Exactly solvable time-dependent potentials generated by Darboux and point transformations

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Stationary Systems and Factorization Method

Factorization method

- Initial system

$$H=-\frac{d^2}{dx^2}+V(x)\,,\quad H|\phi_n\rangle=E_n|\phi_n\rangle\,,\quad E_{n+1}^{(\lambda_M)}=E_n\,,\quad E_0^{(\lambda_M)}=\epsilon\,,\quad n=0,1,\ldots\,.$$

$$H=A^{\dagger}A+\epsilon\,,\quad A=-\frac{d}{dx}+\beta_M(x)\,,$$

$$\epsilon\leq E_0\,,\quad \beta:\mathbb{R}\to\mathbb{R}\,.$$

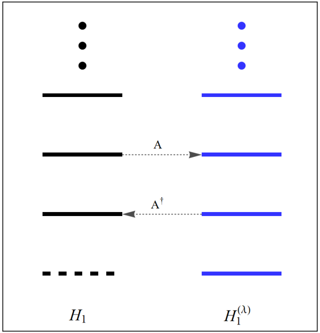
- Final system

$$H_{\lambda_M}=AA^{\dagger}+\epsilon\,,\quad H_{\lambda_M}|\phi_n^{\lambda_M}\rangle=E_n^{(\lambda_M)}|\phi_n^{\lambda_M}\rangle\,,$$

$$H_{\lambda_M}=-\frac{d^2}{dx^2}+V_{\lambda_M}(x)\,,\ V_{\lambda_M}(x)=V(x)+2\beta_M'$$

Spectral information

$$|\phi_{n+1}^{\lambda_M}\rangle=\frac{1}{\sqrt{E_n-\epsilon}}A|\phi_n\rangle\,,\quad A^{\dagger}|\phi_0^{\lambda_M}\rangle=0\,,$$



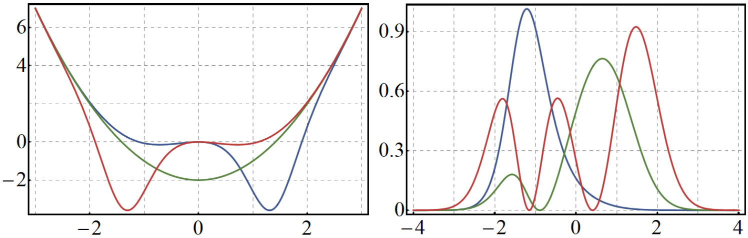
Example: Harmonic oscillator

It is well known that the harmonic oscillator potential $V_1(y) = w_1^2 y^2$ has finite-norm eigenfunctions and eigenvalues of the form

$$\varphi_n(y) = \frac{e^{-w_1 y^2/2}}{\sqrt{2^n n! \sqrt{\pi/w_1}}} H_n(\sqrt{w_1} y) \, , \quad E_n = w_1(2n+1) \, , \quad n = 0, 1, \ldots ,$$

along with the general non-physical solutions $u(y)$ associated with an eigenvalue ϵ,

$$u(y) = e^{-w_1^2 y^2/2} \left[{}_1F_1\left(\frac{1-\epsilon}{4}, \frac{1}{2}, w_1 y^2 \right) + \lambda \sqrt{w_1} y \, {}_1F_1\left(\frac{3-\epsilon}{4} + \frac{1}{2}, \frac{3}{2}, w_1 y^2 \right) \right] .$$



(a) (b)

Figure: (a) New stationary potential potential $V_1^{(\lambda)}(y)$ for $\epsilon = -1$ and $\lambda = 1, -1, 0$. (b) Density distribution for $\epsilon = -1$ and $\lambda = 1$.

Time-dependent phenomena

Consider the Schrödinger operators

$$S_1 = i\partial_t + \partial_{xx} - V_1(x,t)\,, \quad S_2 = i\partial_t + \partial_{xx} - V_2(x,t)\,,$$

altogether with the respective solutions

$$S_1\phi(x,t) = 0\,, \quad S_2\psi(x,t) = 0\,.$$

There exists a operator L such that the intertwining relation [1]

$$LS_1 = S_2 L\,,$$

holds, where

$$L = \ell\left[\partial_x - \frac{\overline{u}_x}{\overline{u}}\right]\,, \quad \ell = \ell(t)\,, \quad \overline{u} = \overline{u}(x,t)\,, \quad i\overline{u}_t = -\overline{u}_{xx} + V_1(x,t)\overline{u}\,.$$

[1] Bagrov, Samsonov and Shekoyan, *Russ. Phys.* **38** (7) 1995.

The intertwining relation allow us to relate the potentials and the solutions as

$$V_2(x,t) = V_1(x,t) + i\frac{\ell_t}{\ell} - 2(\ln \overline{u})_{xx}\,,$$

$$\psi(x,t) = L\phi(x,t)\,, \quad \overline{\psi} = \frac{1}{\ell\,\overline{u}^*}\,.$$

Imposing the reality condition on the potential $V_1(x,t)$ we get

$$\left(\ln\frac{\overline{u}}{\overline{u}^*}\right)_{xxx} = 0\,, \quad \ell(t) = \exp\left(2\int dt\,\mathrm{Im}\left[(\ln\overline{u})_{xx}\right]\right)\,,$$

$$V_2(x,t) = V_1(x,t) - 2\,\partial_{xx}\ln|\overline{u}|\,.$$

Time-dependent oscillators

The initial potential $V_1(x,t) = V_1(x) = x^2$ and a seed function[2] of the form

$$\overline{u}(x,t) = B(t)e^{a(t)x^2} f(x,t)\,, \quad f(x,t) = f(b(t)x)\,, \quad f(x,t) : \mathbb{R} \to \mathbb{R}\,,$$

leads to

$$f(x,t) = e^{-\frac{b(t)^2x^2}{2}} \left[k_a \, {}_1F_1\left(\nu, \frac{1}{2}; b^2(t)x^2\right) + k_b \, b(t) x \, {}_1F_1\left(\nu + \frac{1}{2}, \frac{3}{2}; b^2(t)x^2\right) \right].$$

Thus, the new time-dependent potential, and its solutions are found through

$$V_2(x,t) = V_1(x,t) + i\frac{\ell_t}{\ell} - 2(\ln \overline{u})_{xx}\,,$$

$$\psi(x,t) = L\phi(x,t)\,, \quad \overline{\psi} = \frac{1}{\ell\,\overline{u}^*}\,.$$

[2] K. Zelaya and O. Rosas-Ortiz, *J. Phys.: Conf. Ser.* **839** (2017) 012022.

For $\nu = 1/2$ the new potential is given by

$$V_1(x,t) = x^2 - 2b(t) - 4k_b b(t)\frac{\partial}{\partial x}\left[\frac{e^{-b(t)^2x^2}}{2k_a + \sqrt{\pi}k_b \operatorname{Erf}(b(t)x)}\right],$$

$$b(t) = \frac{c_0}{\sqrt{c_1 + \gamma\cos(4t)}}, \quad 2k_a > \sqrt{\pi}k_b, \quad c_1^2 = \gamma^2 + c_0^4, \quad c_0, c_1, \gamma \in \mathbb{R}.$$

Notice that $\gamma = 0$ leads to

$$b(t) = 1, \quad V_1(x) = x^2 - 2 - 4k_b\frac{\partial}{\partial x}\left[\frac{e^{-x^2}}{2k_a + \sqrt{\pi}k_b \operatorname{Erf}(x)}\right],$$

which is the potential obtained by Mielnik [3].

[3] B. Mielnik, *J. Math. Phys.* **25** (1984) 3387.

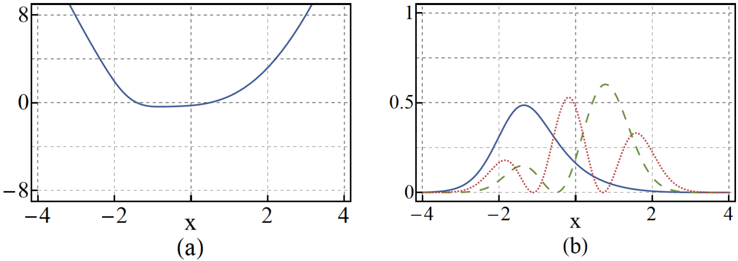


Figure: (a) Time-dependent potential $V_1(x, t)$ and (b) wavefunction modulus $|\overline{\psi}_0(x)|^2$ (solid-blue), $|\psi_1(x)|^2$ (dashed-green), $|\psi_2(x)|^2$ (dotted-red) for $k_a = 2$, $k_b = 1$, $c_0 = 1$ and $c_1 = 10$ at several times t.

A $\mathcal{PT}$ symmetrical potential is obtained for $k_b = 0$,

$$f(x,t) = e^{-\frac{b(t)^2 x^2}{2}} \, k_a \, {}_1F_1\left(\nu, \frac{1}{2}; b^2(t)x^2\right) .$$

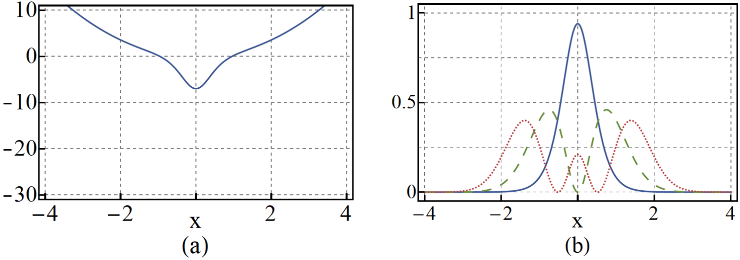


Figure: (a) Time-dependent potential $V_1(x,t)$ and (b) wavefunction modulus $|\overline{\psi}_0(x)|^2$ (solid-blue), $|\psi_1(x)|^2$ (dashed-green), $|\psi_2(x)|^2$ (dotted-red) for $\nu = 2$, $k_a = 2$, $k_b = 0$, $c_0 = 1$ and $c_1 = 10$ at several times t.

Point Transformations

Consider the dimensionless Schödinger equation of the form

$$i\frac{\partial u}{\partial \tau} = -\frac{\partial^2 u}{\partial y^2} + V_1(y)u\,, \quad u = u(y,\tau)\,.$$

By using the appropriate geometrical transformation[4], the previous equation can be mapped into

$$i\frac{\partial \psi}{\partial t} = -\frac{1}{m(t)}\frac{\partial^2 \psi}{\partial x^2} + V_2(x,t)\psi\,, \quad \psi = \psi(x,t)\,.$$

The latter is achieved by means of the general transformation functions

$$y = y(x,t)\,, \quad \tau = \tau(x,t)\,, \quad \overline{\psi}(y,\tau) = K(x,t;\psi)\,.$$

[4] S. Willi-Hans, *Invertible Point Transformations and Nonlinear Differential Equations*, World Scientific Publishing, 1993, Singapore.

By computing the total derivatives $d\overline{\psi}/dx$, $d\overline{\psi}/dt$ and $d^2\overline{\psi}/dx^2$ we obtain the following relations:

$$
\begin{aligned}
\frac{d\overline{\psi}}{dx} &= \frac{\partial K}{\partial \psi}\frac{\partial \psi}{\partial x} + \frac{\partial K}{\partial x} = \frac{\partial y}{\partial x}\frac{\partial \overline{\psi}}{\partial y} + \frac{\partial \tau}{\partial x}\frac{\partial \overline{\psi}}{\partial \tau}\,, \\
\frac{d\overline{\psi}}{dt} &= \frac{\partial K}{\partial \psi}\frac{\partial \psi}{\partial t} + \frac{\partial K}{\partial t} = \frac{\partial y}{\partial t}\frac{\partial \overline{\psi}}{\partial y} + \frac{\partial \tau}{\partial t}\frac{\partial \overline{\psi}}{\partial \tau}\,, \\
\frac{d^2\overline{\psi}}{dx^2} &= 2\frac{\partial^2 K}{\partial \psi \partial x}\frac{\partial \psi}{\partial x} + \frac{\partial^2 K}{\partial \psi^2}\left(\frac{\partial \psi}{\partial x}\right)^2 + \frac{\partial K}{\partial \psi}\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 K}{\partial x^2} \\
&= 2\frac{\partial y}{\partial x}\frac{\partial \tau}{\partial x}\frac{\partial^2 \overline{\psi}}{\partial x \partial \tau} + \left(\frac{\partial y}{\partial x}\right)^2\frac{\partial^2 \overline{\psi}}{\partial y^2} + \left(\frac{\partial \tau}{\partial x}\right)^2\frac{\partial^2 \overline{\psi}}{\partial \tau^2} \\
&\quad + \frac{\partial^2 y}{\partial x^2}\frac{\partial \overline{\psi}}{\partial y} + \frac{\partial^2 \tau}{\partial x^2}\frac{\partial \overline{\psi}}{\partial \tau}\;.
\end{aligned}
$$

From the latter we obtain that the conditions $\partial^2 K/\partial\psi^2 = 0$ and $\partial\tau/\partial x = 0$ remove the undesired term, leading us to the simplified transformation functions

$$K(x,t,\psi) = f(x,t)\psi\,, \quad \tau = \tau(t)\,.$$

After some arrangements we obtain the partial differential equation of the form

$$i\frac{\partial\psi}{\partial t} = -\frac{\tau_t}{y_x^2}\frac{\partial^2\psi}{\partial x^2} + \left[i\frac{y_t}{y_x} - \frac{\tau_t}{y_x^2}\left(2\frac{f_x}{f} - \frac{y_{xx}}{y_x}\right)\right]\frac{\partial\psi}{\partial x} + V_2(x,t)\psi\,,$$

where

$$V_2(x,t) = -i\frac{f_t}{f} + \left(i\frac{y_t}{y_x} + \tau_t\frac{y_{xx}}{y_x^3}\right)\frac{f_x}{f} - \frac{\tau_t}{y_x^2}\frac{f_{xx}}{f} + \tau_t V_1(y(x,t))\,.$$

Form-preserving point transformation

By comparing both Shrödinger equations we obtain the necessary conditions to compute $f(x,t)$, $y(x,t)$ and $\tau(t)$, leading us to[5]

$$y(x,t) = \frac{x+\gamma(t)}{\sigma(t)}, \quad \tau(t) = \int^t \frac{dt'}{\sigma^2(t')},$$

$$\psi(x,t) = e^{i\left(\frac{\dot{\sigma}}{4\sigma}x^2 - \frac{W}{2\sigma}x + \xi\right)} \sqrt{\frac{1}{\sigma}} u(y(x,t), \tau(t)),$$

$$W = \sigma\dot{\gamma} - \dot{\mu}\gamma,$$

with $\sigma(t)$, $\gamma(t)$ and $\xi(t)$ determined from the relation held between the potentials,

$$V_2(x,t) = \frac{V_1(y(x,t))}{\sigma^2} + \frac{\mu\dot{W}_\gamma}{4\sigma}x^2 + \frac{\mu\dot{W}_\gamma}{2\sigma}x + \dot{\xi} - \frac{W^2}{4\sigma^2}.$$

Moreover, the inner product is such that

$$\langle u^{(2)}(\tau) | u^{(1)}(\tau) \rangle = \langle \psi^{(2)}(t) | \psi^{(1)}(t) \rangle.$$

[5] K. Zelaya and O. Rosas-Ortiz, Quantum Nonstationary Oscillators: Invariants, Dynamical Algebras and Coherent States via Point Transformations, Submitted to *Physica Scripta*.

An invariant operator

From the eigenvalue equation associated with the stationary potential $V_1(y)$, we can easily obtain an **invariant operator** related with the time-dependent potential $V_2(x,t)$. That is,

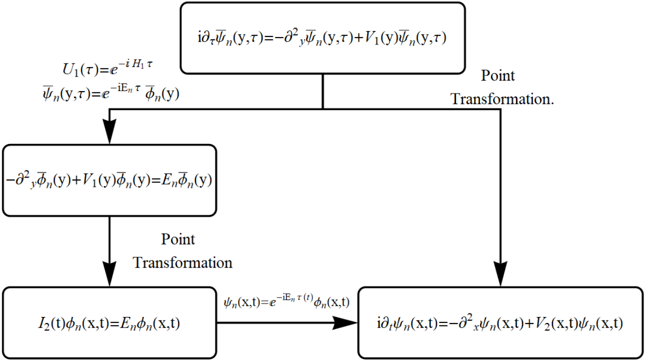
$$-\sigma^2\frac{\partial^2\varphi_n}{\partial x^2}+\sigma\left(i\dot{\sigma}x-iW\right)\frac{\partial\varphi_n}{\partial x}+\left(i\frac{\sigma\dot{\sigma}}{2}+\frac{W^2}{4}-\frac{\dot{\sigma}W}{2}x+\frac{\dot{\sigma}^2}{4}x^2\right)\varphi_n \\ +V_1(y(x,t))\varphi_n=E_n\varphi_n\,,$$

$$\varphi_n(x,t)=e^{-i\left(\frac{\mu W_\mu}{4\sigma}x^2+\frac{\mu W}{2\sigma}x+\eta\right)}\sqrt{\frac{\mu}{\sigma}}\phi_n(y(x,t))\,.$$

Alternatively, the latter can be written as $I_2(t)|\varphi_n(t)\rangle=E_n|\varphi_n(t)\rangle$, with $I_2(t)$ an invariant operator of the form

$$I_2(t)=\sigma^2P_x^2+\frac{\dot{\sigma}^2}{4}X^2-\frac{\sigma\dot{\sigma}}{2}\{X,P_x\}-\frac{\dot{\sigma}W}{2}X+\sigma WP_x+\frac{W^2}{4}+V_1\left(\frac{X+\gamma}{\sigma}\right)\,,$$

with $\{X,P\}=XP+PX$ the anticommutator.


$i\partial_\tau\overline{\psi}_n(y,\tau)=-\partial^2{}_y\overline{\psi}_n(y,\tau)+V_1(y)\overline{\psi}_n(y,\tau)$
$U_1(\tau)=e^{-iH_1\tau}$
$\overline{\psi}_n(y,\tau)=e^{-iE_n\tau}\,\overline{\phi}_n(y)$
Point Transformation.
$-\partial^2{}_y\overline{\phi}_n(y)+V_1(y)\overline{\phi}_n(y)=E_n\overline{\phi}_n(y)$
Point Transformation
$I_2(t)\phi_n(x,t)=E_n\phi_n(x,t)$
$\psi_n(x,t)=e^{-iE_n\tau(t)}\phi_n(x,t)$
$i\partial_t\psi_n(x,t)=-\partial^2{}_x\psi_n(x,t)+V_2(x,t)\psi_n(x,t)$

Consider $V_1(y) = w_1^2 y^2$ as the initial potential and the eigenfunctions

$$\phi_n(y) = \frac{e^{-w_1 y^2/2}}{\sqrt{2^n n! \sqrt{\pi/w_1}}} H_n(\sqrt{w_1} y)\,, \quad E_n = w_1(2n+1)\,, \quad n = 0, 1, \dots,$$

with $\overline{\psi}_n(y,\tau) = e^{-iw_1(2n+1)\tau}\phi_n(y)$. By using the point transformation, and after some arrangements, we obtain the new time-dependent potential

$$V_2(x,t) = 4\Omega^2(t)x^2 + F(t)x\,,$$

together with the conditions held by the transformation functions

$$\ddot{\sigma}(t) + 4\Omega^2(t)\sigma(t) = 4\frac{w_1^2}{\sigma^3(t)}\,, \quad \ddot{\gamma}(t) + 4\Omega^2(t)\gamma(t) = 2F(t)\,,$$

$$\eta(t) = \frac{\gamma W}{4\sigma} - \frac{1}{2}\int^t dt' F(t')\gamma(t')\,.$$

The solutions for the parametric oscillator are given by

$$\psi_n(x,t) = e^{-iw_1(2n+1)\tau(t)}\varphi_n(x,t)\,, \quad \tau(t) = \frac{1}{2w_1}\arctan\left[\frac{W_0}{4w_1}\left(b+2c\frac{q_2}{q_1}\right)\right]\,,$$

where

$$\varphi_n(x,t) = e^{i\left(\frac{\dot\sigma}{4\sigma}x^2-\frac{W}{2\sigma}x+\xi\right)}e^{-\frac{w_1}{2}\left(\frac{x+\gamma}{\sigma}\right)^2}\sqrt{\frac{1}{\sigma}}\frac{H_n\left[\sqrt{w_1}\left(\frac{x+\gamma}{\sigma}\right)\right]}{\sqrt{2^n n!\sqrt{\pi/w_1}}}\,, \quad n=0,1,\ldots\,,$$

are eigenfunctions of the Invariant operator

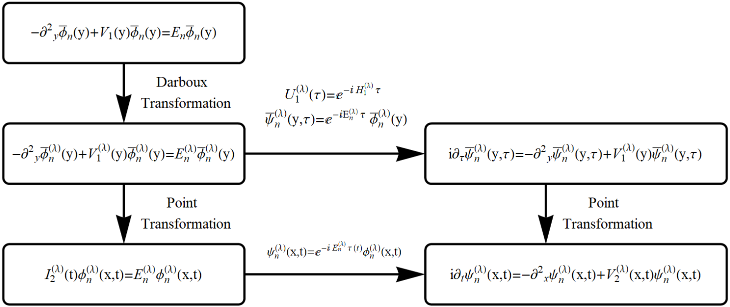
$$\begin{aligned} I_2(t) =& \frac{\sigma^2}{\mu^2}P_x^2 + \left(\frac{W_\mu^2}{4}+w_1^2\frac{\mu^2}{\sigma^2}\right)X^2 + \frac{W_\mu\sigma}{2\mu}\frac{1}{i}(XP+PX) + \frac{\sigma W}{\mu}P_x \\ &+ \left(\frac{W_\mu W}{2}+2w_1^2\frac{\mu\gamma}{\sigma^2}\right)X + \left(\frac{W^2}{4}+w_1^2\frac{\gamma^2}{\sigma^2}\right)\,, \end{aligned}$$

with eigenvalue $w_1(2n+1)$.
It is worth to mention that the parametric oscillator[6] is recovered by fixing $\gamma(t)=F(t)=0$.

[6] H. R. Lewis and W. B. Riesenfled, *J. Math. Phys.* **10** (1969) 1458.

New family of time-dependent potentials

New family of time-dependent potentials



Family of time-dependent oscillators

Initial Stationary model:

$$V_1(y) = w_1^2 y^2 + 2\beta_M'(y)\,, \quad \beta_M = -\frac{\overline{v}_y}{\overline{v}}\,,$$

$$\overline{v}(y) = e^{-w_1^2 y^2/2}\left[{}_1F_1\left(\frac{1-\epsilon}{4}, \frac{1}{2}, w_1 y^2\right) + \lambda\sqrt{w_1}\, y\, {}_1F_1\left(\frac{3-\epsilon}{4} + \frac{1}{2}, \frac{3}{2}, w_1 y^2\right)\right].$$

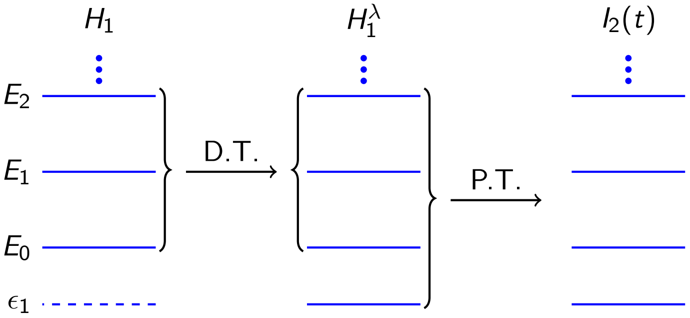
Time-dependent counterpart:

$$V_2(x,t) = \Omega^2(t)x^2 + F(t)x - 2\frac{\partial^2}{\partial x^2}\log \overline{v}(y(x,t))\,,$$

$$y(x,t) = \frac{x+\gamma(t)}{\sigma(t)}\,.$$

$$I_2(t) = \sigma^2 P_x^2 + \left(\frac{\dot{\sigma}^2}{4} + \frac{w_0^2}{\sigma^2}\right)X^2 - \frac{\sigma\dot{\sigma}}{2}\{X, P_x\} + \sigma W P_x$$
$$+ \left(2\frac{\gamma}{\sigma^2} - \frac{W\dot{\sigma}}{2}\right)X + \left(\frac{\gamma^2}{\sigma^2} + \frac{W^2}{4}\right) + 2\sigma^2\mathcal{F}(X,t)\,,$$

$$\mathcal{F}(X,t) = -\frac{\partial^2}{\partial x^2}\ln\overline{v}(y(x,t))\bigg|_{x\to X}\,,$$


H_1
H_1^λ
$I_2(t)$
E_2
E_1
E_0
ϵ_1
D.T.
P.T.

- **Case** $4\Omega^2(t) = \Omega^2$.

$$\sigma^2(t) = c_0 + c_1 \cos(4\Omega t + \phi)\,,$$

with $\phi, c_0, c_1 \in \mathbb{R}$ and $c_0 > c_1$ to produce a zeroes–free and real–valued function $\sigma(t)$ at each times.

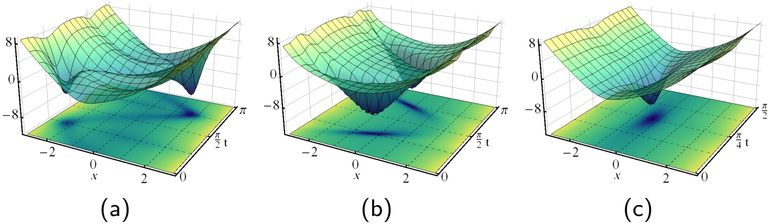


Figure: Time–dependent potential for $4\Omega^2(t) = \Omega^2 = 1$, $c_0 = 3$, $c_1 = 2$, $\epsilon = -5$ and $\lambda = 1.2$. In the panel (a) we have considered the values $\gamma(t) = 0$, whereas in (b) we used $\gamma_0 = 1$, $\gamma_1 = 0$ and in (c) $\gamma_0 = 0$, $\gamma_1 = 1$.

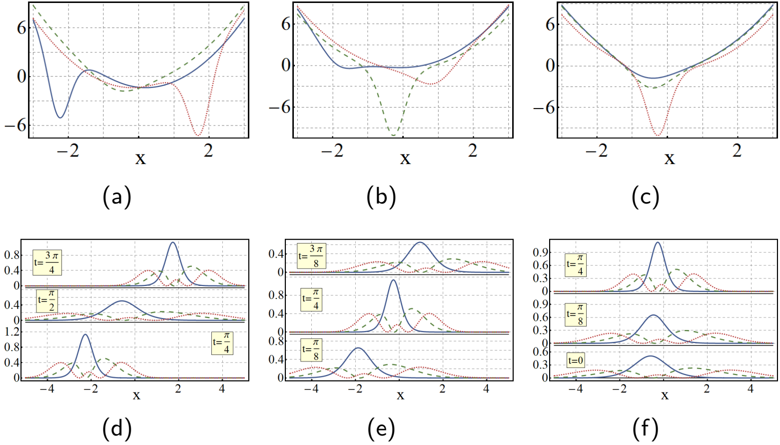


Figure: (First row) The solid–blue, dashed–green and dotted–red curves represent the times $t = \pi/4, \pi/2, 3\pi/4$ for (a), $t = \pi/8, \pi/4, 3\pi/8$ for (b) and $t = 0, \pi/8, \pi/4$ for (c). (Second row) Probability densities $|\psi_n^{(\lambda)}(x,t)|^2$ for n=0(solid–blue), n=1(dashed–green) and n=2(dotted–red).

- **Case** $4\Omega^2(t) = \Omega_1 + \Omega_2 \tanh(kt)$.

$$\sigma^2(t) = 2a\mathrm{Re}(\tilde{\sigma}_1^2(t)) + 2\sqrt{a^2 + \frac{w_1^2}{k^2 g_+^2}}\,|\tilde{\sigma}_1|^2,$$

$$\tilde{\sigma}_1(t) = (1-z)^{-\frac{i}{2}g_+}(1+z)^{-\frac{i}{2}g_-}\ {}_2F_1\left(\left.\begin{array}{c} -i\mu\,,\,1-i\mu \\ 1-ig_+(t) \end{array}\right| \frac{1-z}{2}\right),$$

$$g_\pm = \mu \pm \frac{\Omega_2}{2k^2\mu}\,, \quad \mu = \frac{1}{k}\sqrt{\frac{\Omega_1 + \sqrt{\Omega_1^2 - \Omega_2^2}}{2}}\,, \quad z = \tanh(kt)\,,$$

with ${}_2F_1(a,b;c;z)$ the Hypergeometric function.

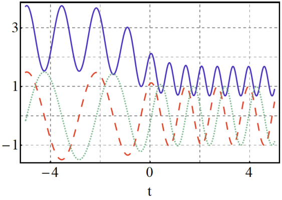
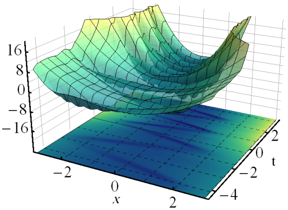
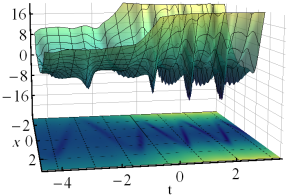


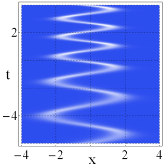
Figure: Behavior of $\sigma(t)$ (solid-blue) for $4\Omega_2^2(t) = \Omega_1 + \Omega_2 \tanh(kt)$ with $k = 1/2$, $\Omega_1 = 15$, $\Omega_2 = 10$ and $a = 1/2$.



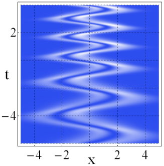
(a)



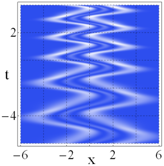
(b)



(c)



(d)



(e)

Figure: Parameters fixed at $w_1 = 1$, $\Omega_1 = 15$, $\Omega_2 = 10$, $\gamma_0 = \gamma_1 = 1/2$, $\lambda = 1$, $a = k = 1/2$, and $\epsilon = -5$.

Conclusions

- A simple and elegant method to construct time-dependent models has been achieved through the point transformations. In this form, the spectral information, among other properties, of a given stationary initial system are inherited to the new time-dependent model in a straightforward way.
- The transformation can be applied in a similar fashion to non-Hermitian models like the Swanson oscillator[7] both in the stationary and non-stationary regime.
- By considering different constrains on the transformation, it is possible to construct time-dependent mass models[8] such as the Caldirola-Kanai oscillator, among others. PDM models are also achievable by considering more complex constrains.
- Point transformations can be extended to construct nonlinear models such as the Schrödinger-Eckhaus equation[9].

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Thanks