Klein four-group and Darboux duality

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1. L. Inzunza and M. S. Plyushchay, Phys. Rev. D 99, 125016, [arXiv:[1902.00538]].

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Discrete symmetries in 1-d QM: Discrete transformations which preserve the 1-d Schrödinger equation shape.

Some quantum systems have:

- *i*) Time reversal symmetry.
- *ii*) Parity symmetry.
- *iii*) Charge Conjugation

Applications: They give us new solutions!→ New systems by means of Darboux transformation (**DT**)[2].

Motivation!

Study and apply discrete symmetry in the Alfaro, Fubini and Furlan model with harmonic term[3], in the context of **D.T**.

2. V. B. Matveev and M. A. Salle, (Springer, Berlin, 1991).
3. V. de Alfaro, S. Fubini and G. Furlan, Nuovo Cim. A **34**, 569 (1976).

Darboux transformation (D.T)

Construct a new system in base of a well known problem...

$H_1 = -\frac{d^2}{dx^2} + V_1 \rightarrow$	$H_{[n]} = -\frac{d^2}{dx^2} + V_1 - 2\ln''(W(\psi_1, \ldots, \psi_n))$
$\psi_\lambda \rightarrow$	$\Psi_{[n],\lambda} = \frac{W(\psi_1,\ldots,\psi_n,\psi_\lambda)}{W(\psi_1,\ldots,\psi_n)} = \mathbb{A}_{[n]}\psi_\lambda$
$E_\lambda \rightarrow$	$E_{[n],\lambda} = E_\lambda, \quad E_{[n],i} = 0 \;\; i = 1, \ldots, n$

Where $\mathbb{A}_{[n]} = A_n A_{n-1} \ldots A_1$ and $A_i = \mathbb{A}_{i-1}\psi_i \frac{d}{dx}\frac{1}{\mathbb{A}_{i-1}\psi_i}$, $i = 0, 1, 2 \ldots, n$ and $A_0 = 1$.

$$\ker \mathbb{A}_{[n]} = \text{span}\{\psi_1, \ldots, \psi_n\}$$

Intertwining relations

$\mathbb{A}_{[n]} H_1 = H_{[n]} \mathbb{A}_{[n]}$ and $\mathbb{A}^\dagger_{[n]} L_{[n]} = H_1 \mathbb{A}^\dagger_{[n]}$.

$(0+1)$ Classical conformal mechanics

The $(0+1)$ conformal mechanics model

$$S = \int \mathcal{L} dt\,, \qquad \mathcal{L} = \frac{1}{2}\left(\dot{q}^2 - \frac{g}{q^2}\right)\,, \qquad g \geq -1/4\,.$$

Noether charges:

$$H_g = \frac{1}{2}\left(p^2 + \frac{g}{q^2}\right)\,, \qquad p = \dot{q}\,,$$

$$D = \frac{qp}{2} - H_g t\,, \qquad K = \frac{q^2}{2} - 2Dt - H_g t^2\,,$$

The Conformal algebra [4]

$$\{D, H_g\} = H_g\,, \qquad \{D, K\} = -K\,, \qquad \{K, H_g\} = 2D\,,$$

4. S. Fedoruk, E. Ivanov and O. Lechtenfeld, J. Phys. A **45**, 173001 (2012) [arXiv:1112.1947 [hep-th]].

Introducing an scale

By doing the change of variables [5]

$$y = \frac{q}{\sqrt{u + vt + wt^2}}\,, \qquad d\tau = \frac{dt}{u + vt + wt^2}\,,$$

The conformal action becomes in

$$S([y]) = \int d\tau \left(y'^2 - \omega^2 y^2 + \frac{g}{y^2} \right) + B.T.$$

with $\omega^2 = \frac{1}{4}(4wu - v^2) > 0$.

5. S. J. Brodsky, G. F. de Teramond, H. G. Dosch and J. Erlich, Phys. Rept. **584** (2015) 1 [arXiv:1407.8131 [hep-ph]].

Noether charges

$$\mathcal{H}_g = \frac{1}{2}\left(p^2 + \omega^2 y^2 + \frac{g}{y^2}\right), \qquad p = y',$$

$$\mathcal{D} = \frac{1}{2}\left(yp\cos(2\omega\tau) + \left(2\omega y^2 - \mathcal{H}_g \omega^{-1}\right)\sin(2\omega\tau)\right),$$

$$\mathcal{K} = \frac{1}{2}\left(y^2\cos(2\omega\tau) - yp\omega^{-1}\sin(2\omega\tau) - \mathcal{H}_g\omega^{-2}\left(\cos(2\omega\tau) - 1\right)\right).$$

They satisfies the Newton Hooke algebra [6,7]:

$$\{\mathcal{H}_g, \mathcal{D}\} = -(\mathcal{H}_g - 2\omega^2\mathcal{K}), \qquad \{\mathcal{H}_g, \mathcal{K}\} = -2\mathcal{D},$$

$$\{\mathcal{D}, \mathcal{K}\} = -\mathcal{K}.$$

6. A. Galajinsky, Nucl. Phys. B **832**, 586 (2010) [arXiv:1002.2290 [hep-th]].
7. K. Andrzejewski, Phys. Lett. B **738**, 405 (2014) [arXiv:1409.3926 [hep-th]].

The "regularized" quantum conformal mechanics model

Quantum generators ($\hbar = 1\,, x = \sqrt{\omega}y\,, \omega = 2$)

$$\mathcal{H}_\nu = -\frac{d^2}{dx^2} + x^2 + \frac{\nu(\nu+1)}{x^2}\,, \qquad \nu \geq -1/2\,,$$

$$\mathcal{C}_\nu^\pm = -\left(\frac{d}{dx} \mp x\right)^2 + \frac{\nu(\nu+1)}{x^2}\,.$$

These generators satisfies the $\mathfrak{sl}(2,\mathbb{R})$ algebra

$$[\mathcal{H}_\nu, \mathcal{C}_\nu^\pm] = \pm 4\mathcal{C}_\nu^\pm\,, \qquad [\mathcal{C}_\nu^-, \mathcal{C}_\nu^+] = 8\mathcal{H}_\nu\,.$$

Solutions[8]:

- $\psi_{\nu,n}(x) = \sqrt{\frac{2n!}{\Gamma(n+l+\frac{3}{2})}}x^{\nu+1}\mathcal{L}_n^{(\nu+1/2)}(x)e^{-x^2/2}$,
- $\widetilde{\psi_{\nu,n}}(x) = \psi_{\nu,n}\int^x \frac{d\zeta}{(\psi_{\nu,n}(\zeta))^2}$.

with $E_{\nu,n} = 4n + 2\nu + 3\,.$

8. A. M. Perelomov, Theor. Math. Phys. **6**, 263 (1971).

The discrete Klein-4 group in the AFF model

The time dependent Schödinger equation

$$\left(-\frac{\partial^2}{\partial x^2} + x^2 + \frac{\nu(\nu+1)}{x^2}\right)\phi(x,t;\nu) = i\frac{\partial}{\partial t}\phi(x,t;\nu)\,,$$

preserve its form under the transformations

$$\rho_1 : \nu \to -\nu - 1\,, \qquad \rho_2 : (x,t) \to (ix,-t)\,,$$

$$\rho_3 = \rho_1\rho_2 = \rho_2\rho_1\,.$$

which satisfies

$$\rho_1^2 = \rho_2^2 = \rho_3^2 = 1\,.$$

$\to K_4 = \{1, \rho_1, \rho_2, \rho_3\}$ is a Klein-4 group.

At the level of stationary Schrödinger equation

$\rho_2 : (x,E) \to (ix,-E)$.

Action of the K-4 Group on the eigenstates

* $\rho_1(\psi_{n,\nu}) = \sqrt{\frac{2n!}{\Gamma(n-\nu-1/2)}}\mathcal{L}_n^{(-\nu-1/2)}(x^2)e^{-x^2/2} := \psi_{n,-\nu-1}$,

  $\rho_1(E_{n,l}) = 4l - 2\nu + 1.$
* $\rho_2(\psi_{n,\nu}) = \sqrt{\frac{2n!}{\Gamma(n+\nu+3/2)}}\mathcal{L}_n^{(\nu+1/2)}(-x^2)e^{x^2/2} := \psi_{-n,\nu}$,

  $\rho_2(E_{n,\nu}) = -E_{n,\nu}$
* $\rho_3(\psi_{n,\nu}) = \sqrt{\frac{2n!}{\Gamma(n-\nu-1/2)}}\mathcal{L}_n^{(-\nu-1/2)}(-x^2)e^{x^2/2} := \psi_{-n,-\nu-1}$,

  $\rho_3(E_{n,-\nu-1}) = -E_{n,-\nu-1}.$

Careful!: In the case $\nu = \ell - 1/2$ with $\ell = 0, 1, \ldots$, the factor $\Gamma(n-\nu-1/2) \to \infty$ when $n < \ell - 1/2$.

The special case $\nu = \ell - 1/2$

By means of the identity

$$\frac{(-\eta)^m}{m!}\mathcal{L}_n^{(m-n)}(\eta) = \frac{(-\eta)^n}{n!}\mathcal{L}_m^{(n-m)}(\eta)\,,$$

on can show the relation

$$\rho_1(\psi_{n,\ell-1/2,m}) = (-1)^n\psi_{l-1/2,n-\ell}\,, \qquad n \geq \ell\,,$$

$$\rho_1(E_n, \ell - 1/2) = E_{n-\ell,\ell-1/2}\,.$$

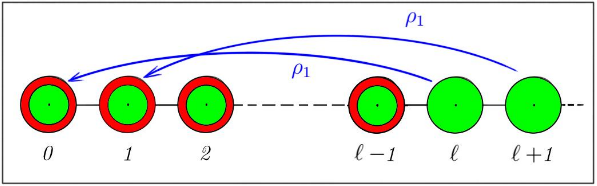

Figure: Action of ρ_1 on eigenstates. The red states are “annihilated”.

If we ignore the normalization constant, we can construct the non-physical solutions

$$\psi_{m,\ell-1/2} := \rho_1 \left(\sqrt{\frac{\Gamma(n+\nu+3/2)}{2m!}} \psi_{m,\ell-1/2} \right) ,$$

$$m = 0, 1 \ldots, \ell - 1 \, .$$

and we had also the relation

$$\psi_{\ell-1/2,\ell-1-m} \propto \rho_2(\widetilde{\psi}_{\ell-1/2,n}) \, ,$$

$$\widetilde{\psi}_{\ell-1/2,n} \propto \rho_2(\psi_{\ell-1/2,\ell-1-m}) \, ,$$

Conformal symmetry as a Darboux chain

For the case $\nu > -1/2$ the kernel of the ladder operators are

$$\ker \mathcal{C}_\nu^{\pm} = \text{span}\{\psi_{\nu,\pm 0}, \psi_{-\nu-1,\pm 0}\}\,,$$

On the other hand, by using **D. T.**

Scheme	System	Intertwining operator
$(\psi_{\nu,0}, \psi_{-\nu-1,0})$	$\mathcal{H}_\nu + 4$	$-\mathcal{C}_\nu^-$
$(\psi_{\nu,-0}, \psi_{-\nu-1,-0})$	$\mathcal{H}_\nu - 4$	$-\mathcal{C}_\nu^+$

which implies

$$\mathcal{C}_\nu^- \phi = -\frac{W(\psi_{\nu,0}\psi_{-\nu-1,0}, \phi)}{W(\psi_{\nu,0}\psi_{-\nu-1,0})}\,,$$

$$\mathcal{C}_\nu^+ \phi = -\frac{W(\psi_{\nu,-0}\,, \psi_{-\nu-1,-0}, \phi)}{W(\psi_{\nu,-0}\,, \psi_{-\nu-1,-0})}\,,$$

where ϕ is an eingestate of $\mathcal{H}_\nu$.

The case $\nu = -1/2$ and the Confluent Darboux transformation (**C.D.T**)

In this case the kernel of the ladder operators are

$$\ker \mathcal{C}^{\pm}_{1/2} = \mathrm{span}\{\psi_{1/2,\pm 0}, \Omega_{-1/2,\pm 0}\}\,,$$

where

$$\Omega_{1/2,\pm 0} = (a_{\pm} - \ln(x))\psi_{-1/2,\pm 0}$$

are Jordan states ($a_{\pm}$ is a constant) which satisfies

$$\mathcal{H}_{\nu}\Omega_{\nu,\pm n} = \psi_{\nu,\pm n}$$

By using the (**C.D.T**)[9]

Scheme	System	Intertwining operator
$(\psi_{1/2,0}, \Omega_{1/2,0})$	$\mathcal{H}_{1/2} + 4$	$-\mathcal{C}^{-}_{1/2}$
$(\psi_{1/2,-0}, \Omega_{-1/2,-0})$	$\mathcal{H}_{1/2} - 4$	$-\mathcal{C}^{+}_{1/2}$

9 A. Schulze-Halberg, Eur. Phys. J. Plus **128** (2013) 68.

The action of $\mathcal{C}_\nu^\pm$ on the complete set of eigenstate

The picture is summarized in the following diagram

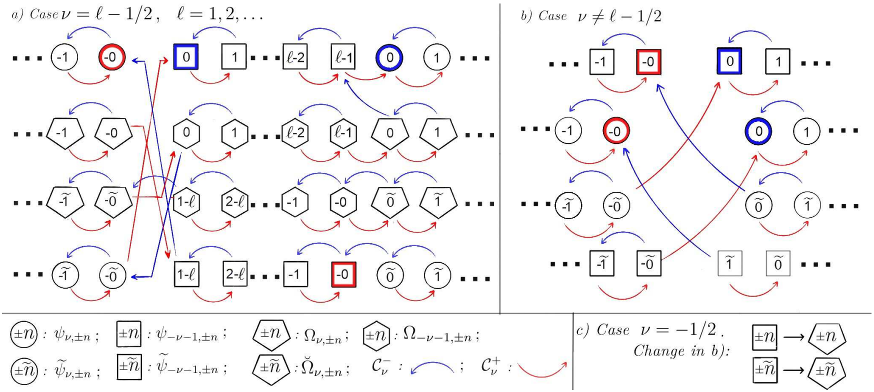

Figure: Action of $\mathcal{C}_\nu^\pm$ on the states. The red/blue market states are annihilated by red/blue arrows.

Selection Rules of states:

i) $\{\alpha_\nu^{KA}\} = (\psi_{\nu,l_1}, \psi_{\nu,l_1+1}, \ldots, \psi_{\nu,l_m}\psi_{\nu,l_m+1})$.

ii) $\{\alpha_\nu^{iso}\} = (\psi_{\nu,-s_1}, \ldots, \psi_{\nu,-s_m})$.

iii) $\{\gamma_\mu\} = (\psi_{-(\mu+m)-1,n_1}, \psi_{\mu+m,n_1-m}, \ldots, \psi_{-(\mu+m)-1,n_N}, \psi_{\mu+m,n_N-m})$.

where $-1/2 < \mu \leq 1/2$.

Note!

* When $\mu = 0$ we have deformations of the half-harmonic oscillator.
* When $\mu = 1/2$ we have $\{\gamma_\mu\} = \{\alpha_{m+1/2}^{KA}\}$ with $l_i = n_i - m - 1$.
* When $\mu = -1/2$ we have $W(\gamma_{-1/2}) = 0 \rightarrow$ Repeated states!.

By means of **D.T** we obtain the systems

Scheme	System	gaps
$\{\alpha_\nu^{KA}\}$	$\mathcal{H}_{\nu+m} + 4m + g_\nu(x)$	$12 + 8k$
$\{\alpha_\nu^{iso}\}$	$\mathcal{H}_{\nu+m} + 2m + f_\nu(x)$	0
$\{\gamma_\mu\}$	$\mathcal{H}_{\mu+m} + 4n + h_{\mu+m}$	$8 + 4k$

where k is the number of adjacent pairs of states in the scheme, and f_ν, g_ν and $h_{\mu+m}$ are rational functions.

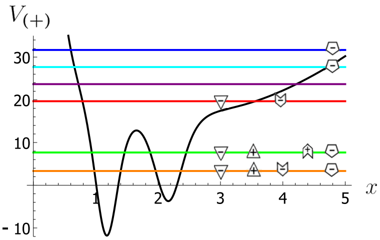

Figure: A rationally extended potential obtained by $\{\alpha_\nu^{KA}\} = (\psi_{\nu,2}, \psi_{\nu,3})$.

The corresponding rational functions has the following proprieties

* g_ν, f_ν, and $h_{\mu+m}$ does not have zeros in $\mathbb{R}^+$.
* g_ν, f_ν, and $h_{\mu+m}$ are zero in $x = 0$ and in $x = \infty$.
* f_ν is a convex function.
* $h_{\mu+\nu}$ does **not** vanish when $\mu = -1/2$!.

$h_{\mu+\nu}$ should by

$W(\{\gamma_\mu\}) = \text{Cons}(\mu)\tilde{W}_{\mu+m}(x)$ where $\text{Cons}(\mu = -1/2) = 0$
$\tilde{W}_{\mu+m}(x) \neq 0$.

The transformation which provide us the system

$$H_{-1/2+m} + 4n + h_{-1/2+m}\,,$$

in reality correspond to the **C.D.T** with the scheme

$$\{\gamma\} = (\psi_{-1/2+m,n_1}, \Omega_{-1/2+m,n_1-m}, \ldots, \psi_{-1/2+m,n_N}\Omega_{-1/2+m,n_N-m})\,,$$

and one can show that

$$\lim_{\mu\to-1/2} \frac{W(\{\gamma_\mu\})}{(\mu+1/2)^N} \propto W(\{\gamma\})\,.$$

Examples

The scheme $(\psi_{-\nu-1,2}, \psi_{\nu,2})$ with $-1/2 < \nu \leq 1/2$ in blue and $(\psi_{-1/2,2}, \Omega_{-1/2,2})$ in red

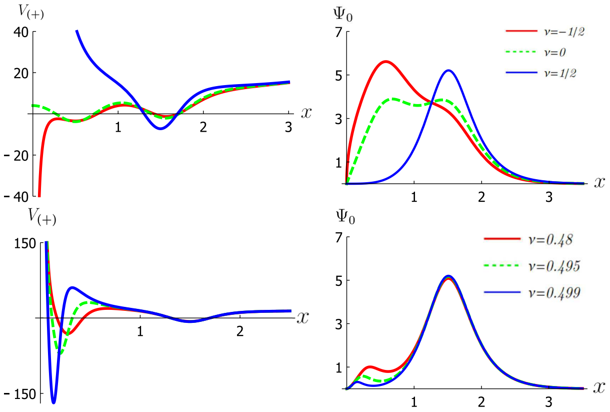

Figure: Potential and ground state in dependence of ν.

Darboux duality

In the non-half-integer ν case[1], for a given scheme

$$\{\Delta_+\} = (\psi_{\nu,k_1}, \ldots, \psi_{\nu,k_{N_1}}, \psi_{-\nu-1,l_1}, \ldots, \psi_{-\nu-1,l_{N_2}})$$

exist a "dual scheme" Δ_- (states $\rho_a(\psi_{\nu,n})$ with $a = 2, 3$ only) which satisfies

$$W(\Delta_+) \propto e^{-(n_N+1)x^2} W(\Delta_-)\,, \qquad N = \max(N_1, N_2)\,.$$

Δ_- is constructed using diagrams like the following

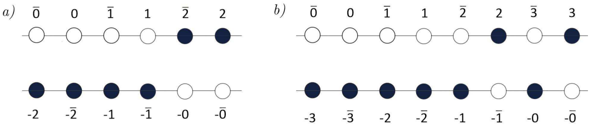

Figure: "Mirror diagrams". The numbers n indicate states $\psi_{\nu,n}$ and $\bar{n}$ indicate states $\psi_{-\nu-1,n}$ states.

[1] In the half integer ν case we include Jordan states.

Spectrum generating operators

By means of **D.T** we have

Scheme	System	Intertwining operator
$\{\Delta_+\}$	$\mathcal{H}_{(+)}$	$A^{\pm}_{(+)}$
$\{\Delta_-\}$	$\mathcal{H}_{(-)}$	$A^{\pm}_{(-)}$

Intertwining operators satisfies

$$A^{-}_{(\pm)}\mathcal{H}_{\nu} = \mathcal{H}_{(\pm)}A^{-}_{(\pm)}\,, \qquad A^{+}_{(\pm)}\mathcal{H}_{(\pm)} = \mathcal{H}_{\nu}A^{+}_{(\pm)}\,,$$

and the relation

$$\mathcal{H}_{(+)} - \mathcal{H}_{(-)} = \Delta E(n_N + 1)\,, \qquad \Delta E = 4\,.$$

One can construct the operators

$$\mathcal{A}_\nu^- = A_{(-)}^- \mathcal{C}_\nu^\pm A_{(-)}^+ \,, \qquad \mathcal{B}_\nu^- = A_{(+)}^- \mathcal{C}_\nu^\pm A_{(+)}^+ \,,$$

$$\mathcal{C}^- = A_{(-)}^- A_{(+)}^+ \,, \qquad \mathcal{C}^+ = A_{(+)}^- A_{(-)}^+ \,,$$

which satisfies

$$[\mathcal{H}_{(\pm)}, \mathcal{F}_a^\pm] = \pm R_a \mathcal{F}_a^\pm \,, \qquad [\mathcal{F}_a^-, \mathcal{F}_a^+] = \mathcal{P}_a(\mathcal{H}_{(\pm)}) \,,$$

a	$\mathcal{F}_a^\pm$	R_a
1	$\mathcal{A}^\pm$	ΔE
2	$\mathcal{B}^\pm$	ΔE
3	$\mathcal{C}^\pm$	$\Delta E(n_N + 1)$

and $\mathcal{P}_a(\zeta)$ are polynomial function in ζ.

Non-linear Newton-Hooke algebras

Constructing dynamics integrals of motions by

$$F_a^{\pm} \to e^{-i\mathcal{H}_{(\pm)}t}\mathcal{F}_a^{\pm}e^{i\mathcal{H}_{(\pm)}t} = e^{\mp Rt}F_a^{\pm}\,.$$

By take the linear combinations

$$\mathfrak{D}_a = \frac{(\mathcal{F}_a^{+} - \mathcal{F}_a^{-})}{2iR_a}\,, \qquad \mathfrak{K}_a = \frac{\mathcal{F}_a^{+} + \mathcal{F}_a^{+} + 2\mathcal{H}_{(\pm)}}{R_a^2}$$

we obtain

$$[\mathcal{H}_{(\pm)}, \mathfrak{D}_a] = i\left(\frac{R_a^2}{2} - \mathcal{H}_{(\pm)}\right)\,, \qquad [\mathcal{H}_{(\pm)}, \mathfrak{K}_a] = -2i\mathfrak{D}_a\,,$$

$$[\mathfrak{D}_a, \mathfrak{K}_a] = \frac{1}{iR_a^3}(\mathcal{P}_a(\mathcal{H}_{(\pm)}) - 2R_a\mathcal{H}_{(\pm)} + 3R_a^3\mathfrak{K}_a)\,.$$

The commutators $[\mathfrak{D}_a, \mathfrak{D}_b]$, $[\mathfrak{D}_a, \mathfrak{K}_b]$ and $[\mathfrak{K}_a, \mathfrak{K}_b]$ are in general different of 0!.

Darboux duality and super conformal algebra

By means of **D. T.** we have

Scheme	System	Intertwining Operators
$\psi_{\nu,0}$	$\mathcal{H}_{\nu+1}+2$	$A^{-}_{(+)}=\frac{d}{dx}+x-\frac{\nu+1}{x}$
$\psi_{\nu,-0}$	$\mathcal{H}_{\nu+m}-2$	$A^{-}_{(-)}=\frac{d}{dx}-x-\frac{\nu+1}{x}$

we can construct

$$\mathcal{H}^{e}_{\nu}=\begin{pmatrix} A_{(+)}A^{\dagger}_{(+)}=\mathcal{H}_{\nu+1}-2\nu-1 & 0 \\ 0 & A^{\dagger}_{(+)}A_{(+)}=\mathcal{H}_{\nu}-2\nu-3 \end{pmatrix},$$

$$\mathcal{H}^{b}_{\nu}=\begin{pmatrix} A_{(-)}A^{\dagger}_{(-)}=\mathcal{H}_{\nu+1}+2\nu+1 & 0 \\ 0 & A^{\dagger}_{(-)}A_{(-)}=\mathcal{H}_{\nu}+2\nu+3 \end{pmatrix},$$

$$\mathcal{E}_n=4n\,,\qquad \mathcal{E}_n=4n+4\nu+6\,.$$

$$\mathcal{H}_\nu^e \rightarrow \text{Exact supersymmetry.}$$

$$\mathcal{H}_\nu^b \rightarrow \text{Broken Supersymmetry.}$$

They are not independent

$$\mathcal{R}_\nu = \frac{1}{4}(\mathcal{H}_\nu^e - \mathcal{H}_\nu^b) = \frac{\sigma_3}{2} - (\nu+1)\mathbb{I}\,.$$

The rest of the generators of the $\mathfrak{osp}(2,2)$ algebra are given by 1

$$\mathcal{Q}_\nu^1 = \begin{pmatrix} 0 & A_{(+)}^- \\ A_{(+)}^+ & 0 \end{pmatrix}, \qquad \mathcal{S}_\nu^2 = \begin{pmatrix} 0 & A_{(-)}^- \\ A_{(-)}^+ & 0 \end{pmatrix},$$

$$\mathcal{Q}_\nu^2 = i\sigma_3 \mathcal{Q}_\nu^1\,, \qquad \mathcal{S}_\nu^1 = i\sigma_3 \mathcal{S}_\nu^1\,,$$

$$\mathcal{G}_\nu^\pm = \begin{pmatrix} \mathcal{C}_{\nu+1}^\pm & 0 \\ 0 & \mathcal{C}_\nu^\pm \end{pmatrix}.$$

The Lie superalgebraic relations

$$[\mathcal{H}^e_\nu, \mathcal{R}_\nu] = [\mathcal{H}^e_\nu, \mathcal{Q}^a_\nu] = 0\,,$$

$$[\mathcal{H}^e_\nu, \mathcal{G}^\pm_\nu] = \pm 4\mathcal{G}^\pm_\nu\,, \qquad [\mathcal{G}^-_\nu, \mathcal{G}^+_\nu] = 8\mathcal{H}^e_\nu - 16\mathcal{R}_\nu\,,$$

$$[\mathcal{H}^e_\nu, \mathcal{S}^a_\nu] = -4i\epsilon^{ab}\mathcal{S}^b_\nu\,, \qquad [\mathcal{R}_\nu, \mathcal{Q}^a_\nu] = -i\epsilon^{ab}\mathcal{Q}^b_\nu\,,$$

$$[\mathcal{R}_\nu, \mathcal{S}^a_\nu] = -i\epsilon^{ab}\mathcal{S}^b_\nu\,,$$

$$[\mathcal{G}^-_\nu, \mathcal{Q}^a_\nu] = 2(\mathcal{S}^a_\nu + i\epsilon^{ab}\mathcal{S}^b_\nu), \qquad [\mathcal{G}^+_\nu, \mathcal{Q}^a_\nu] = -2(\mathcal{S}^a_\nu - i\epsilon^{ab}\mathcal{S}^b_\nu)\,,$$

$$[\mathcal{G}^-_\nu, \mathcal{S}^a_\nu] = 2(\mathcal{Q}^a_\nu - i\epsilon^{ab}\mathcal{Q}^b_\nu)\,,$$

$$[\mathcal{G}^+_\nu, \mathcal{S}^a_\nu] = -2(\mathcal{Q}^a_\nu + i\epsilon^{ab}\mathcal{Q}^b_\nu)\,,$$

$$\{\mathcal{Q}^a_\nu, \mathcal{Q}^b_\nu\} = 2\delta^{ab}\mathcal{H}^e_\nu\,, \qquad \{\mathcal{S}^a_\nu, \mathcal{S}^b_\nu\} = 2\delta^{ab}(\mathcal{H}^e_\nu - 4\mathcal{R}_\nu)\,,$$

$$\{\mathcal{Q}^a_\nu, \mathcal{S}^b_\nu\} = \delta^{ab}(\mathcal{G}^+_\nu + \mathcal{G}^-_\nu) + i\epsilon^{ab}(\mathcal{G}^+_\nu - \mathcal{G}^-_\nu)\,.$$

The Klein-4 group and superconformal mechanics

Transformation ρ_1 First, one can see that $f = f^{-1}$ defined as

$$\mathcal{H}^e_\nu \to \mathcal{H}^e_\nu - 4\mathcal{R}_\nu = \mathcal{H}^b_\nu\,, \qquad \mathcal{R}_\nu \to -\mathcal{R}_\nu\,,$$

$$\mathcal{G}^\pm_\nu \to \mathcal{G}^\pm_\nu\,, \qquad \mathcal{Q}^1_\nu \to -\mathcal{S}^1_\nu\,,$$

$$\mathcal{Q}^2_\nu \to \mathcal{S}^2_\nu\,, \qquad \mathcal{S}^1_\nu \to -\mathcal{Q}^1_\nu\,, \qquad \mathcal{S}^2_\nu \to \mathcal{Q}^2_\nu\,,$$

is an automorphism. Then, the application on the generators ρ_1 correspond to

$$\rho_1(\mathcal{O}_\nu) = \sigma_1 f(\mathcal{O}_{\nu-1})\sigma_1\,.$$

Note

* For $\nu \neq -1/2$, the transformed generator satisfies de superconformal algebra, but the new Hamiltonian is in broken phase.
* For $\nu = -1/2$ the transformed Hamiltonian is just $\sigma_1(\mathcal{H}^e_{-1/2})\sigma_1$ which is in unbroken phase.

Transformation ρ_2: By directly application we have

$$\rho_2(\mathcal{H}^e_\nu) = -\mathcal{H}^b_\nu\,, \quad \rho_2(\mathcal{G}^\pm_\nu) = -\mathcal{G}^\mp_\nu\,, \quad \rho_2(\mathcal{R}_\nu) = \mathcal{R}_\nu\,,$$

$$\rho_2(\mathcal{Q}^1_\nu) = -i\mathcal{S}^1_\nu\,, \qquad \rho_2(\mathcal{Q}^2_\nu) = -i\mathcal{S}^2_\nu\,,$$

$$\rho_2(\mathcal{S}^1_\nu) = -i\mathcal{Q}^1_\nu\,, \qquad \rho_2(\mathcal{S}^2_\nu) = -i\mathcal{Q}^2_\nu\,,$$

Note:

The generators satisfies the conformal algebra, but the "Hamiltonian" of the system has negatives energies (not physical).

Summary

i) Klein 4 group is related with the conformal symmetry.

ii) The action of ladder operators on eigenstates (conformal symmetry) can be understood as Darboux transformations.

iii) Half integer values of ν have special proprieties at the level of eigenstates.

iv) Application: new rationally extended potentials and spectrum generating ladder operators.

Figure: The mountain and the man...

Thank you very much!