Quantum Localisation on the Circle

7th International Workshop on New Challenges in Quantum Mechanics: Integrability and Supersymmetry

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- Except if $\widehat{\alpha}$ stands for the 2π-periodic discontinuous angle function,

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- However, for $\widehat{\alpha}$ Self-Adjoint (SA), $spec(\widehat{\alpha}) \subset [0, 2\pi]$, the CCR $[\widehat{\alpha}, \widehat{p}_\alpha] = \mathrm{i}\hbar I$ does not hold for SA quantum angular momentum $\widehat{p}_\alpha = -\mathrm{i}\hbar\frac{\partial}{\partial\alpha}$.

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- Instead, one has

$$[\widehat{\alpha}, \widehat{p}_\alpha] = i\hbar I \left[1 - 2\pi \sum_n \delta(\alpha - 2n\pi)\right] \,. \tag{2}$$

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- We revisit the problem of the quantum angle through coherent state (CS) quantisation, which is a particular method belonging to covariant integral quantisation (S.T. Ali, J.-P. Antoine, and J.-P. Gazeau, *Coherent States, Wavelets and their Generalizations*).

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- One of our aims is to build acceptable angle operators from the classical angle function through a consistent and manageable quantisation procedure.

Covariant integral quantisation - general scheme

UFABC

- Let G be a Lie group with left Haar measure $d\mu(g)$ and $g \mapsto U(g)$ a UIR of G in $\mathcal{H}$. For $\rho \in B(\mathcal{H})$, suppose the following operator is defined in a weak sense:

$$R := \int_G \rho\left(g\right) \mathrm{d}\mu\left(g\right), \rho\left(g\right) := U\left(g\right) \rho U^{\dagger}\left(g\right)$$

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- That is, the family of operators $\rho(g)$ provides a resolution of the identity

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- which is covariant in the sense that

$$U(g) A_f U^\dagger(g) = A_{U(g)f}\,, \; (U(g)f)(g') = f(g^{-1}g')$$

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- The interesting case is when X is a symplectic manifold (e.g., co-adjoint orbit of G) and can be viewed as the phase space for the dynamics.
- Given a quasi-invariant measure ν on X, one has for a global Borel section $\sigma : X \to G$ a unique quasi-invariant measure $\nu_\sigma(x)$.
- Let U be a square-integrable UIR, and ρ a density operator such that $c_\rho := \int_X \mathrm{tr}\,(\rho\,\rho_\sigma(x))\,\mathrm{d}\nu_\sigma(x) < \infty$ with $\rho_\sigma(x) := U(\sigma(x))\rho U(\sigma(x))^\dagger$.

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- Covariance holds in the sense $U(g) A_f^\sigma U(g)^\dagger = A^{\sigma_g}_{\mathcal{U}_l(g)f}$, where $\sigma_g(x) = g\sigma(g^{-1}x)$ with $\mathcal{U}_l(g)f(x) = f\left(g^{-1}x\right)$.
- For $\rho = |\eta\rangle\langle\eta|$, we are working with CS quantisation, where the CS's are defined as $|\eta_x\rangle := |U(\sigma_g(x))\eta\rangle$.

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- Finally, given a UIR χ of V and a UIR L of S, one can construct an irreducible representation $(v, s) \mapsto {}^{\chi L}U(v, s)$ of G induced by the representation $\chi \otimes L$ of $V \rtimes S_0$.

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- Given $\eta \in \mathcal{H} = L^2(\mathcal{O}^*, d\nu)$, one constructs a family $\eta_{\mathbf{p},\mathbf{q}}$: $\eta_{\mathbf{p},\mathbf{q}}(k) = \left({}^{\chi L}U(\sigma(\mathbf{p}, \mathbf{q}))\eta\right)(k)$

- If one can prove $\int_X \mathrm{d}\mu(\boldsymbol{p},\boldsymbol{q})\langle\,\phi\,|\,\eta_{\boldsymbol{p},\boldsymbol{q}}\,\rangle_{\mathcal{H}}\langle\,\eta_{\boldsymbol{p},\boldsymbol{q}}\,|\,\psi\,\rangle_{\mathcal{H}} = c_\eta\langle\,\phi\,|\,\psi\,\rangle$ where $\phi,\psi : \mathcal{O}^* \to \mathbb{C}$ and $0 < c_\eta < \infty$

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- The semiclassical portrait of the operator A_f is defined as $\check{f}(\boldsymbol{p},\boldsymbol{q}) = \dfrac{1}{c_\eta}\int_X \mathrm{d}\mu(\boldsymbol{p}',\boldsymbol{q}')f\,(\boldsymbol{p}',\boldsymbol{q}')\left|\langle\eta_{\boldsymbol{p}',\boldsymbol{q}'}|\eta_{\boldsymbol{p},\boldsymbol{q}}\rangle\right|^2\,.$

The Euclidean group E(2)

- Now $G = \mathrm{E}(2)$, where $V = \mathbb{R}^2$ and $S = \mathrm{SO}(2)$, so $\mathrm{E}(2) = \mathbb{R}^2 \rtimes \mathrm{SO}(2) = \{(\boldsymbol{r}, \theta)\,,\ \boldsymbol{r} \in \mathbb{R}^2\,,\ \theta \in [0, 2\pi)\}$, with composition $(\boldsymbol{r}, \theta)(\boldsymbol{r}', \theta') = (\boldsymbol{r} + \mathcal{R}(\theta)\boldsymbol{r}', \theta + \theta')$.

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- $V^* = \mathbb{R}^2$, $\mathcal{O}^* = \{\boldsymbol{k} = \mathcal{R}(\theta)\boldsymbol{k}_0 \in \mathbb{R}^2 \,|\, \mathcal{R}(\theta) \in \mathrm{SO}(2)\} \simeq \mathbb{S}^1$

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- The stabilizer under the coadjoint action $\mathrm{Ad}^{\#}_{\mathrm{E}(2)}$ is $H_0 = \left\{(\boldsymbol{x}, 0) \in \mathrm{E}(2) \,|\, \hat{\boldsymbol{c}} \cdot \boldsymbol{x} = 0,\ \hat{\boldsymbol{c}} \in \mathbb{R}^2,\ \|\hat{\boldsymbol{c}}\| = 1,\ \text{fixed}\right\}$.

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- The classical phase space $X \equiv T^*\mathbb{S}^1 \simeq \mathbb{R} \times \mathbb{S}^1$ carries coordinates (p, q) and has symplectic measure $\mathrm{d}p \wedge \mathrm{d}q$.
- The UIR of E(2) are $L^2(\mathbb{S}^1, \mathrm{d}\alpha) \ni \psi(\alpha) \mapsto \left(U(\boldsymbol{r}, \theta)\psi\right)(\alpha) = e^{\mathrm{i}(r_1 \cos\alpha + r_2 \sin\alpha)}\psi(\alpha - \theta)$.

Theorem

Given the unit vector $\hat{\boldsymbol{c}} \in \mathbb{R}^2$ *and the corresponding subgroup* H_0*, there exists a family of affine sections* $\sigma : \mathbb{R} \times \mathbb{S}^1 \to E(2)$ *defined as* $\sigma(p, q) = (\mathcal{R}(q)(\boldsymbol{\kappa} p + \boldsymbol{\lambda}), q)$ *, where* $\boldsymbol{\kappa}, \boldsymbol{\lambda} \in \mathbb{R}^2$ *are constant vectors, and* $\hat{\boldsymbol{c}} \cdot \boldsymbol{\kappa} \neq 0$*.*

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From the section $\sigma(p, q)$, the representation $U(\mathbf{r}, \theta)$, and a vector $\eta \in L^2(\mathbb{S}^1, \mathrm{d}\alpha)$, we define the family of states
$|\eta_{p,q}\rangle = U(\sigma(p, q))|\eta\rangle$.

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Given the unit vector $\hat{\mathbf{c}} \in \mathbb{R}^2$ and the corresponding subgroup H_0, there exists a family of affine sections $\sigma : \mathbb{R} \times \mathbb{S}^1 \to E(2)$ defined as $\sigma(p,q) = (\mathcal{R}(q)(\boldsymbol{\kappa} p + \boldsymbol{\lambda}), q)$, where $\boldsymbol{\kappa}, \boldsymbol{\lambda} \in \mathbb{R}^2$ are constant vectors, and $\hat{\mathbf{c}} \cdot \boldsymbol{\kappa} \neq 0$.

From the section $\sigma(p,q)$, the representation $U(\mathbf{r}, \theta)$, and a vector $\eta \in L^2(\mathbb{S}^1, \mathrm{d}\alpha)$, we define the family of states $|\eta_{p,q}\rangle = U(\sigma(p,q))|\eta\rangle$.

Theorem

The vectors $\eta_{p,q}$ form a family of coherent states for $E(2)$ which resolves the identity on $L^2(\mathbb{S}^1, \mathrm{d}\alpha)$, $I = \int_{\mathbb{R}\times\mathbb{S}^1} \frac{\mathrm{d}p\,\mathrm{d}q}{c_\eta} \,|\,\eta_{p,q}\,\rangle\langle\,\eta_{p,q}\,|$, if $\eta(\alpha)$ is <u>admissible</u> in the sense that $\operatorname{supp}\eta \in (\gamma - \pi, \gamma) \bmod 2\pi$, and $0 < c_\eta := \frac{2\pi}{\kappa} \int_{\mathbb{S}^1} \frac{|\eta(q)|^2}{\sin(\gamma - q)}\,\mathrm{d}q < \infty$.

- Given the family of coherent states $|\eta_{p,q}\rangle$, we apply the linear map $f \mapsto A_f^\sigma = \int_{\mathbb{R}\times\mathbb{S}^1} \frac{\mathrm{d}p\,\mathrm{d}q}{c_\eta} f(p,q)\,|\,\eta_{p,q}\,\rangle\langle\,\eta_{p,q}\,|$ to classical observables $f(p,q)$.

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- For $f(p,q) = u(q)$ with $u(q+2\pi) = u(q)$, A_u is the multiplication operator $(A_u\psi)(\alpha) = (E_{\eta;\gamma} * u)\,(\alpha)\,\psi(\alpha)$ where $E_{\eta;\gamma}(\alpha) := \frac{2\pi}{\kappa c_\eta}\frac{|\eta(\alpha)|^2}{\sin(\gamma-\alpha)}$, supp $E_{\eta;\gamma} \subset (\gamma-\pi,\gamma)$ is a probability distribution on the interval $[-\pi,\pi]$.

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- For the momentum $f(p,q) = p$,
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- The classical limit is obtained by considering the large κ limit with the condition $|\eta(\alpha)|^2 \to \delta(\alpha-\gamma+\pi/2)$. Then $\check{f}(q,p) = f(q,p) + o(1/\kappa)$.

- For the 2π-periodic and discontinuous angle function $a(\alpha)=\alpha$ for $\alpha\in[0,2\pi)$, we get the multiplication operator $(E_{\eta,\gamma}*a)(\alpha)=\alpha+2\pi(1-\int_{-\pi}^{\alpha}E_{\eta;\gamma}(q)\,\mathrm{d}q)-\int_{\gamma-\pi}^{\gamma}q\,E_{\eta,\gamma}(q)\,\mathrm{d}q\,.$

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- We choose a specific section with $\lambda = 0$, $\gamma = \pi/2$ and as fiducial vectors the family $\eta^{(s,\epsilon)}(\alpha)$ of periodic smooth even functions, $\mathrm{supp}\eta = [-\epsilon, \epsilon]$ mod 2π, parametrized by $s > 0$ and $0 < \epsilon < \pi/2$,

$$\eta^{(s,\epsilon)}(\alpha) = \frac{1}{\sqrt{\epsilon e_{2s}}}\,\omega_s\left(\frac{\alpha}{\epsilon}\right) \quad \text{where} \quad e_s := \int_{-1}^{1} \mathrm{d}x\,\omega_s(x)\,.$$

and

$$\omega_s(x) = \begin{cases} \exp\left(-\dfrac{s}{1-x^2}\right) & 0 \le |x| < 1\,, \\ 0 & |x| \ge 1\,, \end{cases}$$

are smooth and compactly supported test functions.

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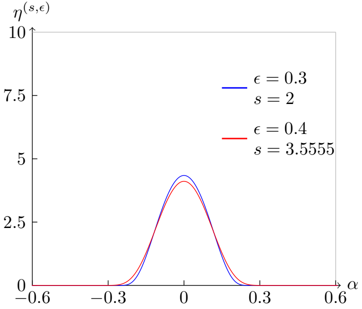
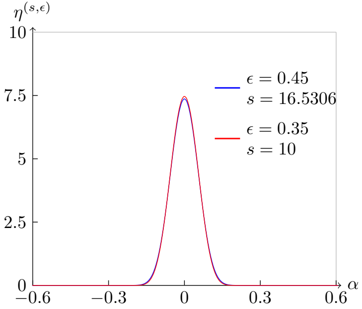
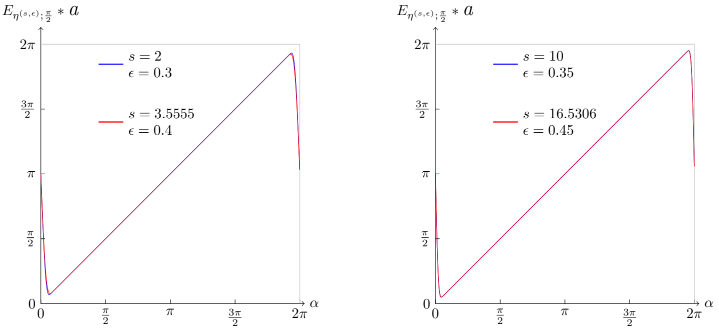


(a) $\tau = 22.22$



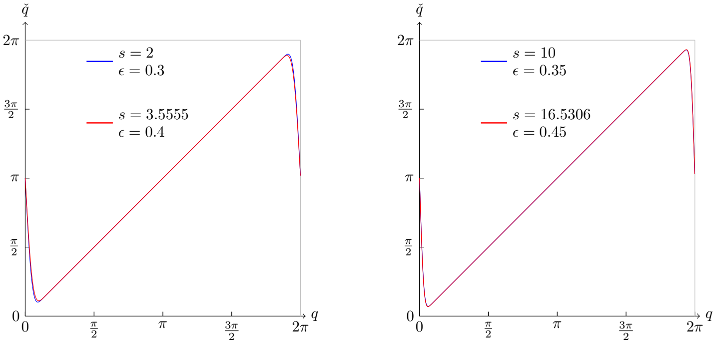
(b) $\tau = 81.63$

Figure: Plots of $\eta^{(s,\epsilon)}$ for various values of $\tau = \dfrac{s}{\epsilon^2}$.



(a) $\tau = 22.22$ (b) $\tau = 81.63$

Figure: Plots of $\left(E_{\eta^{(s,\epsilon)};\frac{\pi}{2}} * a\right)(\alpha)$ for various values of $\tau = \dfrac{s}{\epsilon^2}$.



(a) $\tau = 22.22$ (b) $\tau = 81.63$

Figure: Plots of the lower symbol $\check{q}(q)$ of the angle operator $A_{\boldsymbol{a}}$ for various values of $\tau = \dfrac{s}{\epsilon^2}$.

Angle-angular momentum: commutation relations and Heisenberg inequality

- For $\lambda = 0$ and $\psi(\alpha) \in L^2(\mathbb{S}^1, \mathrm{d}\alpha)$, we find the non-canonical CR $\left([A_p, A_{\mathfrak{a}}]\psi\right)(\alpha) = -\mathrm{i}\left(1 - 2\pi E_{\eta;\gamma}(\alpha)\right)\psi(\alpha)$

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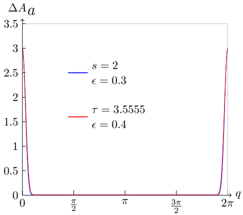
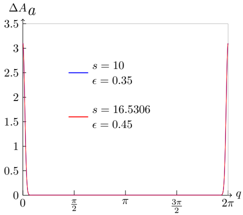
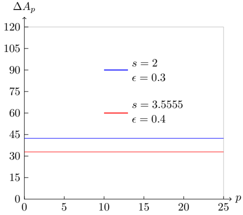
- The uncertainty relation for A_p and A_{a}, with the coherent states $\eta_{p,q}$, is $\Delta A_p\, \Delta A_{\mathsf{a}} \geqslant \frac{1}{2}\left|\langle\, \eta_{p,q} \left|[A_p, A_{\mathsf{a}}]\right| \eta_{p,q}\,\rangle\right|$.



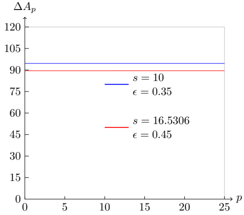
(a) $\tau = 22.22$



(b) $\tau = 81.63$

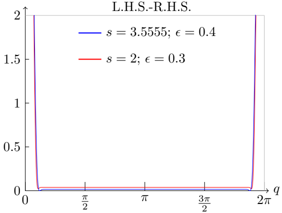


(c) $\tau = 22.22$



(d) $\tau = 81.63$

Figure: Plots of the dispersions ΔA_a and ΔA_p with respect to the coherent state $|\,\eta_{p,q}^{(s,\epsilon)}\,\rangle$ for various values of $\tau = \dfrac{s}{\epsilon^2}$.



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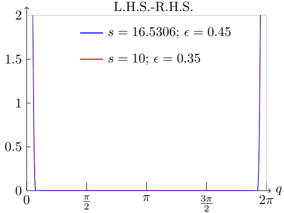

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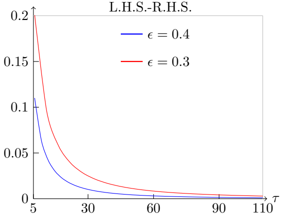

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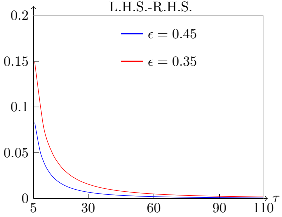

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Figure: Plots of the difference L.H.S.-R.H.S. of the uncertainty relation with respect to the coherent state $\left|\eta_{p,q}^{(s,\epsilon)}\right\rangle$ for various values of $\tau = \frac{s}{\epsilon^2}$.

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UFABC

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- The translation covariance mod 2π $U(\theta) A^{\sigma}_f U^{\dagger}(\theta) = A^{\sigma}_{f(\cdot - \theta)}$ is a quantum expression of the transition map between different charts on the circle.

obrigado!

Figure: UFABC Campus in Santo André, São Paulo