself. Thus,

$$\cos\Theta\, S|f\rangle = \cos\Theta\, f(\theta,\phi) = \cos\theta\, f(\theta,\phi) = g(\theta,\phi) \in \mathfrak{D}\,.$$

$$\begin{aligned}\langle\theta,\phi|\widehat{\cos\Theta}|f\rangle &= \langle\theta,\phi|S^{-1}\cos\Theta S|f\rangle = \langle\theta,\phi|S^{-1}\, g(\theta,\phi)\rangle = g(\theta,\phi)\\ &= \cos\theta\, f(\theta,\phi) = \cos\theta\langle\theta,\phi|f\rangle\,.\end{aligned}$$

Since any continuous oper, on $\mathfrak{G}$ may be extended to a weakly continuous oper. on $\mathfrak{G}^\times$ using the duality formula, then, we conclude that

$$\boxed{\widehat{\cos\Theta}|\theta,\phi\rangle = \cos\theta\,|\theta,\phi\rangle} \qquad 0 \le \theta < \pi,\ 0 \le \phi < 2\pi$$

A similar result for the coordinate ϕ may be obtained.

From

$$Y_{l_1}^{m_1}(\theta,\phi)\, Y_{l_2}^{m_2}(\theta,\phi) = \frac{1}{\sqrt{2\pi}} \sum_{L,M} \langle l_1\, 0\, l_2\, 0 | L\, 0\rangle \, \langle l_1\, m_1\, l_2\, m_2 | L\, M\rangle \, Y_L^M(\theta,\phi)\,,$$

with $M = m_1 + m_2,\; L$ s.t. $|l_1 - l_2| \leq L \leq l_1 + l_2$ and $|M| \leq L$.
In particular, we have:

$$Y_1^1(\theta,\phi)\, f(\theta,\phi) = \frac{1}{\sqrt{2\pi}} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} f_{l,m} \sqrt{l+1/2} \sum_{L,M} c_{CG}\, Y_L^M(\theta,\phi)\,, \quad \forall f(\theta,\phi) \in \mathfrak{D}$$

An estimation : $\quad ||\, Y_1^1(\theta,\phi)\, f(\theta,\phi) ||_p^2 \leq ||\, f(\theta,\phi) ||_{p+1}^2$

Thus, the multiplication by $Y_1^1(\theta,\phi)$ is a linear continuous operator on $\mathfrak{D}$

Since $Y_1^1(\theta,\phi) = -e^{i\phi} \sin\theta/(2\sqrt{\pi})$ the oper. $\sin\Theta\, e^{i\Phi}$ is continuous on $\mathfrak{D}$
We have proven that $\sin^{-1}\Theta$ is also continuous on $\mathfrak{D}$, then, so is $e^{i\Phi}$. Hence

$$\boxed{\widehat{e^{i\Phi}} := S^{-1}\, e^{i\Phi}\, S} \quad \text{continuous on } \mathfrak{G}$$

We may extend this oper. to a weakly continuous linear mapping on $\mathfrak{G}^\times$, s.t.

$$\boxed{\widehat{e^{i\Phi}}\, |\theta,\phi\rangle = e^{i\phi}\, |\theta,\phi\rangle}$$

Conclusions

- RHS exhibit a strict relation with Lie algebras and UEAs, which is not shared by Hilbert spaces.
- RHS can indeed be constructed such that generators and elements of the UEA of Lie algebras can be represented as well behaved continuous operators where problems of domains, that characterize unbounded operators, do not apply.
- A fundamental property of RHS is indeed related to the fact that they support repr. of locally compact Lie groups
- In particular, when the repr. is irreducible, as in the cases here considered, the space of the oper. defined on the RHS is isomorphic to the UEA, so that all properties of the algebra can be transferred to its repr. on RHS, which acquire all the symmetries of the algebra.

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