Freezable quantum states or bound states in the continuum for time dependent potentials

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Confluent supersymmetry in quantum mechanics

Our starting point is the one-dimensional stationary Schrödinger equation. We can write it in the form

$$\Psi'' + (E - V_0)\,\Psi \;=\; 0.$$

In order to apply a n-th order confluent SUSY transformation, we first determine $n+1$ functions $u_0, u_1, ..., u_n$, that solve the following system of equations,

$$\begin{aligned} u_0'' + (\lambda - V_0)\,u_0 &= 0 \\ u_j'' + (\lambda - V_0)\,u_j &= -u_{j-1}, \quad j = 1, ..., n, \end{aligned}$$

we assume $\lambda \neq E$.

Confluent supersymmetry in quantum mechanics

Once the system has been solved, we take a solution Ψ of our initial Schrödinger equation and construct the following functions Φ_n, χ_n and $\chi_n^{\perp}$:

$$\Phi_n = \frac{W_{u_0,\ldots,u_{n-1},\Psi}}{W_{u_0,\ldots,u_{n-1}}}, \qquad \chi_n = \frac{W_{u_0,\ldots,u_n}}{W_{u_0,\ldots,u_{n-1}}}, \qquad \chi_n^{\perp} = \frac{W_{u_0,\ldots,u_{n-2}}}{W_{u_0,\ldots,u_{n-1}}},$$

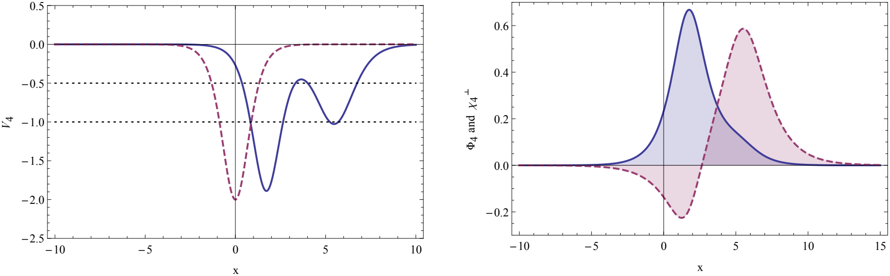
where the symbol W stands for the Wronskian of the functions in its index. Then, Φ_n, χ_n and $\chi_n^{\perp}$ are solutions to the following Schrödinger equations

$$\Phi_n'' + (E - V_n)\,\Phi_n = 0 \qquad\qquad \chi_n'' + (\lambda - V_n)\,\chi_n = 0.$$

The transformed potential V_n is given by the expression

$$V_n = V_0 - 2\,\frac{d^2}{dx^2}\,\log\left(W_{u_0,u_1,\ldots,u_{n-1}}\right).$$

Confluent supersymmetry in quantum mechanics



Left: Fourth-order SUSY partner of the Pöschl-Teller potential (blue). The Pöschl-Teller potential is plotted as reference (purple). Right: Eigenfunctions Φ_4 and $\chi_4^\perp$ corresponding to the potential on the left.

A Contreras-Astorga and A Schulze-Halberg, J. Phys. A: Math. Theor. 50 (2017)

On integral and differential representations supersymmetry algorithm

The system of equations,

$$\begin{aligned} u_0'' + (\lambda - V_0)\, u_0 &= 0 \\ u_j'' + (\lambda - V_0)\, u_j &= -u_{j-1}, \quad j = 1, ..., n, \end{aligned}$$

can be solved using two techniques: variation of constants (integration) and using parametric derivatives.
The first of these representations can be constructed by means of the variations-of-constants formula :

$$u_j = \hat{u} - u_0 \int^x \left(\int^t u_0\, u_{j-1}\, ds \right) \frac{1}{u_0^2}\, dt, \quad j = 1, ..., n-1,$$

where $\hat{u}$ stands for any solution of the first equation.

On integral and differential representations supersymmetry algorithm

An alternative representation for the transformation functions involves parametric derivatives with respect to λ. Assuming that u_0 is a function of the two variables x and λ, we have

$$u_j \;=\; \sum_{k=0}^{j-1} \frac{\partial \hat{u}_k}{\partial \lambda^k} + \frac{1}{j!}\frac{\partial u_0}{\partial \lambda^j}, \quad j = 1, ..., n-1,$$

where $\hat{u}_k$, $k = 0, ..., n-2$, stand for arbitrary solutions of the first equation of the system, including the trivial zero solution.
Note that the both representations are not equivalent, but related to each other.

Second order case

Let us call $u_0 = u_a$ and the second LI solution u_b. Moreover, $v_{VC} = u_1$ obtained by integration and $v_{DF} = u_1 = \partial_\lambda u_0$ obtained by differentiation.

Since both v_{VC} and v_{DF} are particular solutions of the nonhomogeneous equation, their difference must be a solution to the homogeneous equation. Equivalently, there are two parameters d_1 and d_2, such that the equation

$$d_1\ u_a + d_2\ u_b \ = \ v_{DF} - v_{VC},$$

is fulfilled. Differentiation with respect x yields

$$d_1\ (u_a)_x + d_2\ (u_b)_x \ = \ (v_{DF})_x - (v_{VC})_x.$$

Thus, d_1 and d_1 are given by:

$$d_1 \ = \ W_{v_{DF}-v_{VC},u_b}(x_0,\lambda), \qquad d_2 \ = \ W_{u_a,v_{DF}-v_{VC}}(x_0,\lambda).$$

Jordan chains of second order

Furthermore, the following interesting result was obtained

$$\int_{x_0}^{x} u^2(t,\lambda)dt = W_{u,u_\lambda}(x_0,\lambda) - W_{u,u_\lambda}(x,\lambda).$$

This last equation shows how to integrate the square of a function that is solution of a Schrödinger equation using a Wronskian and vice versa.

This identity is useful when we are interested, for example, in finding normalization constants, probabilities in an interval or integrals of special functions.

A Contreras-Astorga and A Schulze-Halberg, J. Phys. A: Math. Theor. 48 (2015)

The radial oscillator system

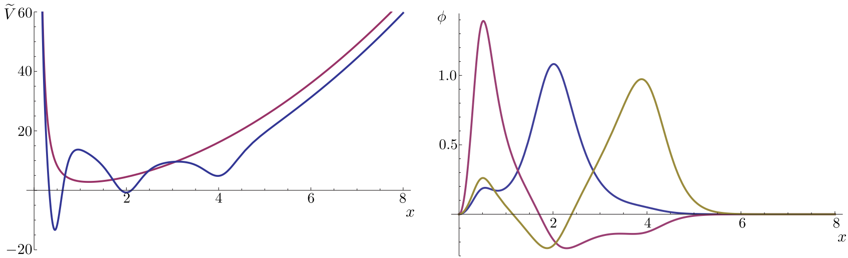


Figure: On the left, the radial oscillator potential (purple curve) and its second order SUSY partner $\widetilde{V}$ (blue curve). On the right its first three eigenfunctions.

Time-dependent systems

Consider a one dimension time independent Schrödinger equation in the spatial variable y as

$$\psi''(y) + \left(E - \widetilde{V}_0\right)\psi(y) = 0$$

where $\widetilde{V}_0 = \widetilde{V}_0(y)$ and a solution ψ are known. Now let us take arbitrary functions $A = A(t)$ and $B = B(t)$ and let the variable y be defined in terms t and x as:

$$y(x,t) = x\exp\left[4\int A(t)dt\right] + 2\int B(t)\exp\left[4\int A(t)dt\right]dt$$

then the function

$$\phi(x,t) = \psi \exp\left\{-i\left[Ax^2 + Bx + E\int \exp\left[8\int A dt\right] dt + \int \left[2iA + B^2\right] dt\right]\right\},$$

is solution of the equation

$$i\frac{\partial}{\partial t}\phi(x,t) + \frac{\partial^2}{\partial x^2}\phi(x,t) - V_0(x,t)\phi(x,t) = 0.$$

The last equation is a **time dependent Schrödinger equation** where the potential is given by

$$V_0(x,t) = \widetilde{V}_0 \exp\left[8\int A dt\right] + \left[\frac{d}{dt}A - 4A^2\right] x^2 + \left[\frac{d}{dt}B - 4AB\right] x.$$

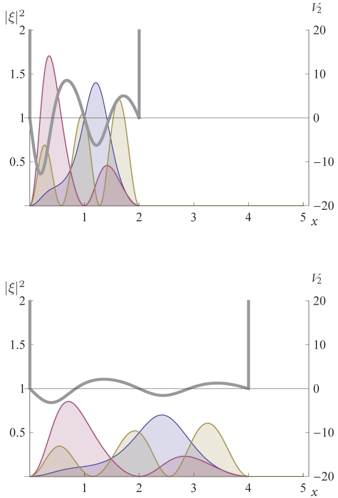


Figure: A confluent SUSY partner of the infinite square-well potential with a moving barrier, at $t = 1/4$, $t = 1/2$, $t = 3/4$, $t = 1$.

Photonic systems with 2D landscapes of complex refractive index

Consider a monochromatic light beam traveling in a medium with refractive index $n = n(X, Y, Z)$. The Maxwell equation are

$$\nabla \times \vec{E} = i\omega\mu_0\vec{H}, \quad \nabla \times \vec{H} = -i\omega\epsilon\vec{E}.$$

Let us write the electric field $\vec{E}$ as $\vec{E} = \exp(ikn_0 Z)\left(\vec{\psi}_T + \hat{a}_Z\psi_Z\right)$ where $k = 2\pi/\lambda$ is the wave number, n_0 is a reference value of the index of refraction. Introducing scaled variables $x = X/x_0$, $y = Y/x_0$, $z = Z/2\ell$, where x_0 size of the beam in the transverse and ℓ is the diffraction length. Then Maxwell eqs simplify to:

$$i\partial_z\vec{\psi}_T^{(0)} + \nabla_\perp^2\vec{\psi}_T^{(0)} + 2n_0k^2x_0^2\delta n\vec{\psi}_T^{(0)} = 0.$$

Note, each component of $\vec{\psi}_T$ satisfies a Schrödinger equation.

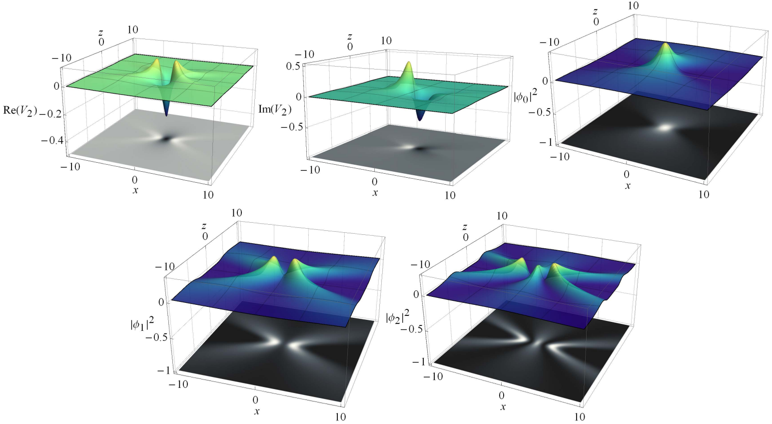
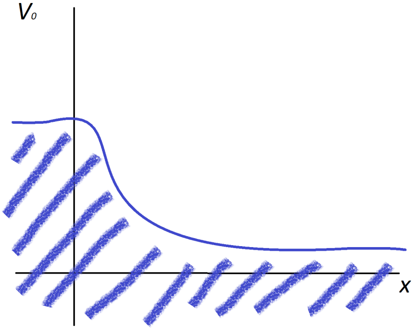


Figure: Localized defect in a homogenous crystal. Plots of the real (top left) and imaginary (top center) parts of V_2, where V_2 is a PT-symmetric potential. Moreover, the absolute value squared of three solutions are plotted: $|\phi_0|^2$ (top right), $|\phi_1|^2$ (bottom left) and $|\phi_2|^2$ (bottom right).

A Contreras-Astorga and V Jakubský, Phys Rev A (2019)

Bound states in the continuum

Given a potential V_0, expression of the confluent SUSY partner and the missing state are:



$$V_2 = V_0 - \left[\ln\left(\omega_0 + \int_{x_0}^{x} u_0^2 ds\right)\right]''$$

$$\psi_\epsilon = \frac{u_0}{\omega_0 + \int_{x_0}^{x} u_0^2 ds}$$

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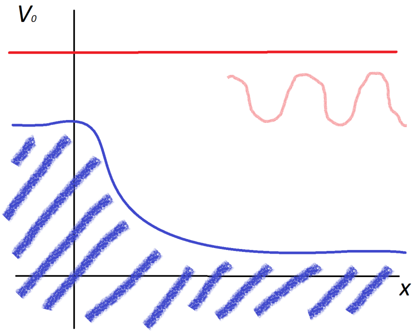

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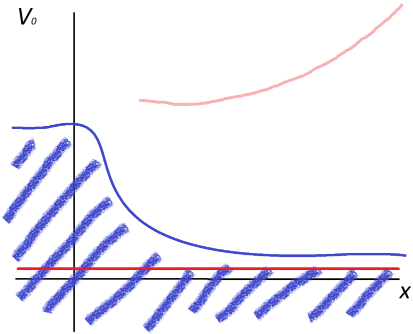

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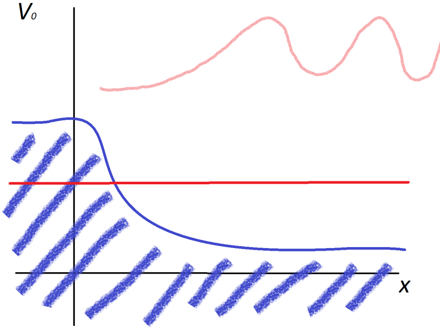


Freezable quantum states

Recipe

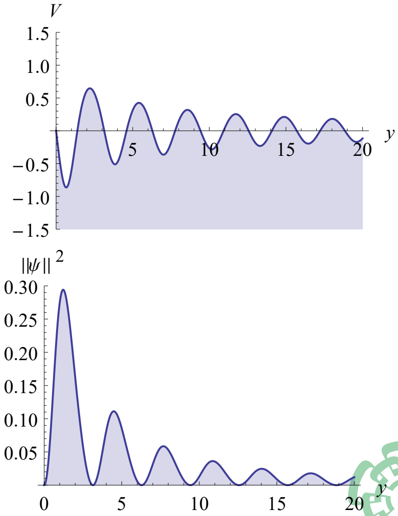
1. Create a time independent potential with a BIC using the confluent SUSY. Note, for more than one BIC you can use a higher-order SUSY.
2. Add dynamics with the change of variable: $y = x/t$. The result is a time dependent system, when $t \to -\infty$, $V(x,t) \to 0$.
3. When $t \in (-\infty, t_0)$, use $V(x,t)$ and a vector potential $\mathbf{A} = 0$.
4. At $t \in (t_0, \infty)$ you can freeze the evolving state (BIC?), just fix $V = V(x, t_0)$ and $\mathbf{A} = -\partial_x \theta \mathbf{e}_x$, where θ is the position dependent phase of the BIC $\phi(x, t_0)$.
5. The **Freezable State** will now be an eigenstate of $H = (-\partial_x + iA_x)^2 + V(x, t_0)$,

Example

Let us take the free particle in the interval $x \in (0, \infty)$.

We can generate a confluent SUSY parter with a transformation function $u_0 = \sin(kx)$.

The SUSY partner will have a BIC with eigenvalue $\epsilon = k^2$.



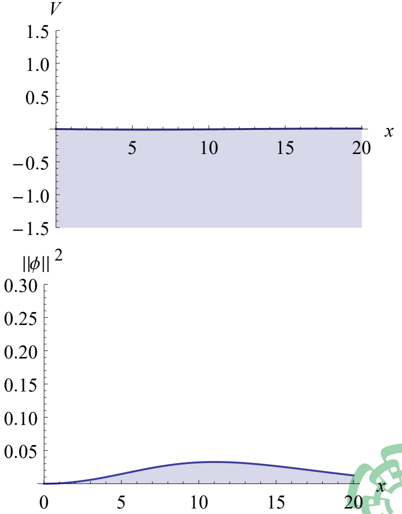
Example

We can add dynamics with the change $y = x/(4t+1)$.

The wave function now depends on x and t: $\psi_\epsilon(y) \to \phi(x,t)$, same happens to the potential.

The dynamic equation is $i\partial_t\phi + \partial_x^2\phi - V\phi = 0$.

$t = -2$



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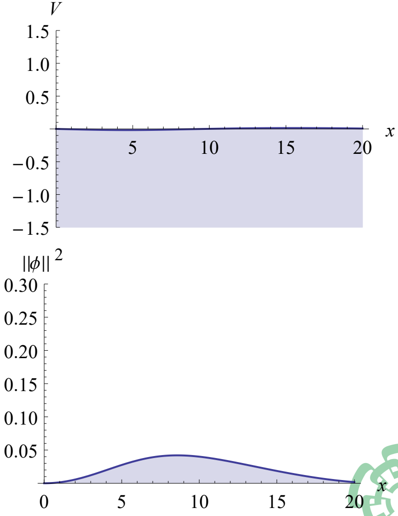
$t = -1.5$



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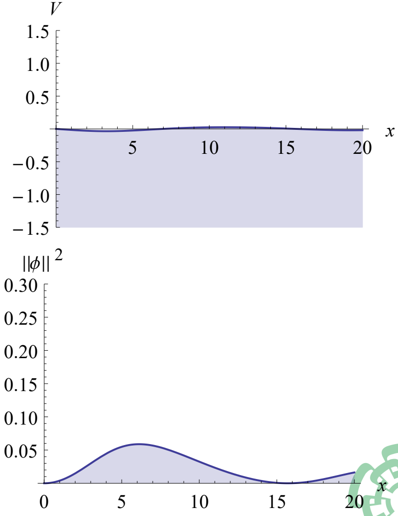
$t = -1$



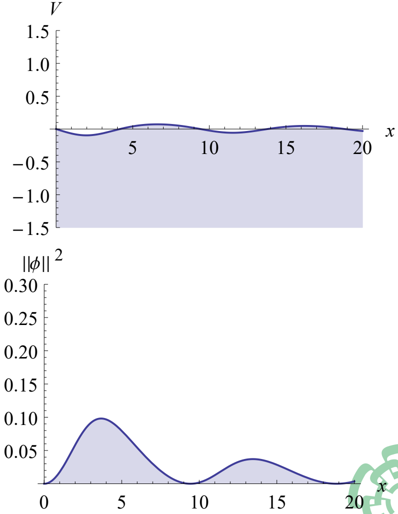
Example

We can add dynamics with the change $y = x/(4t+1)$.

The wave function now depends on x and t: $\psi_\epsilon(y) \to \phi(x,t)$, same happens to the potential.

The dynamic equation is $i\partial_t\phi + \partial_x^2\phi - V\phi = 0$.

$t = -0.5$

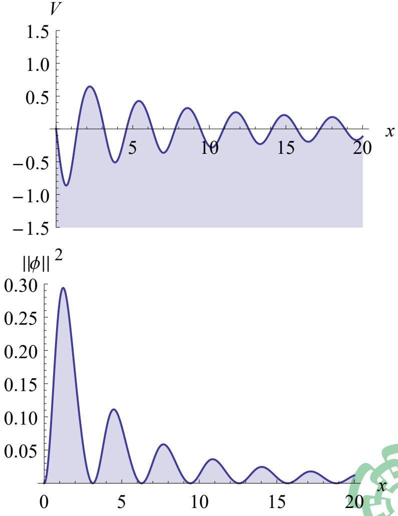


Example

We can add dynamics with the change $y = x/(4t+1)$.

The wave function now depends on x and t: $\psi_\epsilon(y) \to \phi(x,t)$, same happens to the potential.

The dynamic equation is $i\partial_t\phi + \partial_x^2\phi - V\phi = 0$.



$t = 0$

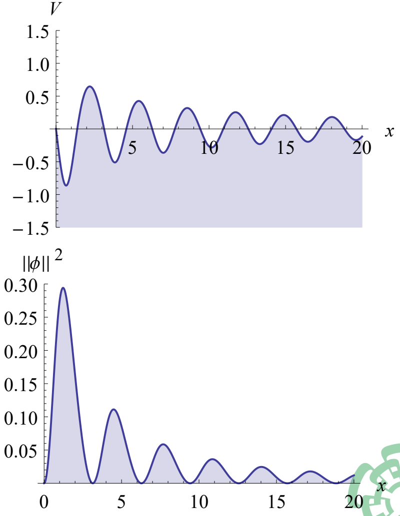
Example

We decide to freeze the state ϕ at $t \geq 0$.

The potential now is fixed, $V = V(x, 0)$. Also we add vector potential $\mathbf{A} = -\partial_x \theta \mathbf{e}_x$, where θ is the position dependent phase of the BIC $\phi(x, t_0)$.

The **Freezable State** will now be an eigenstate of $H = (-\partial_x + iA_x)^2 + V(x, t_0)$.

$t \geq 0$



Conclusions

In this work, we constructed time dependent potentials via supersymmetric quantum mechanics. The generated potentials

have a quantum state with the property that after a certain time t_0, when the potential does not longer change, the evolving state becomes a bound state in the continuum, its probability distribution freezes.