Quantum Field Theory

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Universidad de Salamanca

Taller de Altas Energías 2019, Benasque

A list of topics

* From particles to fields to scattering amplitudes
* Perturbation theory and diagrams
* Loops and divergences
* Renormalization

Schedule

* **Lectures**:
  - Monday 9th, 9:00 to 11:00 (with a 5 minutes break)
  - Tuesday 10th, 9:00 to 10:00
  - Wednesday 11th, 11:30 to 12:30

* **Tutorials**:
  - Monday 9th, 16:00 to 17:00
  - Wednesday 11th, 17:00 to 18:00

Tutor: Jorge Alda.

A sample of textbooks

* L. Álvarez-Gaumé & M.A. Vázquez-Mozo, "An Invitation to Quantum Field Theory", Springer 2012.
* M.E. Peskin & D.V. Schroeder, "An Introduction to Quantum Field Theory", Perseus Books 1995.
* C. Quigg, "Gauge theories of the strong, weak, and electromagnetic interactions" (2nd edition), Princeton University Press 2013.
* M. Schwartz, "Quantum field theory and the standard model", Cambridge 2014.

A note about conventions:

✱ We use the "**mostly minus**" metric (a.k.a. West Coast metric):

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

✱ Unless otherwise said, **natural units** are used throughout:

$$\hbar = c = 1$$

✱ We use **Heaviside-Lorentz** electromagnetic units:

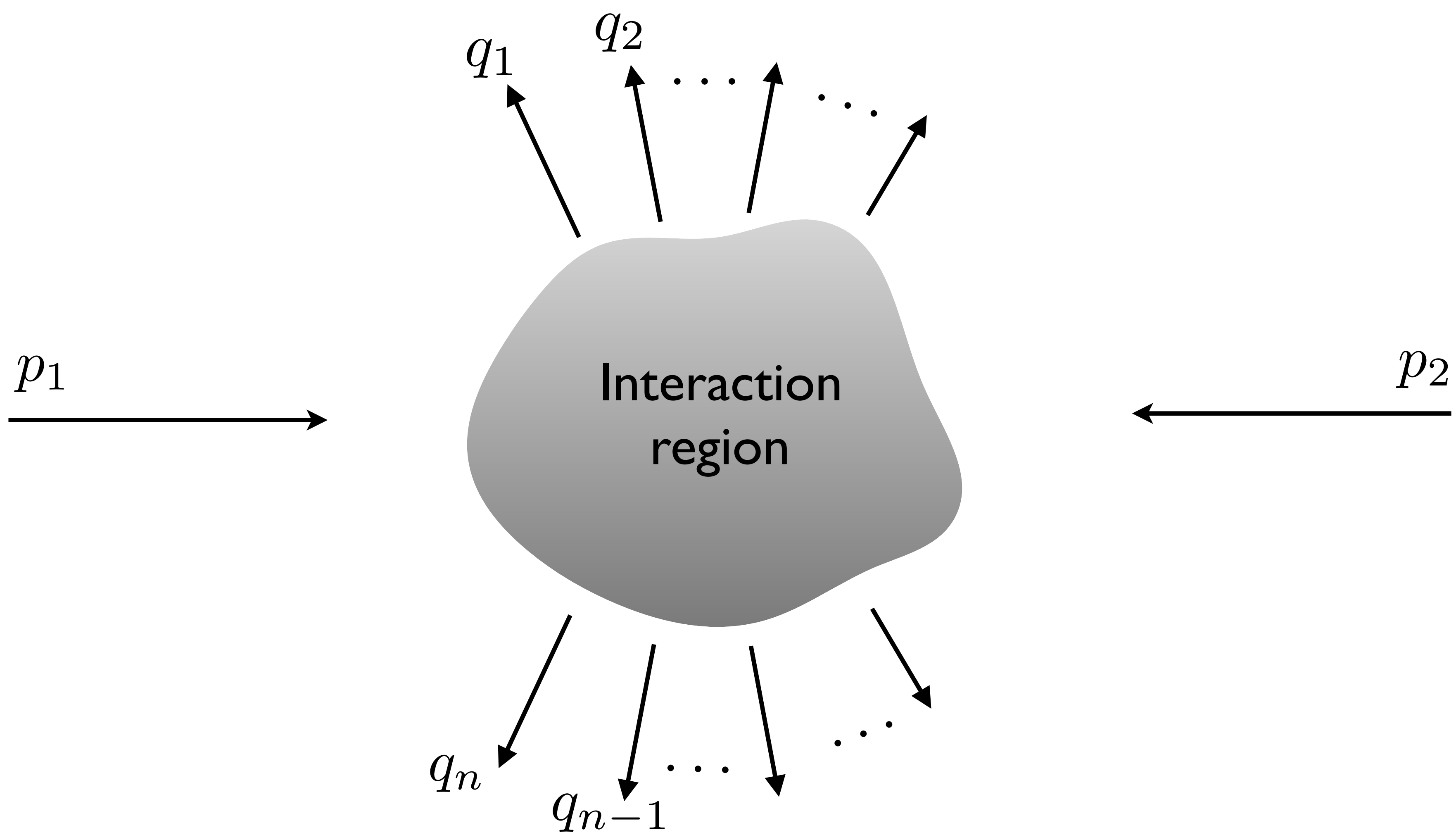
$$\begin{aligned} \boldsymbol{\nabla} \cdot \mathbf{E} &= \rho \\ \boldsymbol{\nabla} \cdot \mathbf{B} &= 0 \\ \boldsymbol{\nabla} \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\ \boldsymbol{\nabla} \times \mathbf{B} &= \mathbf{j} + \frac{\partial \mathbf{E}}{\partial t} \end{aligned}$$

$$\alpha = \frac{e^2}{4\pi}$$ (fine structure constant)

Part I

From elementary particles to quantum fields

Elementary particles are studied through **scattering experiments**, typically



Quantum mechanics, even relativistic, is **not enough** to describe these **high energy** experiments…

Let us consider the **relativistic quantum** evolution of a **localized, single-particle wave packet**:

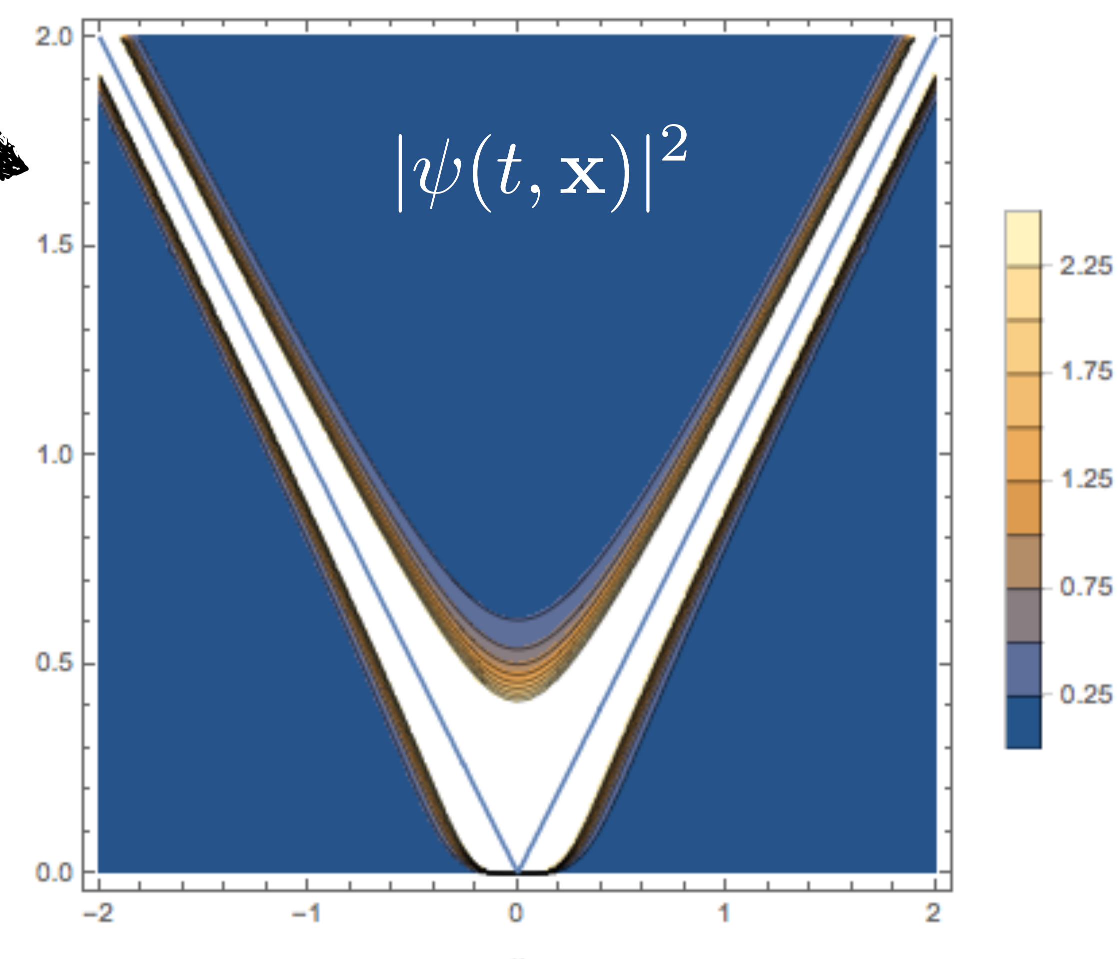
$$\psi(t,\mathbf{x}) = e^{-it\overbrace{\sqrt{-\nabla^2+m^2}}^{H=\sqrt{p^2+m^2}}}\delta^{(3)}(\mathbf{x}) = \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}-it\sqrt{k^2+m^2}}$$

positive frequency

$$= \frac{1}{4\pi^2}\int_0^\infty k^2 dk \int_{-1}^{1} d(\cos\theta) e^{ik|\mathbf{x}|\cos\theta - it\sqrt{k^2+m^2}} = \frac{1}{2\pi^2|\mathbf{x}|}\int_0^\infty kdk\ \sin(k|\mathbf{x}|)e^{-it\sqrt{k^2+m^2}}$$

The integral can be regularized by $t \to t - i\epsilon$, to give

$$\psi(t,\mathbf{x}) = -\frac{i}{2\pi^2}\frac{m^2 t}{t^2-\mathbf{x}^2}K_2\left(im\sqrt{t^2-\mathbf{x}^2}\right)$$



The probability $|\psi(t,\mathbf{x})|^2$ **spills** outside the light-cone

Causality is violated!

Let us consider the **relativistic quantum** evolution of a **localized, single-particle wave packet**:

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$$\psi(t,\mathbf{x}) = e^{-it\overbrace{\sqrt{-\nabla^2+m^2}}^{H=\sqrt{p^2+m^2}}}\delta^{(3)}(\mathbf{x}) = \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}-it\sqrt{k^2+m^2}}$$

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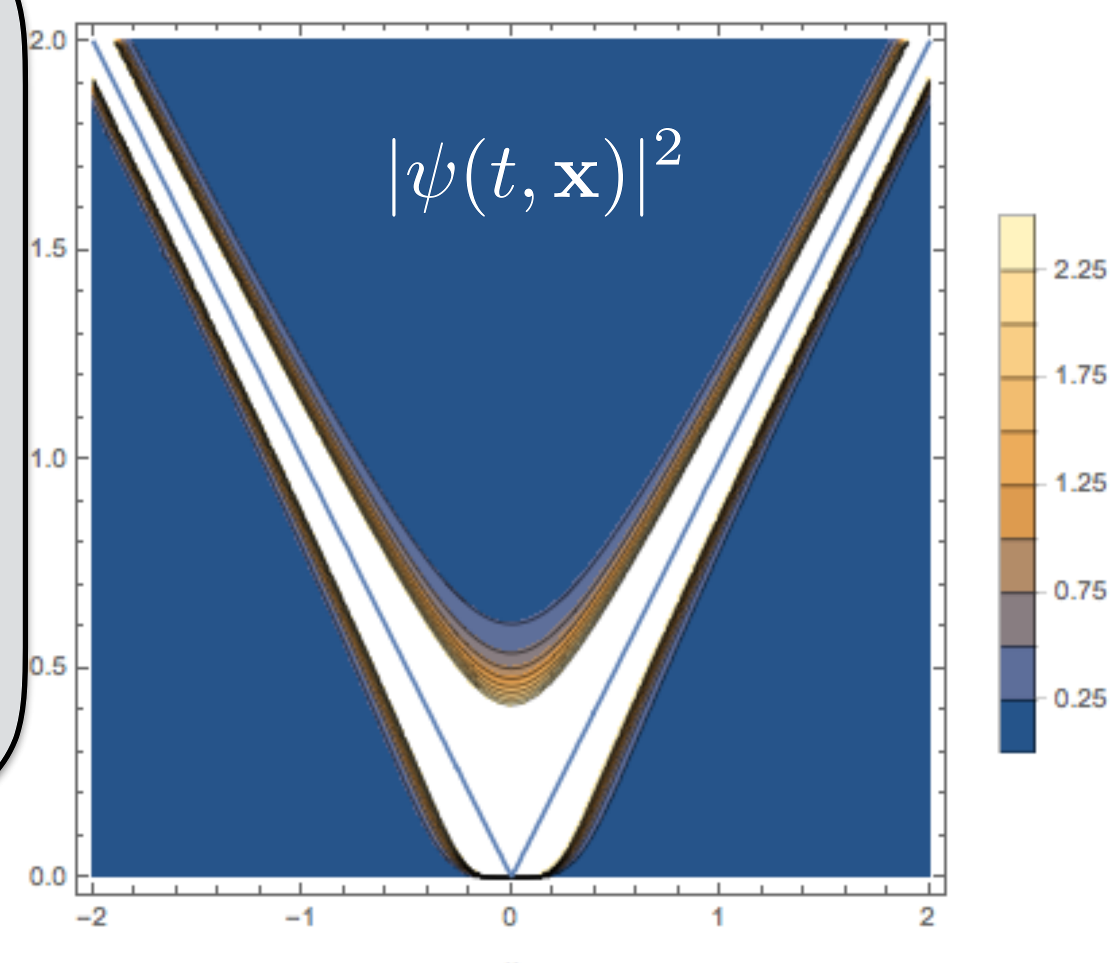
What we are computing is the **propagator** of a relativistic particle:

$$\psi(t,\mathbf{x}) = \langle \mathbf{x}|e^{-itH}|\mathbf{0}\rangle \equiv G(t,\mathbf{x};0,\mathbf{0})$$

Relativistic quantum mechanics propagates states **outside** the light cone

$$G(t',\mathbf{x}';t,\mathbf{x}) \neq 0 \quad \text{when} \quad (t'-t)^2-(\mathbf{x}'-\mathbf{x})^2 < 0$$

Causality is violated!



But when $t^2 - \mathbf{x}^2 < 0$ there are **frames** in which $(t, \mathbf{x})$ **happens before** $(0, \mathbf{0})$. Thus,

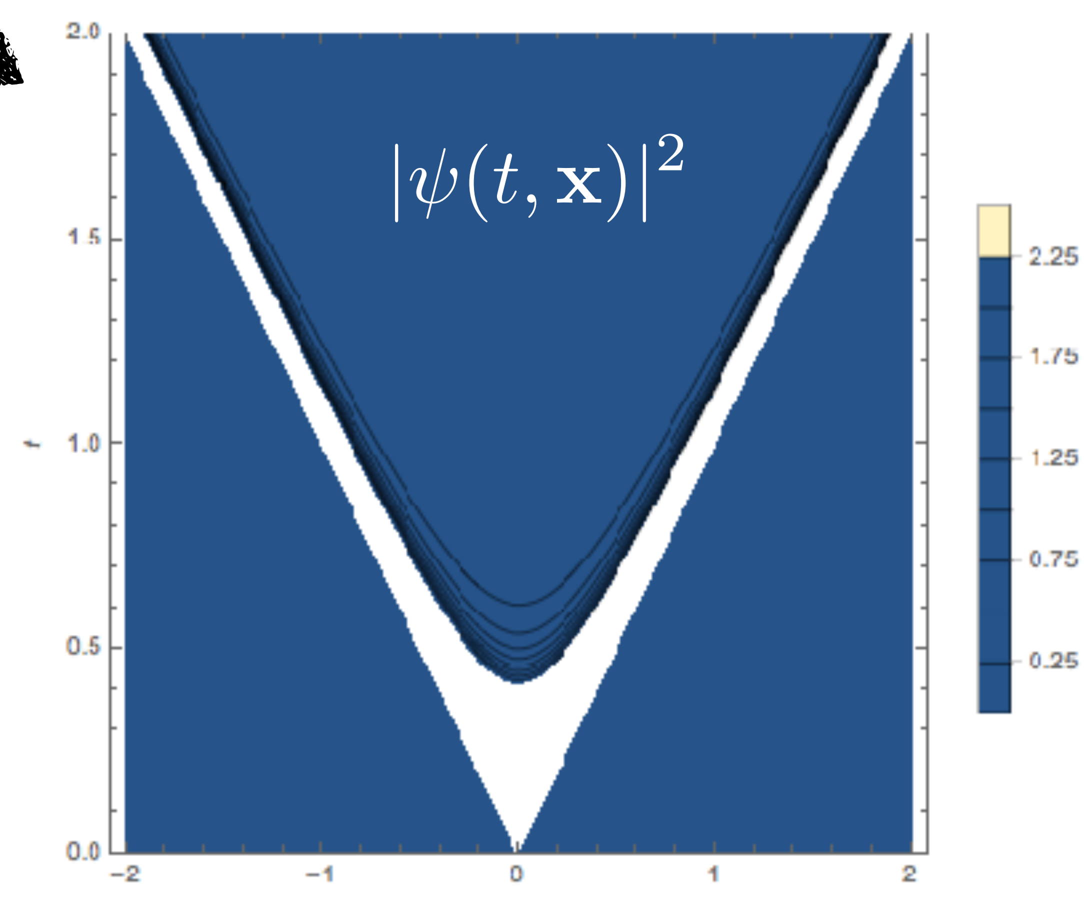
$$\psi(t,\mathbf{x}) = \begin{cases} \langle \mathbf{x}|e^{-itH}|\mathbf{0}\rangle \quad \text{when} \quad t^2 - \mathbf{x}^2 > 0 \\ \\ \langle \mathbf{x}|e^{-itH}|\mathbf{0}\rangle + \langle \mathbf{0}|e^{itH}|\mathbf{x}\rangle = 2\mathrm{Re}\,\langle \mathbf{x}|e^{-itH}|\mathbf{0}\rangle \quad \text{when} \quad t^2 - \mathbf{x}^2 < 0 \end{cases}$$

But since we have computed

$$\langle \mathbf{x}|e^{-itH}|\mathbf{0}\rangle = -\frac{i}{2\pi^2}\frac{m^2 t}{t^2 - \mathbf{x}^2} K_2\left(im\sqrt{t^2 - \mathbf{x}^2}\right)$$

the result is

$$\psi(t,\mathbf{x}) = -\frac{i}{2\pi^2}\frac{m^2 t}{t^2 - \mathbf{x}^2} K_2\left(im\sqrt{t^2 - \mathbf{x}^2}\right)\theta(t^2 - \mathbf{x}^2)$$



Causality is restored!

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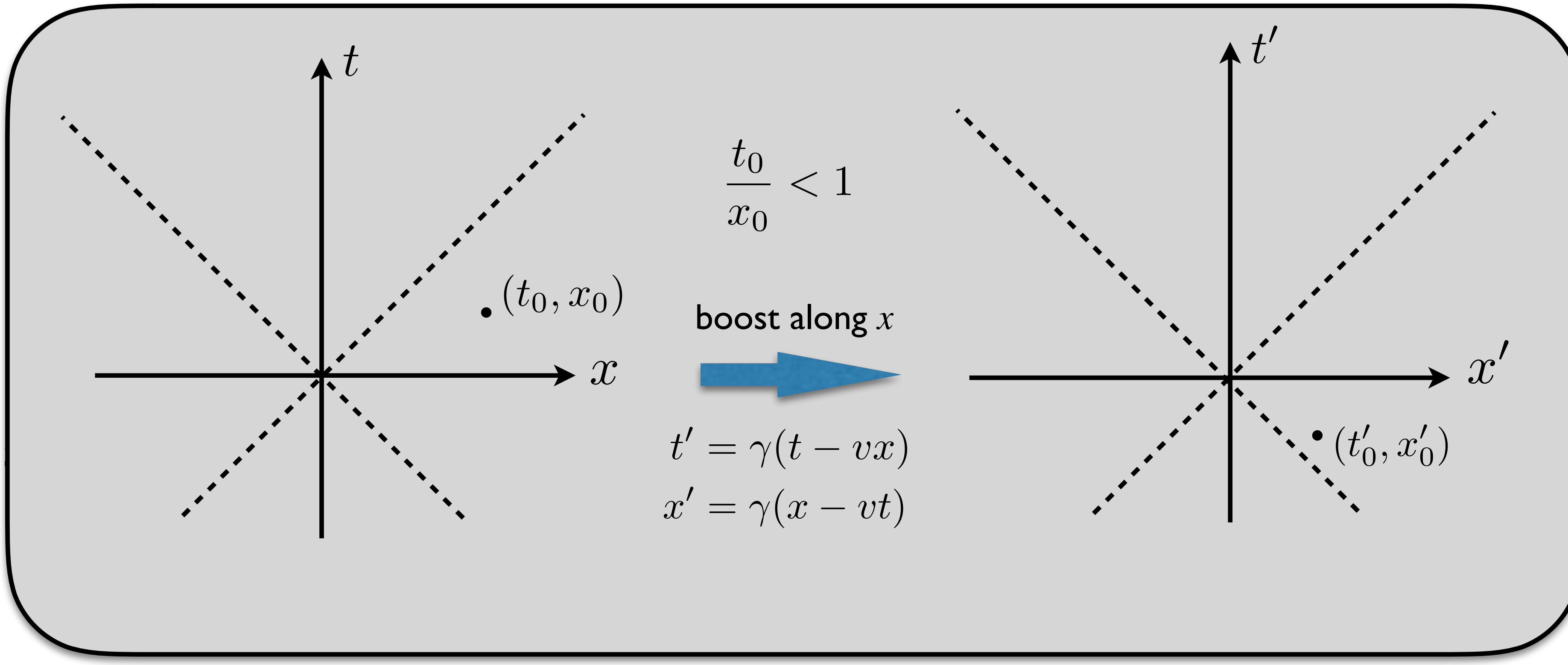

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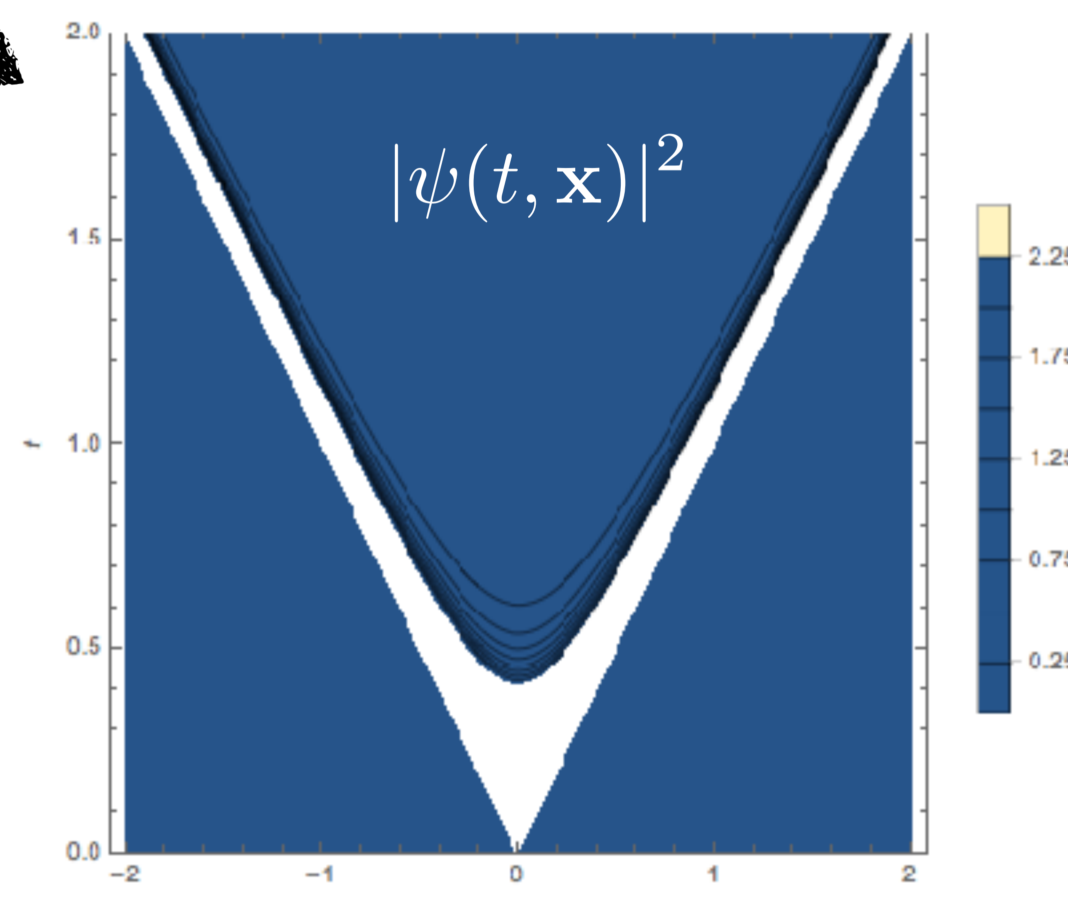

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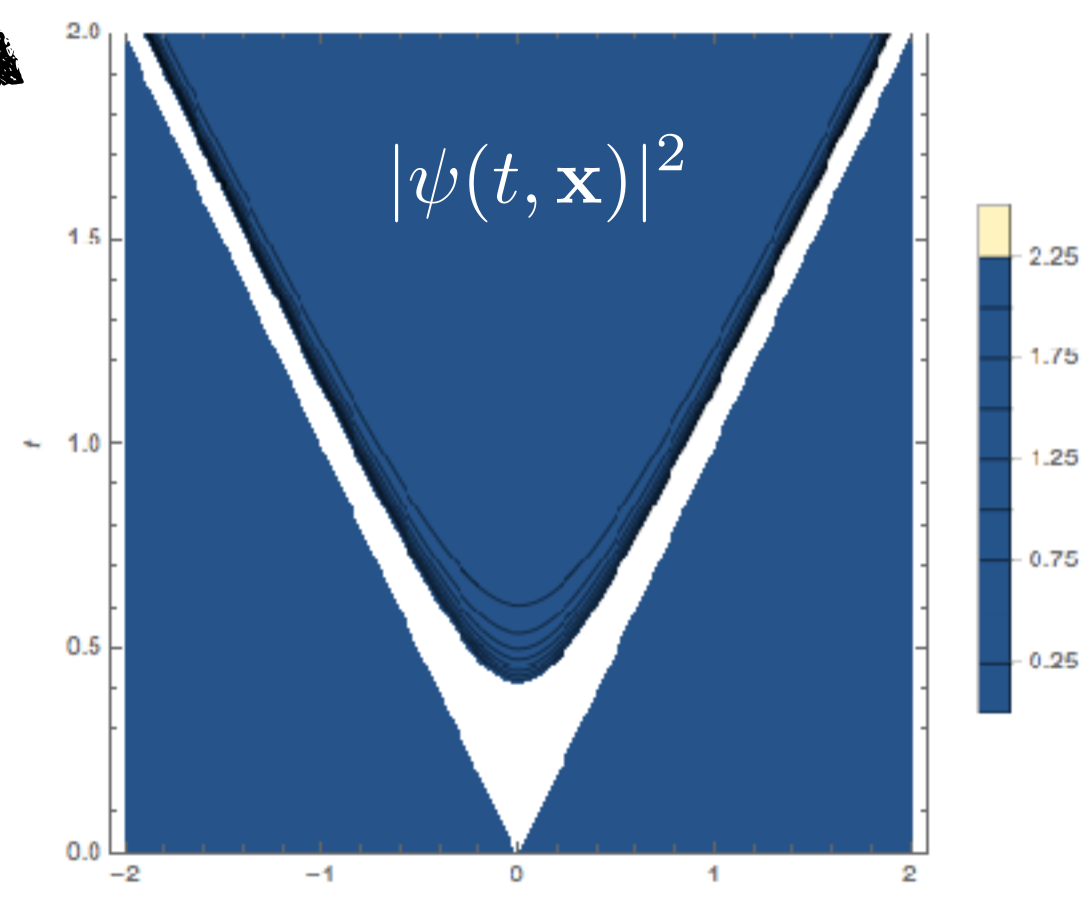

Causality is restored!

We have to allow **particles travelling backward in time**!!

Their wave functions are

$$\psi(t,\mathbf{x})_{\Downarrow} = \langle \mathbf{0}|e^{itH}|\mathbf{x}\rangle = \langle \mathbf{x}|e^{-itH}|\mathbf{0}\rangle^{*} = \psi(t,\mathbf{x})^{*}_{\Uparrow}$$

Ernst Stückelberg
(1905-1984)

Richard Feynman
(1918-1988)

Thus, under any **global** $\mathrm{U}(1)$ **symmetry**

$$\psi(t,\mathbf{x})_{\Uparrow} \longrightarrow e^{iq\theta}\psi(t,\mathbf{x})_{\Uparrow} \quad \Rightarrow \quad \psi(t,\mathbf{x})_{\Downarrow} \longrightarrow e^{-iq\theta}\psi(t,\mathbf{x})_{\Downarrow}$$

these particles have **oposite charges**, $q_{\Downarrow} = -q_{\Uparrow}$ (but the same mass! $H_{\Uparrow,\Downarrow} = \sqrt{-\nabla^2 + m^2}$)

To **restore causality** we are forced to introduce **antiparticles**!!

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States moving backward in time can be reinterpreted as **negative frequency states** with **reversed** momentum, propagating **forward** in time:

$$\psi(t,\mathbf{x})_{\Downarrow} = \int \frac{d^3k}{(2\pi)^3} e^{-i\mathbf{k}\cdot\mathbf{x}+it\sqrt{k^2+m^2}}$$

$$= \int \frac{d^3k}{(2\pi)^3} e^{i(-\mathbf{k})\cdot\mathbf{x}-it(-\sqrt{k^2+m^2})}$$

negative frequency

$\to e^{-iq\theta}\psi(t,\mathbf{x})_{\Downarrow}$

mass! $H_{\Uparrow,\Downarrow} = \sqrt{-\nabla^2+m^2}$)

To **restore causality** we are forced to introduce **antiparticles**!!

Switching on **interactions**, charge conservation allows the creation of **particle-antiparticle pairs**, provided **enough energy** is available.

For example, localizing particle below their **Compton wavelength**

$$\Delta x \sim \frac{1}{m} \xrightarrow{\Delta x \Delta p \sim 1} \Delta p \sim m \longrightarrow \Delta E \sim m$$

and due to energy **quantum fluctuations** the creation of particle-antiparticle pairs **cannot be prevented**.

$$\Downarrow$$

We have to **give up** the **single-particle description**!

Thus, **relativistic quantum mechanics** is a **dead end** for high energy particle physics…

To handle many particles, **second quantization** seems the best approach, introducing **creation-annihilation operators** for particles with **on-shell momentum** p

$$a(p),\ \ a(p)^\dagger \quad \xrightarrow[\omega_{\mathbf{p}}=\sqrt{\mathbf{p}^2+m^2}]{p^2=m^2} \quad \begin{aligned} &[a(p),a(p')^\dagger]=(2\pi)^3(2\omega_{\mathbf{p}})\delta^{(3)}(\mathbf{p}-\mathbf{p}') \\ &[a(p),a(p')]=[a(p)^\dagger,a(p')^\dagger]=0 \end{aligned}$$

Lorentz invariant (exercise)

(Multi-)particle states are obtained from the **Poincaré-invariant vacuum** $|0\rangle$

$$|p\rangle=a(p)^\dagger|0\rangle \quad \longrightarrow \quad \langle p|p'\rangle=(2\pi)^3(2\omega_{\mathbf{p}})\delta^{(3)}(\mathbf{p}-\mathbf{p}')$$

Lorentz invariant (exercise)

$$|f\rangle=\int\left[\prod_{i=1}^{n}\frac{d^3p_i}{(2\pi)^3}\frac{1}{2\omega_{\mathbf{p}_i}}\right]f(\mathbf{p}_1,\ldots,\mathbf{p}_n)a(p_1)^\dagger\ldots a(p_n)^\dagger|0\rangle$$

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Lorentz invariant (exercise)

$$\mathscr{U}(\Lambda)|0\rangle = e^{-ia\cdot P}|0\rangle = |0\rangle$$

$$\mathscr{U}(\Lambda)a(p)\mathscr{U}(\Lambda)^\dagger = a(\Lambda p)$$

$$\mathscr{U}(\Lambda)|p\rangle = |\Lambda p\rangle$$

where $\mathscr{U}(\Lambda)\in \mathrm{SO(1,3)}$

$${}_1,\ldots,\mathbf{p}_n)a(p_1)^\dagger\ldots a(p_n)^\dagger|0\rangle$$

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Lorentz invariant (exercise)

Free fields are **linear combinations** of creation-annihilation operators. E.g., for a free **Hermitian** scalar field

$$\phi(x) = \phi(x)^\dagger \longrightarrow \phi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_{\mathbf{p}}}\Big[f(x,\mathbf{p})a(p) + f(x,\mathbf{p})^* a(p)^\dagger\Big]$$

Imposing the **equations of motion**,

$$(\Box + m^2)\phi(x) = 0 \longrightarrow f(x,\mathbf{p}) = e^{-i\omega_{\mathbf{p}}t + i\mathbf{p}\cdot\mathbf{x}}$$

$$p^2 = m^2 \longrightarrow \omega_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2}$$

The free quantum field satisfies:

* **Equal-time** canonical commutation relations

$$[\phi(t,\mathbf{x}), \dot{\phi}(t,\mathbf{x}')] = i\delta^{(3)}(\mathbf{x}-\mathbf{x}'), \qquad [\phi(t,\mathbf{x}), \phi(t,\mathbf{x}')] = [\dot{\phi}(t,\mathbf{x}), \dot{\phi}(t,\mathbf{x}')] = 0$$

* **Microcausality**

$$[\phi(x), \phi(x')] = 0 \qquad \text{when} \qquad (x-x')^2 < 0$$

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positive frequency

$$\phi(x) = \phi(x)^\dagger \quad \longrightarrow \quad \phi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_{\mathbf{p}}}\left[e^{-i\omega_{\mathbf{p}}t+i\mathbf{p}\cdot\mathbf{x}}a(p) + e^{i\omega_{\mathbf{p}}t-i\mathbf{p}\cdot\mathbf{x}}a(p)^\dagger\right]$$

negative frequency

Imposing the **equations of motion**,

$$(\Box + m^2)\phi(x) = 0 \quad \longrightarrow \quad f(x,\mathbf{p}) = e^{-i\omega_{\mathbf{p}}t+i\mathbf{p}\cdot\mathbf{x}}$$

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The **many-particle** Fock states **diagonalize** the free field **Hamiltonian**

IR divergence

$$H=\frac{1}{2}\int d^3x\left[\dot{\phi}^2+(\boldsymbol{\nabla}\phi)^2+m^2\phi^2\right] \xrightarrow{\text{(exercise)}} H=\frac{1}{2}\int\frac{d^3p}{(2\pi)^3}\left[a(p)^\dagger a(p)+(2\pi)^3\omega_{\mathbf{p}}\delta^{(3)}(\mathbf{0})\right]$$

$$=\int\frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_{\mathbf{p}}}\left[\omega_{\mathbf{p}}a(p)^\dagger a(p)\right]+E_0$$

$$\boxed{E_0=\langle 0|H|0\rangle=\frac{V}{(2\pi)^3}\sum_{\mathbf{p}}\frac{1}{2}\omega_{\mathbf{p}}}$$

UV divergence

Subtracting the (divergent) zero-point energy E_0

$$[a(p),a(p')^\dagger]=(2\pi)^3(2\omega_{\mathbf{p}})\delta^{(3)}(\mathbf{p}-\mathbf{p}')$$

$$H|p\rangle=\frac{1}{2}\int\frac{d^3k}{(2\pi)^3}a(k)^\dagger a(k)a(p)^\dagger|0\rangle=\omega_{\mathbf{p}}|p\rangle$$ (exercise)

$$H|p_1,\ldots,p_n\rangle\equiv H\,a(p_1)^\dagger\ldots a(p_n)^\dagger|0\rangle=\left(\sum_{i=1}^{n}\omega_{\mathbf{p}_i}\right)|p_1,\ldots,p_n\rangle$$ (exercise)

Particles are the **low-lying excitations** of **quantum fields**

A **particle** is characterized by a number of **"Casimirs"**:

✱Poincaré group: $\begin{cases} P_\mu P^\mu = m^2 \\ \\ W_\mu W^\mu = -m^2 s(s+1) \\ W_\mu W^\mu \neq 0 \text{ for } m=0 \\ \text{(generically)} \end{cases}$

$P_\mu P^\mu = m^2 \longrightarrow$ **Mass**

$W_\mu W^\mu \longrightarrow$ **Spin**

$$W^\mu = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}J_{\nu\alpha}P_\beta$$

Pauli-Lubański vector

$$W^0 = -\mathbf{J}\cdot\mathbf{P}$$

$\longrightarrow$ helicity

For $m \neq 0$, in the rest frame

$$W^0 = 0, \;\; \mathbf{W} = m\mathbf{J}$$

✱Internal symmetry groups: $\begin{cases} \text{electric charge} \\ \vdots \end{cases}$

To do particle physics, we have to **choose** the appropriate **interpolating field:**

✱It **transforms correctly** (i.e., the right value for the "Casimirs")

✱It **creates** the corresponding particle out of the **vacuum**:

$$\langle 0|\phi(x)|p\rangle \neq 0$$

The x-dependence is **fixed** by the Poincaré invariance of the vacuum

$$\langle 0|\phi(x)|p\rangle = \langle 0|e^{iP\cdot x}\phi(0)e^{-iP\cdot x}|p\rangle = \langle 0|\phi(0)|p\rangle e^{-ip\cdot x}$$

The fields can be **canonically normalized**, such that:

* **Scalar field**: $\langle 0|\phi(0)|p\rangle = 1$

* **Dirac field**: $\begin{cases} \langle 0|\psi_\alpha(0)|p,\sigma;0\rangle = u_\alpha^{(\sigma)}(p) \\ \langle 0|\overline{\psi}_\alpha(0)|0;p,\sigma\rangle = \overline{v}_\alpha^{(\sigma)}(p) \end{cases}$

* **Photon field**: $\langle 0|A_\mu(0)|p,\lambda\rangle = \varepsilon_\mu^{(\lambda)}(p)$

Any properly normalized interpolating field **does the job**, provided it satisfies **microcausality**

$$[\phi(x),\phi(x')] = 0 \qquad \text{when} \qquad (x-x')^2 < 0$$

(resp. anticommutators for Fermi fields)

→ Borchers classes

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The fields can be **canonically normalized**, such that:

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To describe massive scalar particles, instead of $\phi(x)$ we can also use

$$\Phi(x) = -\frac{1}{m^2}\Box\phi(x)$$

$$\langle 0|\Phi(x)|p\rangle = -\frac{1}{m^2}\Box\langle 0|\phi(x)|p\rangle = -\frac{1}{m^2}\Box e^{-ip\cdot x} = e^{-ip\cdot x}$$

and

$$[\Phi(x),\Phi(x')] = \frac{1}{m^4}\Box_x\Box_{x'}[\phi(x),\phi(x')] = 0$$

for $(x-x')^2 < 0$

Borchers classes

Still, to do particle physics we need to **introduce interactions**…

In interacting field theories, **particles** still emerge as **weakly coupled excitations:**

dimensionless
effective couplings

$$\mathscr{L}(\phi,\partial\phi)=\mathscr{L}(\phi,\partial\phi)_{\rm free}+\sum_i g_i\mathscr{O}_i(\phi,\partial\phi)$$

nonquadratic terms

where g_i is **"small"** and

$$\langle\Omega|\phi(x)|p\rangle=\phi(\mathbf{p})e^{-ip\cdot x}$$

ground state of
the **full** theory

one-particle
wave function

Thus:

* Particles are identified by **quantizing** the **free theory**.

* Interactions are treated in **perturbation theory**.

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Examples

∗**QED**: electrons+photons

$$\mathscr{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \overline{\psi}(\gamma^\mu\partial_\mu - m)\psi - qA_\mu\overline{\psi}\gamma^\mu\psi$$

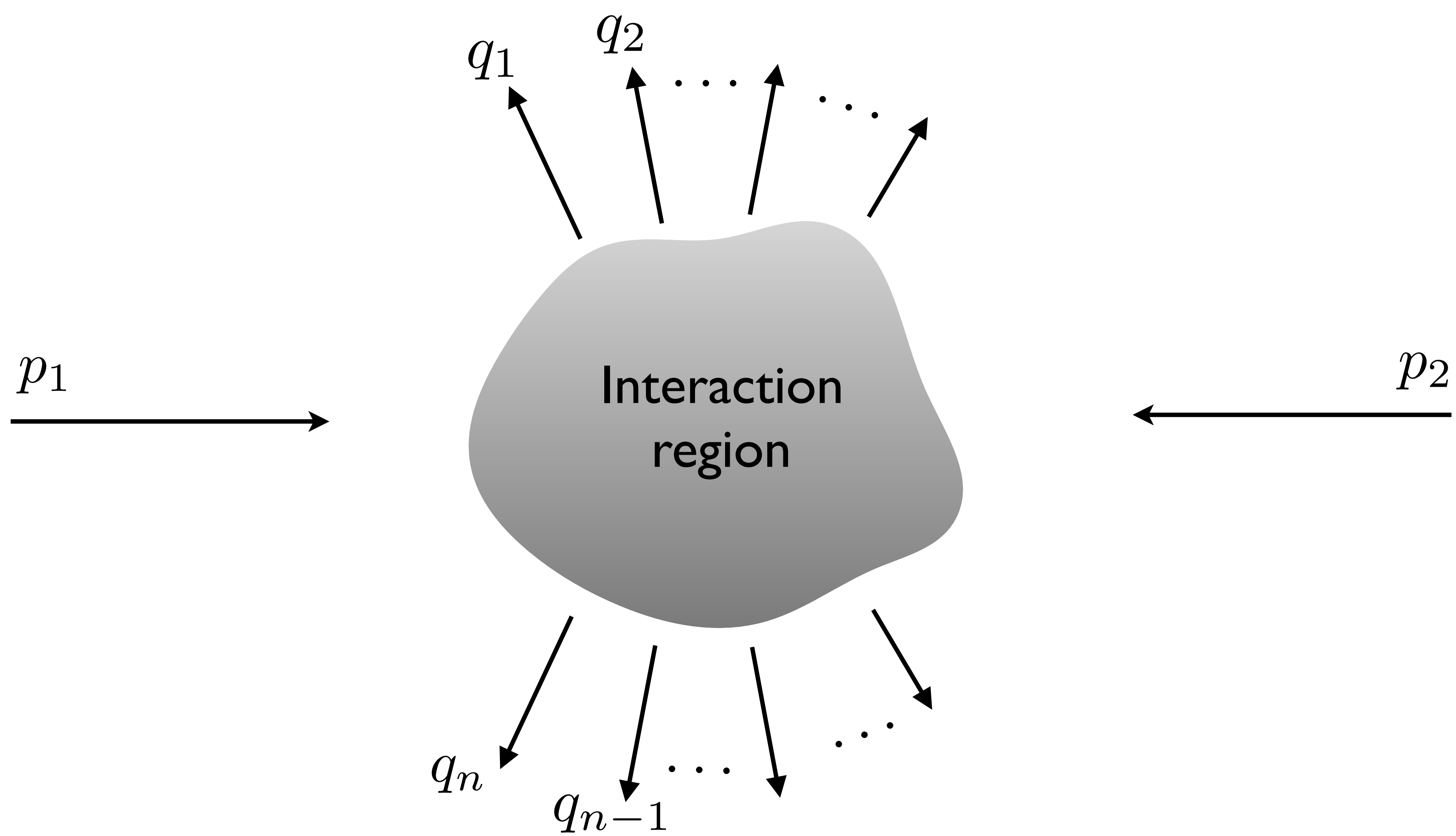
∗**QCD** (**high energies**): quarks+gluons

$$\mathscr{L} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + \sum_{f=1}^{6}\overline{Q}_f(\gamma^\mu\partial_\mu - m_f)Q_f - gA^a_\mu\sum_{f=1}^{6}\overline{Q}_f T^a\gamma^\mu Q_f$$

∗**QCD** (**low energies**): pions (+nucleons)

$$\mathscr{L} = \frac{1}{2}\mathrm{tr}\left(\partial_\mu\boldsymbol{\pi}\partial^\mu\boldsymbol{\pi}\right) - \frac{1}{3f_\pi^2}\mathrm{tr}\Big(\partial_\mu\boldsymbol{\pi}[\boldsymbol{\pi},[\boldsymbol{\pi},\partial^\mu\boldsymbol{\pi}]]\Big) + \ldots$$

∗ Interactions are treated in **perturbation theory**.



A **scattering** experiment is characterized by its initial (*in*) and final (*out*) multiparticle state:

$$|p_1, p_2\rangle_{\text{in}} \qquad\qquad |q_1, q_2, \ldots, q_{n-1}, q_n\rangle_{\text{out}}$$

Both are **Heisenberg-picture** (i.e., time-independent) **states** in a very complicated **interacting theory**.

Our aim is to **compute** the **probability amplitude**:

$$S(i \longrightarrow f) = {}_{\text{out}}\langle q_1, \ldots, q_n | p_1, p_2\rangle_{\text{in}}$$

$$|p_1, p_2\rangle_{\rm in} \qquad\qquad |q_1, q_2, \ldots, q_{n-1}, q_n\rangle_{\rm out}$$

These states can also be seen as belonging to the **free, multiparticle Fock space**

$$|p_1, p_2\rangle, |q_1, q_2, \ldots, q_n\rangle \in \mathscr{F} \equiv \bigoplus_{n=0}^{\infty} \mathscr{H}_1 \otimes \overset{(n)}{\cdots} \otimes \mathscr{H}_1$$

The scattering experiment is then described by the ***S*-matrix operator**

$$S : \mathscr{F} \longrightarrow \mathscr{F}$$

$$S(i \longrightarrow f) = {}_{\rm out}\langle q_1, \ldots, q_n | p_1, p_2\rangle_{\rm in} \equiv \langle q_1, \ldots, q_n | S | p_1, p_2\rangle$$

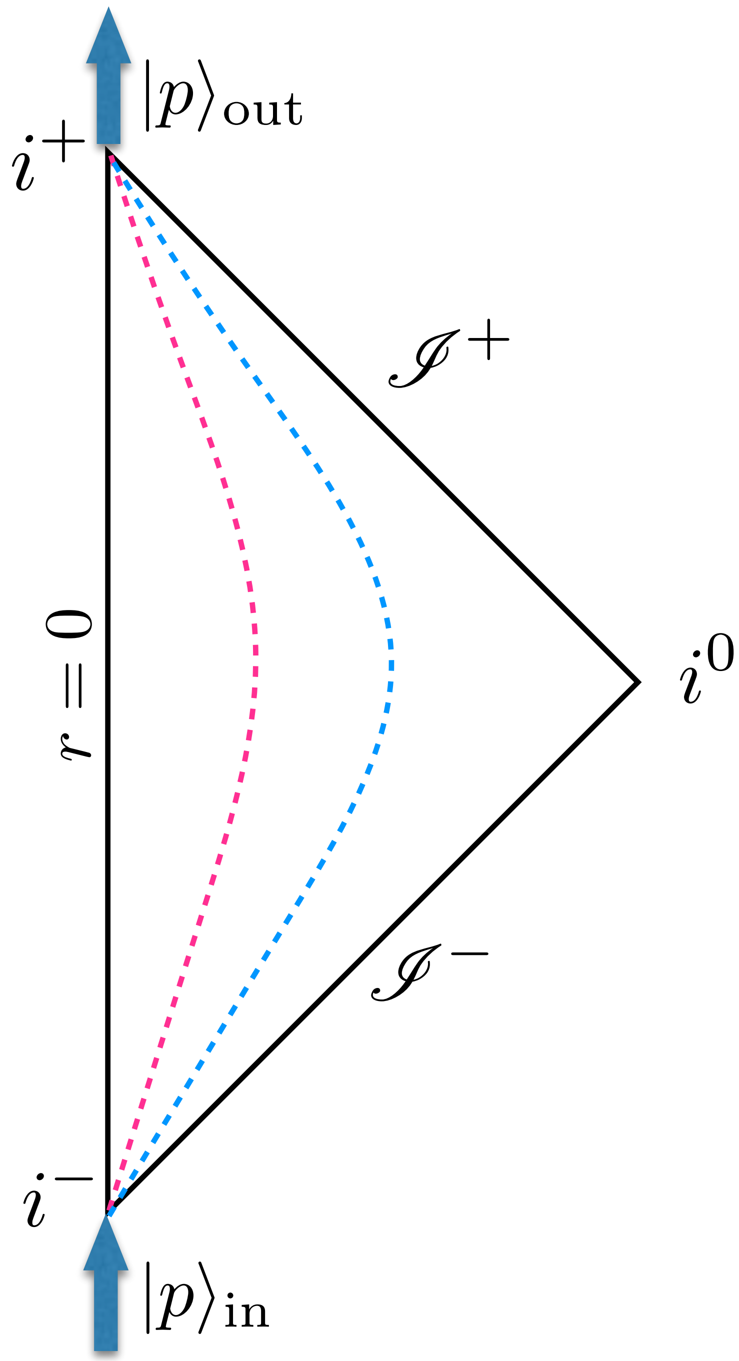
The ***S*-matrix** operator satisfies a number of **properties**:

* **Unitarity**: $S^\dagger = S^{-1}$

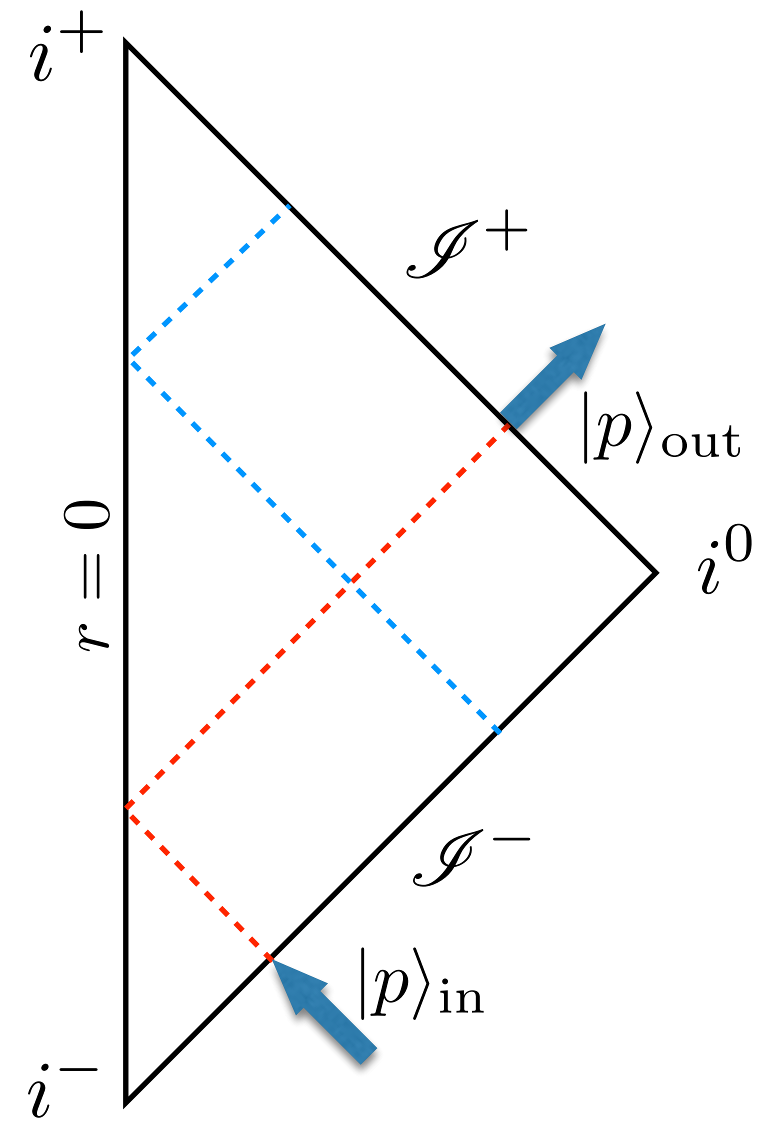
* **Lorentz invariance**: $\mathscr{U}(\Lambda) S \mathscr{U}(\Lambda)^\dagger = S$ with $\Lambda \in \mathrm{SO}(1,3)$

* $\langle q_1, \ldots, q_n | S | p_1, p_2\rangle$ is **analytic** in the external momenta.

The ***S*-matrix** is a kind of **holographic** quantity in **Minkowski** space-time: *in*- and *out*-states **live** on its **boundary**.



massive states



massless states

We can isolate **nontrivial scattering** in the S-matrix by writing

$$S = \mathbf{1} + iT$$

so the matrix elements have the structure

invariant amplitude

$$\begin{aligned}\langle q_1, \ldots, q_n | S | p_1, p_2 \rangle &= \langle q_1, \ldots, q_n | p_1, p_2 \rangle + \langle q_1, \ldots, q_n | iT | p_1, p_2 \rangle \\ &= \langle q_1, \ldots, q_n | p_1, p_2 \rangle + (2\pi)^4 \delta^{(4)}\left(p_1 + p_2 - \sum_{i=1}^{n} q_i \right) i\mathscr{M}_{i\to f}\end{aligned}$$

In terms of the **invariant amplitude**, the **differential cross section** is given by

$$d\sigma = \frac{|i\mathscr{M}_{i\to f}|^2}{\underbrace{4\omega_{\mathbf{p}_1}\omega_{\mathbf{p}_2}|\mathbf{v}_1 - \mathbf{v}_2|}_{\text{observer dependent incoming flux}}} (2\pi)^4 \delta^{(4)}\left(p_1 + p_2 - \sum_{i=1}^{n} q_i \right) \underbrace{\prod_{k=1}^{n} \frac{d^3 q_k}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{q}_k}}}_{\text{phase space factor}}$$

We can isolate **nontrivial scattering** in the S-matrix by writing

In the case of **particle decay**,

$$|p\rangle \longrightarrow |q_1,\ldots,q_n\rangle$$

the **decay width** is given by

$$d\Gamma = \frac{|i\mathscr{M}_{i\to f}|^2}{2\omega_{\mathbf{p}}}(2\pi)^4\delta^{(4)}\left(p-\sum_{i=1}^{n}q_i\right)\prod_{k=1}^{n}\frac{d^3q_k}{(2\pi)^3}\frac{1}{2\omega_{\mathbf{q}_k}}$$

so t

$\langle q_1,$

invariant amplitude

$$\sum_{i=1}^{n} q_i\Bigg)\, i\mathscr{M}_{i\to f}$$

In ter … given by

$$d\sigma = \underbrace{\frac{|i\mathscr{M}_{i\to f}|^2}{4\omega_{\mathbf{p}_1}\omega_{\mathbf{p}_2}|\mathbf{v}_1-\mathbf{v}_2|}}_{\text{observer dependent incoming flux}}(2\pi)^4\delta^{(4)}\left(p_1+p_2-\sum_{i=1}^{n}q_i\right)\underbrace{\prod_{k=1}^{n}\frac{d^3q_k}{(2\pi)^3}\frac{1}{2\omega_{\mathbf{q}_k}}}_{\text{phase space factor}}$$

The computation of the S-matrix in terms of the interacting field theory is done using the **Lehmann-Symanzik-Zimmermann (LSZ) reduction formula**.

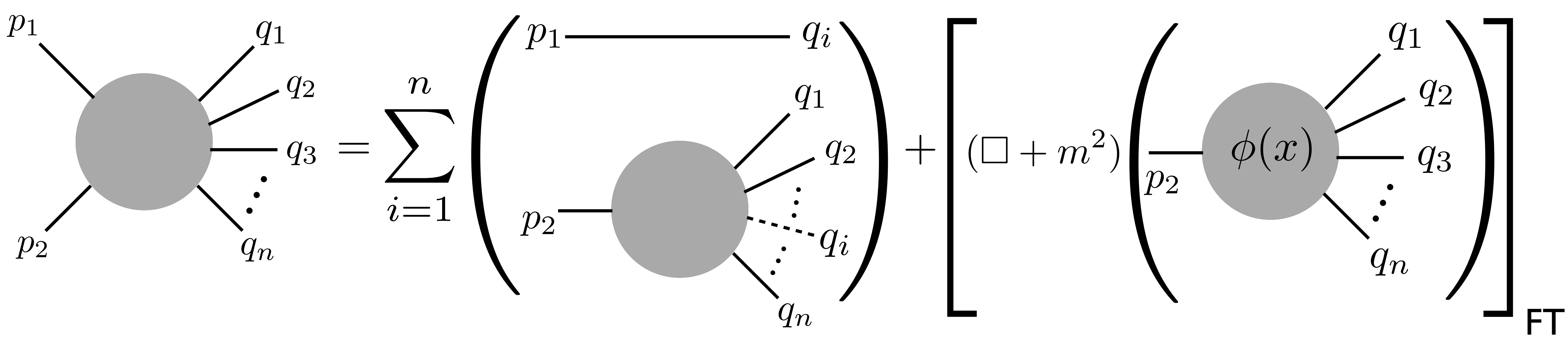
Harry Lehmann
(1924-1998)

Kurt Symanzik
(1923-1983)

Wolfhart Zimmermann
(1928-2016)

$$
\begin{aligned}
{}_{\text{out}}\langle q_1,\ldots,q_n|p_1,p_2\rangle_{\text{in}} = \sum_{i=1}^{n}(2\pi)^3(2\omega_{\mathbf{q}_i})\delta^{(3)}(\mathbf{q}_i-\mathbf{p}_1)\,{}_{\text{out}}\langle q_1,\ldots,\widehat{q_i},\ldots,q_n|p_2\rangle_{\text{in}} \\
+iZ^{-1/2}\int d^4x\,e^{-ip_1\cdot x}(\Box+m^2)\,{}_{\text{out}}\langle q_1,\ldots,q_n|\phi(x)|p_2\rangle_{\text{in}}
\end{aligned}
$$

Symbolically:



Iterating the procedure, we **trade** all incoming and outgoing particles by **time-ordered** field insertions:

$$T[\phi(x)\phi(y)] = \theta(x^0-y^0)\phi(x)\phi(y) + \theta(y^0-x^0)\phi(y)\phi(x)$$

$$\langle q_1,\ldots,q_n|S|p_1,p_2\rangle = \text{disconected terms}$$

$$+i(Z^{-1/2})^{n+2}\int d^4x_1 d^4x_2 e^{-ip_1x_1-ip_2x_2}\int d^4y_1\ldots d^4y_n e^{iq_1y_1+\ldots+iq_ny_n}$$

$$\times(\Box+m^2)_{x_1}(\Box+m^2)_{x_2}(\Box+m^2)_{y_1}\ldots(\Box+m^2)_{y_n}\langle\Omega|T[\phi(x_1)\phi(x_2)\phi(y_1)\ldots\phi(y_n)]|\Omega\rangle$$

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Fourier transforms

$$\langle q_1, \ldots, q_n | S | p_1, p_2 \rangle = \text{disconected terms}$$

$$+ i(Z^{-1/2})^{n+2} \int d^4x_1 d^4x_2 e^{-ip_1x_1 - ip_2x_2} \int d^4y_1 \ldots d^4y_n e^{iq_1y_1 + \ldots + iq_ny_n}$$

$$\times (\Box + m^2)_{x_1} (\Box + m^2)_{x_2} (\Box + m^2)_{y_1} \ldots (\Box + m^2)_{y_n} \langle \Omega | T[\phi(x_1)\phi(x_2)\phi(y_1) \ldots \phi(y_n)] | \Omega \rangle$$

Inverse free propagators

S-matrix amplitudes are computed in terms of **time-ordered** (amputated) **correlation functions**

$$G(x_1, \ldots, x_n) = \langle \Omega | T[\phi(x_1) \ldots \phi(x_n)] | \Omega \rangle$$

We have **reduced** the problema of computing **scattering observables** to that of evaluating **time-ordered (Green) correlation functions**

QFT $\longrightarrow$ $\langle\Omega|T[\phi(x_1)\ldots\phi(x_n)]|\Omega\rangle$

$$\Big\downarrow \text{LSZ}$$

$$\langle q_1,\ldots,q_{n-2}|S|p_1,p_2\rangle$$

$$\Big\downarrow$$

$$i\mathscr{M}_{i\to f}$$

$$\Big\downarrow$$

$\dfrac{d\sigma}{d\Omega}$ $\longrightarrow$ **Experiment**

We need a **method** to **compute** time-ordered Green functions in an interacting theory.

This **procedure** to compute scattering amplitudes might seem like **overkilling**…

* **S-matrix approach** (1960s): use the mathematical properties of the S-matrix (Lorentz invariance, analyticity, unitarity and crossing) to "bootstrap" the result

String theory

Geoffrey Chew
(1924-2019)

Lately, a number of developments have **revived** the **"S-matrix spirit"**:

* **Double copy** (KLT relations, BCJ color-kinematics duality,…): construct gravitational amplitudes "algebraically" from gauge theory amplitudes, "gravity = (gauge)^2".

* **Britto-Cachazo-Feng-Witten (BCFW) recursion relations**: express scattering amplitudes in terms of amplitudes with fewer particles.

* **Amplituhedron** (Arkani-Hamed & Trnka): find underlying mathematical structures (positive Grassmannian) in the amplitude from where properties like unitarity and locality

Once we managed to identify the **weakly coupled degrees of freedom** in our system, we can treat interactions in **perturbation theory**

"small"

$$S[\phi] = S_0[\phi] + \lambda S_{\rm int}[\phi]$$

quadratic in the field

higher orders

Time-ordered correlation functions can be written in terms of **functional integrals**

$$\langle\Omega|T[\phi(x_1)\ldots\phi(x_n)]|\Omega\rangle = \frac{\displaystyle\int \mathscr{D}\phi\,\phi(x_1)\ldots\phi(x_n)e^{iS_0[\phi]+i\lambda S_{\rm int}[\phi]}}{\displaystyle\int \mathscr{D}\phi\,e^{iS_0[\phi]+i\lambda S_{\rm int}[\phi]}}$$

When **λ** is "small", we can **expand** the right-hand side in **powers** of the **coupling constant.** The resulting series involves functional integrals with the **structure**

free theory (Fock) vacuum

$$\int \mathscr{D}\phi\,\mathscr{O}[\phi(x_1),\ldots,\phi(x_n)]e^{iS_0[\phi]} \sim \langle 0|T\Big[\mathscr{O}[\phi(x_1),\ldots,\phi(x_n)]\Big]|0\rangle$$

These **free** time-ordered Green functions are **computable** using the **Wick theorem**!

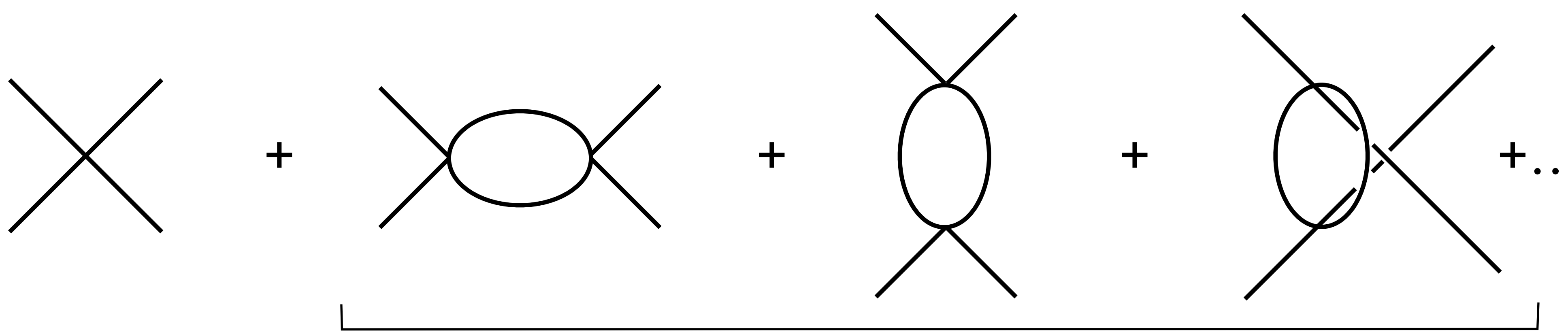
All Wick contractions at each order in perturbation theory can be **systematically** computed using **Feynman diagrammatics**

Each theory is characterized by its **propagators** and **vertices**. For ϕ^4

Richard Feynman
(1918-1988)

To compute the contribution to a process at **order** $\boldsymbol{n}$ in perturbation theory, we construct **all possible connected amputated diagrams** with $\boldsymbol{n}$ **vertices** and as many **external legs** as **particles involved**

For example, for a **two-to-two** particle scattering

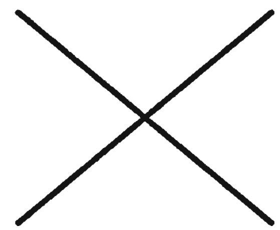


$n = 1$ $\qquad\qquad$ $n = 2$

The **contribution** of each terms is **computed** using the **Feynman rules**.

For a ϕ^4 scalar theory, the **Feynman rules** in **momentum space** are

$$\Longrightarrow \quad \frac{i}{p^2-m^2+i\varepsilon}$$



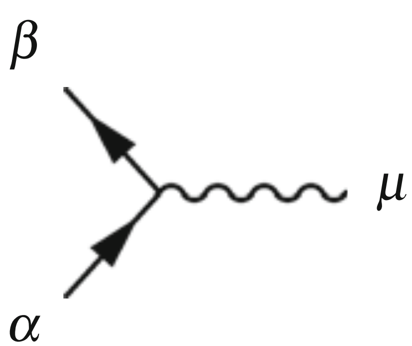
$$\Longrightarrow \quad -i\lambda$$

Richard Feynman
(1918-1988)

whereas for **QED**

$$\Longrightarrow \quad \left(\frac{i}{\not{p}-m+i\varepsilon}\right)_{\beta\alpha}$$

$$\Longrightarrow \quad \frac{-i\eta_{\mu\nu}}{p^2+i\varepsilon}$$



$$\Longrightarrow \quad -ie\gamma^{\mu}_{\beta\alpha}.$$

Incoming fermion:

$\Longrightarrow \quad u_\alpha(\mathbf{p},s)$

Incoming antifermion:

$\Longrightarrow \quad \overline{v}_\alpha(\mathbf{p},s)$

Outgoing fermion:

$\Longrightarrow \quad \overline{u}_\alpha(\mathbf{p},s)$

Outgoing antifermion:

$\Longrightarrow \quad v_\alpha(\mathbf{p},s)$

Incoming photon:

$\Longrightarrow \quad \varepsilon_\mu(\mathbf{p})$

Outgoing photon:

$\Longrightarrow \quad \varepsilon_\mu(\mathbf{p})^*$

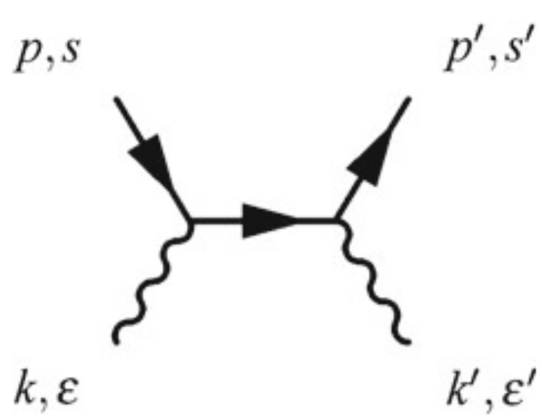
+ **integration** over internal momenta, **a delta function momentum conservation** at each **vertex**, a **factor of –1** for each fermion loop, and a **combinatorial factor**.

As an example, for **Compton scattering**

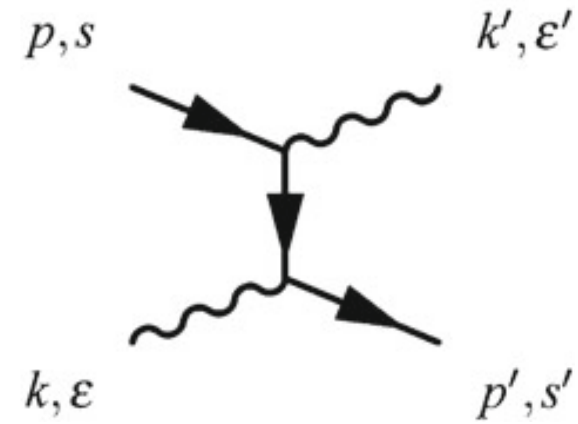
$$\gamma(k,\varepsilon) + e^-(p,s) \longrightarrow \gamma(k',\epsilon') + e^-(p',s')$$

the invariant amplitude at leading $\mathcal{O}(e^2)$ is given by

$$i\mathscr{M}_{i\to f} =$$



$+$



Remember:

$$\not{A} \equiv A_\mu \gamma^\mu$$

$$p^2 = p'^2 = m^2$$

$$k^2 = k'^2 = 0$$

$$k \cdot \varepsilon(\mathbf{k}) = k' \cdot \varepsilon(\mathbf{k}') = 0$$

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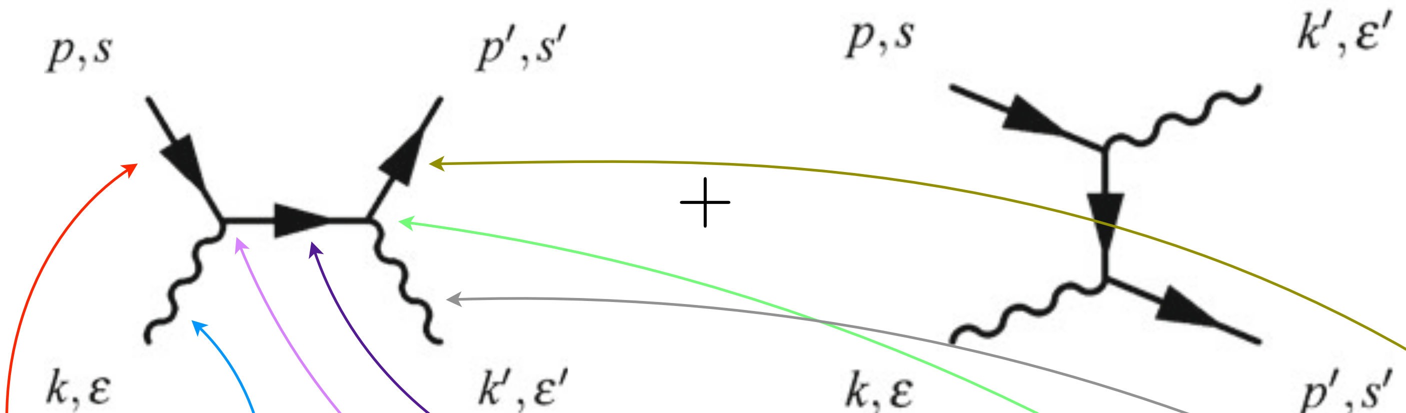
$i\mathscr{M}_{i\to f} =$



$$= u_\alpha(\mathbf{p},s)\varepsilon_\mu(\mathbf{k})(-ie\gamma^\mu_{\beta\alpha})\left(\frac{i}{\not{p}+\not{k}-m}\right)_{\sigma\beta}(-ie\gamma^\nu_{\lambda\sigma})\varepsilon'_\nu(\mathbf{k}')^*\overline{u}_\lambda(\mathbf{p}',s')$$

$$+ u_\alpha(\mathbf{p},s)\varepsilon'_\mu(\mathbf{k}')^*(-ie\gamma^\mu_{\beta\alpha})\left(\frac{i}{\not{p}-\not{k}-m}\right)_{\sigma\beta}(-ie\gamma^\nu_{\lambda\sigma})\varepsilon_\nu(\mathbf{k})\overline{u}_\lambda(\mathbf{p}',s')$$

$$= -ie^2\overline{u}(\mathbf{p}',s')\not{\varepsilon}'(\mathbf{k}')^*\frac{\not{p}+\not{k}+m}{(p+k)^2-m^2}\not{\varepsilon}(\mathbf{k})u(\mathbf{p},s)$$

$$-ie^2\overline{u}(\mathbf{p}',s')\not{\varepsilon}(\mathbf{k})\frac{\not{p}-\not{k}+m}{(p-k)^2-m^2}\not{\varepsilon}'(\mathbf{k}')^*u(\mathbf{p},s)$$

In the **low energy limit** $\mathbf{p}^2, \mathbf{p}'^2, \mathbf{k}^2, \mathbf{k}'^2 \ll m^2$ the invariant amplitude is

$$i\mathscr{M}_{i\to f} = \frac{ie^2}{m}\Big[\varepsilon(\mathbf{k})\cdot\varepsilon'(\mathbf{k}')\Big]\overline{u}(\mathbf{p}',s')\frac{\not{k}}{|\mathbf{k}|}u(\mathbf{p},s)$$

(exercise)

If our experiment is **blind** to the **electron spin**, we have to **average** over the **incoming electron spin** and **sum** over the **spin of the outgoing electron**

$$\overline{|i\mathscr{M}_{i\to f}|^2} = \frac{1}{2}\left(\frac{e^2}{m|\mathbf{k}|}\right)^2|\varepsilon(\mathbf{k})\cdot\varepsilon'(\mathbf{k}')^*|^2\sum_{s=\pm\frac{1}{2}}\sum_{s'=\pm\frac{1}{2}}|\overline{u}(\mathbf{p}',s')\not{k}u(\mathbf{p},s)|^2$$

(exercise)

average over incoming $e-$ spin

sum over outgoing $e-$ spin

$$= 4e^4|\varepsilon(\mathbf{k})\cdot\varepsilon'(\mathbf{k}')^*|^2$$

For an electron at **rest**, the differential cross section is

$$\frac{d\sigma}{d\Omega} = \frac{3e^4}{48\pi m^2}|\varepsilon(\mathbf{k})\cdot\varepsilon'(\mathbf{k}')^*|^2$$

Part II

Divergences and their cure

With all this, it seems we have a complete **recipe** to do particle physics:

∗ Identify the **weakly coupled** degrees of freedom.

∗ **Choose** an appropriate **interpolating field**.

∗ Write an **interacting** field theory compatible with the **symmetries** of the system.

∗ **Compute** the **correlation functions** in **perturbation theory**.

∗ Use the **LSZ** reduction formula to evaluate **perturbatively** the ***S*-matrix elements** and **cross sections.**

With all this, it seems we have a complete **recipe** to do particle physics:

∗ Identify the **weakly coupled** degrees of freedom.

∗ **Choose** an appropriate **interpolating field**.

∗ Write a ... em.

∗ **Comp** ...

∗ Use t ... **atrix elemen** ...

Warning!

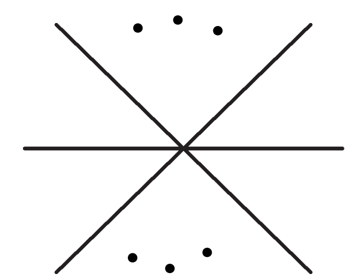
A particle may **exist** in the theory even if there is **no field** associated with it. Particles can appear as **poles** (i.e., **bound states**) in the Green functions of **other fields**.

However, when computing ***S*-matrix elements** for these fields, we need to introduce them through their corresponding **interpolating fields**.

The problem comes when computing **quantum corrections**…

Restoring the powers of $\hbar$, the Feynman rules of a ϕ^n are

 $\Longrightarrow$ $\dfrac{i\hbar}{p^2 - m^2 + i\varepsilon}$

 $\Longrightarrow$ $-i\dfrac{\lambda}{\hbar}$

The power of $\hbar$ of a diagram with E external lines, I internal propagators, and V vertices is

$$\#(\hbar) = I - V$$

while the **number of loops** in the diagram is

global conservation delta function

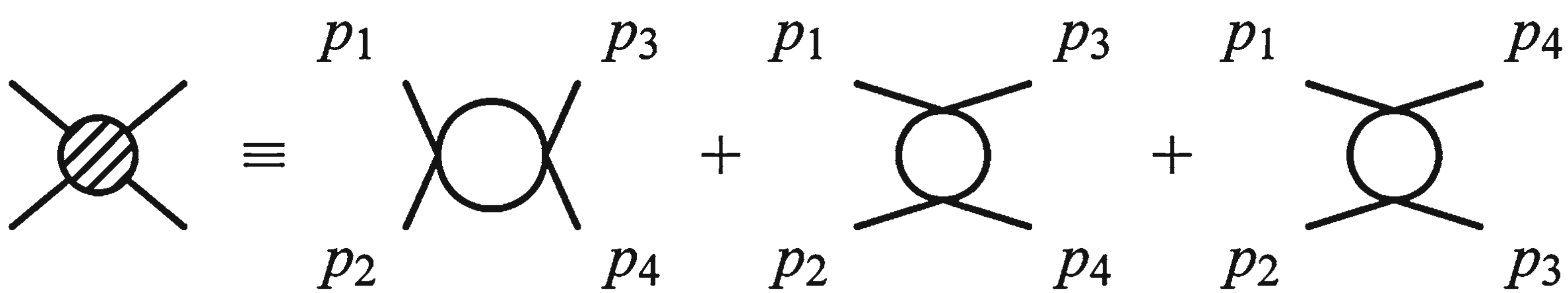
$$L = I - (V - 1) = I - V + 1$$

\# of integrations

\# of independent delta functions

Thus, $\#(\hbar) = I - V = L - 1$ and an ***L*-loop diagram** scales as $\hbar^{L-1}$

The problem lies in that **loop diagrams** frequently give **divergent** results.



$$= \frac{\lambda^2}{2}\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2-m^2+i\epsilon}\left[\frac{1}{(k+p_1+p_2)^2-m^2+i\epsilon}\right.$$

$$\left.+\frac{1}{(k+p_1+p_3)^2-m^2+i\epsilon}+\frac{1}{(k+p_1+p_4)^2-m^2+i\epsilon}\right]$$

These integrals are **logarithmically divergent**

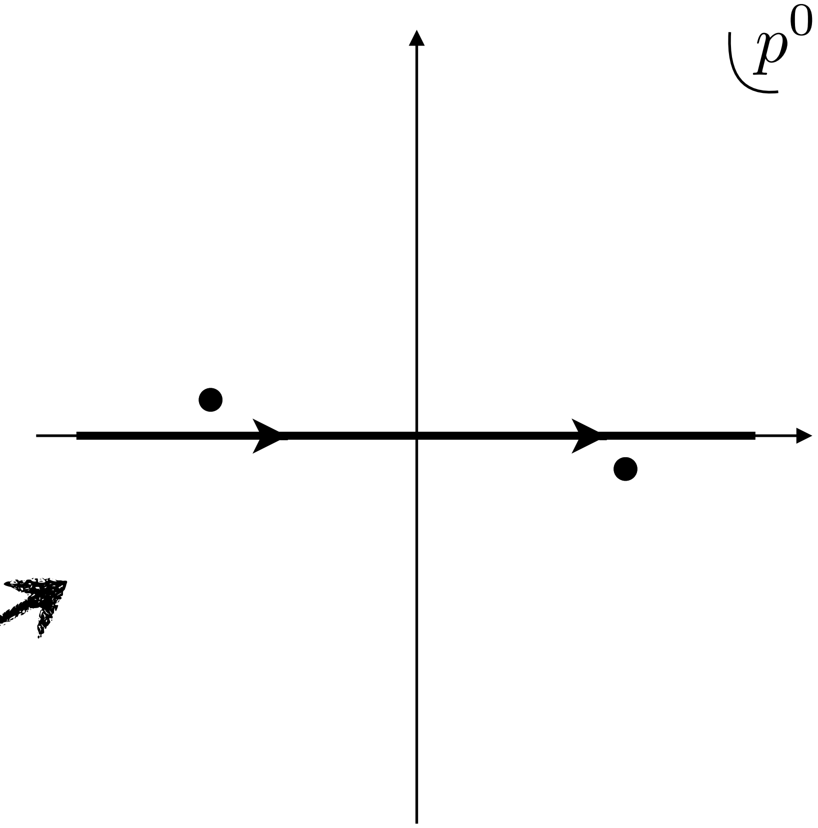
$$\int\frac{\overbrace{d^4k}^{\sim k^3dk}}{(2\pi)^4}\frac{1}{\underbrace{k^2-m^2+i\epsilon\;(k+p_1+p_2)^2-m^2+i\epsilon}_{\sim k^4}}\sim\int^{\infty}\frac{dk}{k}\longrightarrow\infty$$

To avoid meaningless results, we need to **regularize** our theory

Let us look at a **typical Feynman integral**:

$$I = \int \frac{d^4p}{(2\pi)^4}\frac{1}{p^2 - m^2 + i\epsilon} \qquad \left(\sim \int^{\infty} p\,dp\right)$$

$$p^0 = \pm\sqrt{\mathbf{p}^2 + m^2} \mp i\epsilon$$



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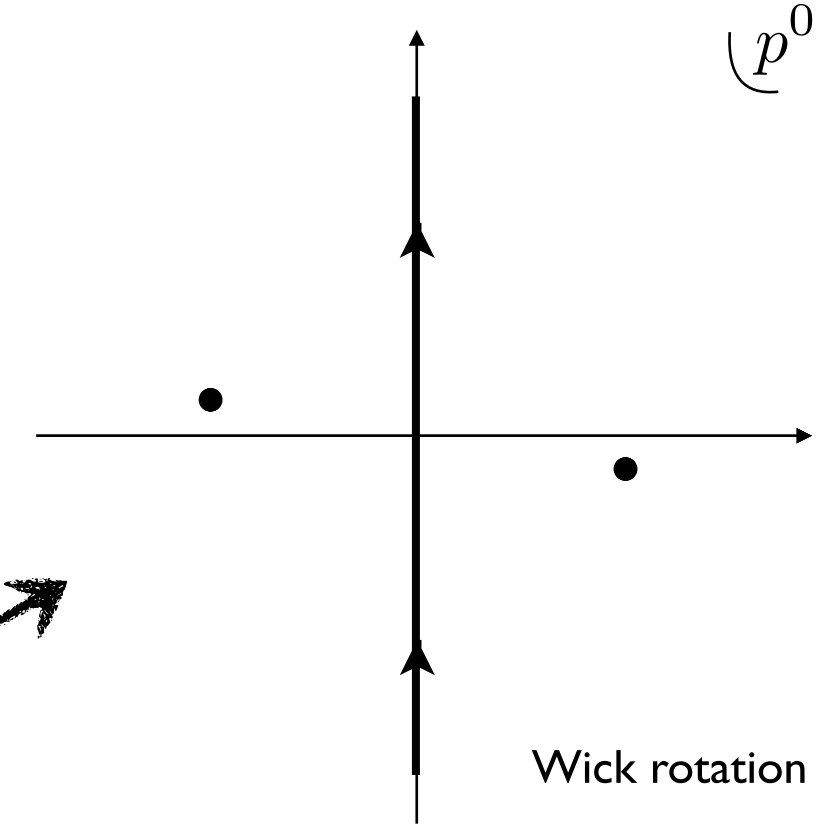

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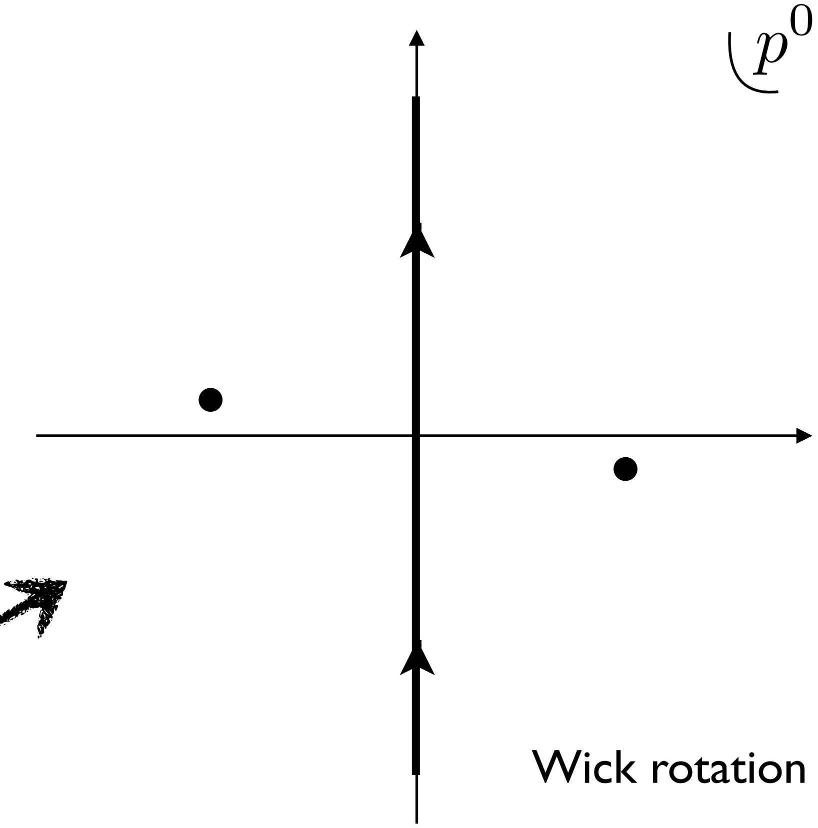

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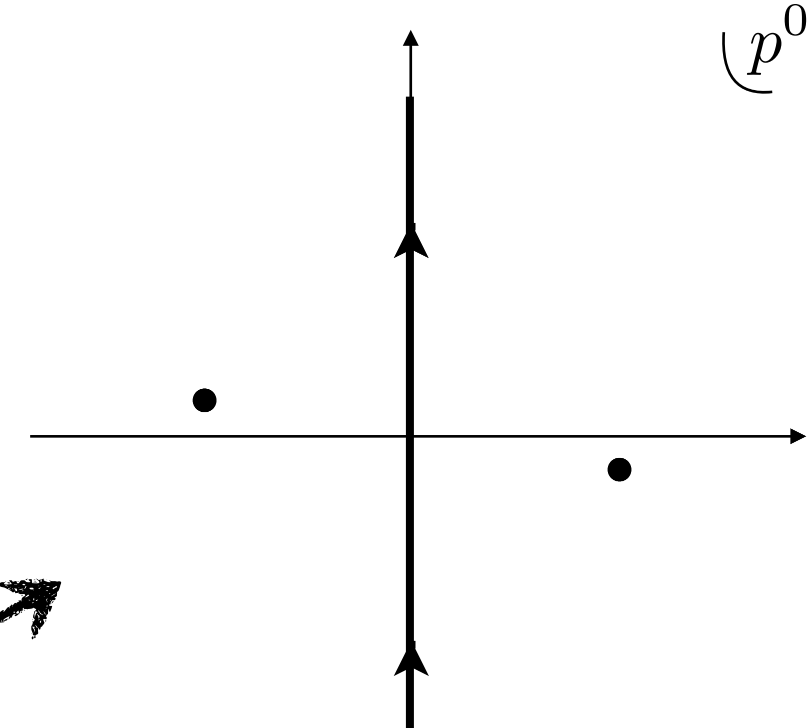


Wick rotation

There are many **ways** to **make sense** of this. For example:

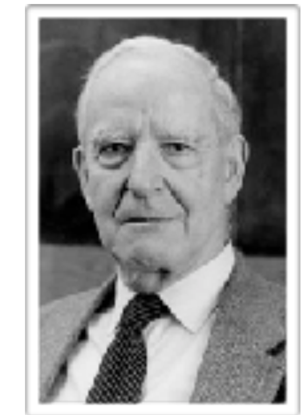
- **Sharp momentum cutoff** Λ

$$I(\Lambda) = -i \int\limits_{|\ell_E|<\Lambda} \frac{d^4\ell_E}{(2\pi)^4}\frac{1}{\ell_E^2 + m^2} \sim \Lambda^2$$

This method, however, **breaks Lorentz** and **gauge invariance**.

- **Pauli-Villars method:** introduce a number of fictitious fields with large masses M_i and whose propagators have the **"wrong" sign**

Wolfgang Pauli (1900-1958)



Felix Villars (1921-2002)

$$I(M_i) = \int \frac{d^4p}{(2\pi)^4}\left(\frac{1}{p^2-m^2+i\epsilon} - \sum_{i=1}^{n}\frac{g_i}{p^2-M_i^2+i\epsilon}\right)$$

$$= -i\int \frac{d^4\ell_E}{(2\pi)^4}\left(\frac{1}{\ell_E^2+m^2} - \sum_{i=1}^{n}\frac{g_i}{\ell_E^2+M_i^2}\right)$$

Pauli-Villars regularization is **Lorentz and gauge invariant**, but rather cumbersome.

- **Dimensional regularization:** define the **Feynman integrals in d dimensions** and continue d to complex values.

$$I(d) = \int \frac{d^dp}{(2\pi)^d}\frac{1}{p^2-m^2+i\epsilon} = -i\int\frac{d^d\ell_E}{(2\pi)^d}\frac{1}{\ell_E^2+m^2}$$

This requires the introduction of an **energy scale μ** to preserve the **dimensions** of the coupling constant. E.g., for a scalar ϕ^4 theory $\lambda \longrightarrow \mu^{4-d}\lambda$

Dimensional regularization **preserves Lorentz and gauge invariance**, but one has to be **careful** when working with **chiral theories**!

Once the theory is **regularized**, we can compute **finite** scattering amplitudes

external momenta

$$i\mathscr{M} = f(p_i;\lambda,m,\Lambda)$$

couplings

masses

cutoff

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external momenta

$$i\mathscr{M} = f(p_i;\lambda,m,\Lambda) \xrightarrow{\Lambda\to\infty} \infty$$

couplings masses cutoff

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external momenta

$$i\mathscr{M} = f(p_i; \lambda, m, \Lambda) \xrightarrow{\Lambda \to \infty} \infty$$

couplings masses cutoff

Hendrik A. Kramers
(1894-1952)

To handle the theory, we introduce the notion of **renormalization**:

* Only **measurable** quantities are **physical**.

* The **quantities** appearing in the **Lagrangian** (masses, couplings, fields, etc.) are **unphysical auxiliary parameters**.

$$\boxed{\phi_0(x) = \sqrt{Z(\Lambda)}\phi(x)}$$

* Divergences are **“absorbed”** in these unphysical parameters

$$i\mathscr{M} = f\Big(p_i; \lambda_0(\Lambda), m_0(\Lambda), \Lambda\Big) \xrightarrow{\Lambda \to \infty} f(p_i; \lambda, m)$$

renormalized quantities

* The relation between renormalized and physical quantities is fixed by the **(physical) definition** of the latter (**renormalization conditions**).

Let us apply this program to a **scalar** ϕ^4 theory. The **renormalized Lagrangian** is

$$\boxed{\phi_0(x) = \sqrt{Z(\Lambda)}\phi(x)}$$

$$\begin{aligned}\mathscr{L}_{\rm ren} &= \frac{1}{2}\partial_\mu\phi_0\partial^\mu\phi_0 - \frac{m_0(\Lambda)^2}{2}\phi_0^2 - \frac{\lambda(\Lambda)}{4!}\phi_0^4 \\ &= \frac{1}{2}Z(\Lambda)\partial_\mu\phi\partial^\mu\phi - \frac{m_0(\Lambda)^2 Z(\Lambda)}{2}\phi^2 - \frac{\lambda(\Lambda)Z(\Lambda)^2}{4!}\phi^4\end{aligned}$$

It can be rewritten in terms of the **finite**, **renormalized**, **masses** and **couplings** as

$$\mathscr{L}_{\rm ren} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4 + \frac{1}{2}\delta_Z\partial_\mu\phi\partial^\mu\phi - \frac{\delta_m}{2}\phi^2 + \frac{\delta_\lambda}{2}\phi^4$$

counterterms

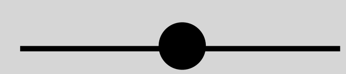
where

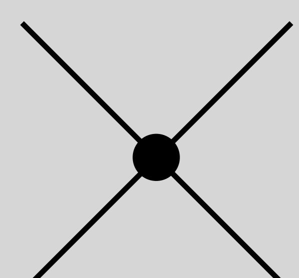
$$Z(\Lambda) = 1 + \delta_Z(\Lambda)$$

$$m_0(\Lambda)^2 Z(\Lambda) = m^2 + \delta_m(\Lambda)$$

$$\lambda_0(\Lambda)Z(\Lambda)^2 = \lambda + \delta_\lambda(\Lambda)$$

Feynman rules for counterterms


$= i(p^2\delta_Z - \delta_m)$


$= -i\delta_\lambda$

$$\boxed{\mathscr{L}_{\rm ren} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4 + \frac{1}{2}\delta_Z\partial_\mu\phi\partial^\mu\phi - \frac{\delta_m}{2}\phi^2 + \frac{\delta_\lambda}{2}\phi^4}$$

By construction, **quantities** computed from the **renormalized Lagrangian** are **finite**. Renormalization can now be **systematically** implemented:

- **Regularize** the theory.

- **Compute** loop diagrams using the Lagrangian

$$\mathscr{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4$$

- Fix the **counterterms** to eliminate the **divergences** at each loop level.

- **Evaluate physical quantities** in terms of finite **renormalized parameters**.

- **Compute amplitudes**

Let us look at it **hands-on**: ϕ^4 at one loop.

At one loop there are two divergent diagrams by **power counting**:

 $\sim \Lambda^2$  $\sim \log\Lambda$

Using a **hard cutoff**, we have

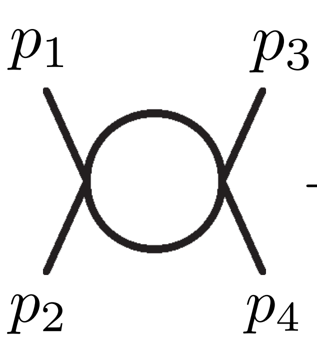
p 

$$= -\frac{i\lambda}{2}\int^{\Lambda}\frac{d^4p}{(2\pi)^4}\frac{i}{p^2-m^2+i\epsilon} = -\frac{i\lambda}{2}\int_{|\ell_E|<\Lambda}\frac{d^4\ell_E}{(2\pi)^4}\frac{1}{\ell_E^2+m^2}$$

$$= -\frac{im^2\lambda}{32\pi^2}\left[\frac{\Lambda^2}{m^2}-\log\left(\frac{\Lambda^2}{m^2}\right)\right] + \text{finite piece}$$

Mandelstam variables

$$s=(p_1+p_2)^2$$
$$t=(p_1-p_3)^2$$
$$u=(p_1-p_4)^2$$



$$+\text{crossed} \;=\; \frac{i\lambda^2}{32\pi^2}\int_0^1 dx\left\{\log\left[\frac{\Lambda^2}{m^2-x(1-x)s}\right]+\log\left[\frac{\Lambda^2}{m^2-x(1-x)t}\right]\right.$$

$$\left.+\log\left[\frac{\Lambda^2}{m^2-x(1-x)u}\right]\right\} + \text{finite piece}$$

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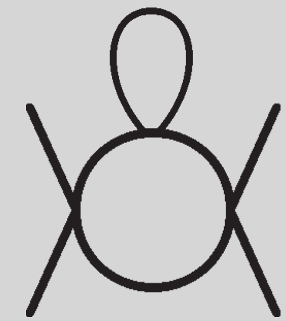
Diagrams with subdivergences



are dealt with by renormalizing the divergent subdiagram.

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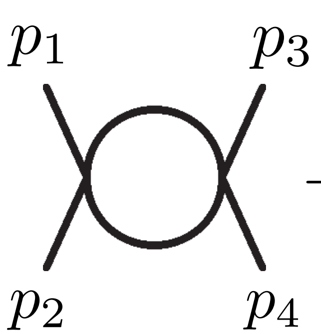

+crossed

$$= \frac{i\lambda^2}{32\pi^2}\int_0^1 dx\left\{\log\left[\frac{\Lambda^2}{m^2-x(1-x)s}\right]+\log\left[\frac{\Lambda^2}{m^2-x(1-x)t}\right]\right.$$

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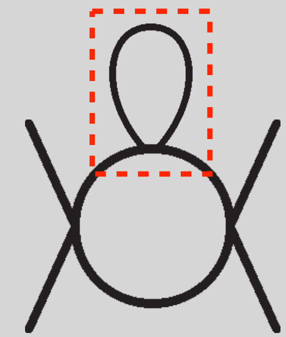

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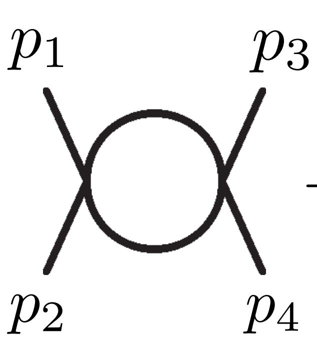

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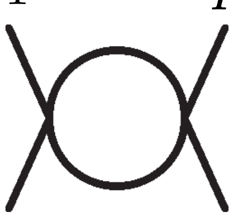
p_1 p_3



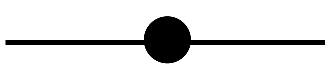
p_2 p_4

$$+\text{crossed} \;=\; \frac{i\lambda^2}{32\pi^2}\int_0^1 dx\left\{\log\left[\frac{\Lambda^2}{m^2-x(1-x)s}\right]+\log\left[\frac{\Lambda^2}{m^2-x(1-x)t}\right]\right.$$

$$\left.+\log\left[\frac{\Lambda^2}{m^2-x(1-x)u}\right]\right\}+\text{finite piece}$$

$$= -\frac{im^2\lambda}{32\pi^2}\left[\frac{\Lambda^2}{m^2} - \log\left(\frac{\Lambda^2}{m^2}\right)\right] + \text{finite piece}$$

From this result we can **identify** two of the **counterterms** at **one loop**:



$$= i(p^2\delta_Z - \delta_m) \quad \Longrightarrow \quad \begin{cases} \delta_Z\Big|_{1-\text{loop}} = 0 \\ \delta_m\Big|_{1-\text{loop}} = -\dfrac{m^2\lambda}{32\pi^2}\left[\dfrac{\Lambda^2}{m^2} - \log\left(\dfrac{\Lambda^2}{\mu^2}\right)\right] \end{cases}$$

where we have introduced an **arbitrary energy scale** μ. The **"bare"**, cutoff-dependent **mass at one loop** to be

$$Z(\Lambda) = 1 + \delta_Z(\Lambda) \quad \Longrightarrow \quad Z(\Lambda) = 1$$

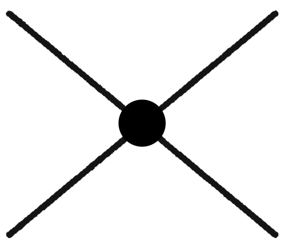
no field renormalization at one loop!

and

$$m_0(\Lambda)^2 Z(\Lambda) = m^2 + \delta_m(\Lambda) \quad \Longrightarrow \quad m_0(\Lambda)^2 = m^2\left\{1 - \frac{\lambda}{32\pi^2}\left[\frac{\Lambda^2}{m^2} - \log\left(\frac{\Lambda^2}{\mu^2}\right)\right]\right\}$$

$$+\text{crossed} = \frac{i\lambda^2}{32\pi^2}\int_0^1 dx\left\{\log\left[\frac{\Lambda^2}{m^2-x(1-x)s}\right]+\log\left[\frac{\Lambda^2}{m^2-x(1-x)t}\right]\right.$$
$$\left.+\log\left[\frac{\Lambda^2}{m^2-x(1-x)u}\right]\right\}+\text{finite piece}$$

The logarithmic **divergence** is **cancelled** by choosing the **counterterm**



$$= -i\delta_\lambda \quad\Longrightarrow\quad \delta_\lambda\Big|_{1-\text{loop}} = \frac{3\lambda^2}{32\pi^2}\log\left(\frac{\Lambda^2}{\mu^2}\right)$$

where again μ is an **arbitrary energy scale**.

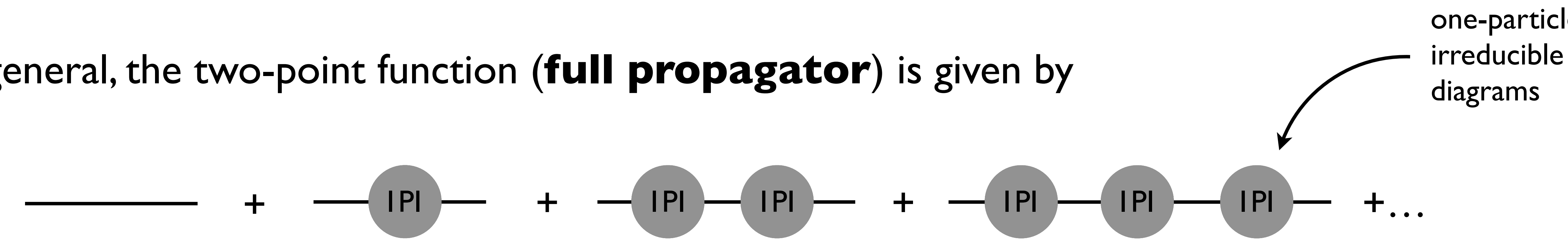
The "**bare**" coupling constant at **one-loop** is:

$$\lambda_0(\Lambda)Z(\Lambda)^2 = \lambda + \delta_\lambda(\Lambda) \quad\xrightarrow[Z(\Lambda)=1]{}\quad \lambda_0(\Lambda) = \lambda + \frac{3\lambda^2}{32\pi^2}\log\left(\frac{\Lambda^2}{\mu^2}\right)$$

Warning!!! Renormalized quantities are **not** necessarily **physical**!

Physical quantities are **defined operationally**. Let us look at the **mass**.

In general, the two-point function (**full propagator**) is given by



$$= \frac{i}{p^2-m^2} + \frac{i}{p^2-m^2}\Pi(p^2)\frac{i}{p^2-m^2} + \frac{i}{p^2-m^2}\Pi(p^2)\frac{i}{p^2-m^2}\Pi(p^2)\frac{i}{p^2-m^2} + \ldots$$

$$= \frac{i}{p^2-m^2}\sum_{n=0}^{\infty}\left[\Pi(p^2)\frac{i}{p^2-m^2}\right]^n = \frac{i}{p^2-m^2}\frac{1}{1-\Pi(p^2)\frac{i}{p^2-m^2}}$$

$$= \frac{i}{p^2-m^2-i\Pi(p^2)}$$

We can define the **physical mass** as the **pole** of the **full propagator**

physical mass

$$m^2_{\text{phys}} - m^2 - i\Pi(m^2_{\text{phys}}) = 0$$

renormalized mass

mass renormalization condition

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Physical quantities are **defined operationally**. Let us look at the **mass**.

In general, the two-point function (**full propagator**) is given by

one-particle irreducible diagrams

$+\dots$

$\frac{i}{\quad - m^2} + \dots$

In fact, we **also** have to require that the **residue** at the pole **equals** ***i***

$$p^2 - m^2 - i\Pi(p^2) = \left(1 - i\frac{d\Pi}{dp^2}\bigg|_{p^2=m^2_{\rm phys}}\right)(p^2 - m^2_{\rm phys}) + \dots$$

thus,

$$\frac{d\Pi}{dp^2}\bigg|_{p^2=m^2_{\rm phys}} = 0$$

$$= \frac{i}{p^2 - m^2 - i\Pi(p^2)}$$

We can define the **physical mass** as the **pole** of the **full propagator**

physical mass

$$m^2_{\rm phys} - m^2 - i\Pi(m^2_{\rm phys}) = 0$$

renormalized mass

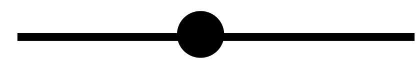
mass renormalization condition

$$\boxed{m^2_{\text{phys}} - m^2 - i\Pi(m^2_{\text{phys}}) = 0}$$

From our loop **calculation**,

$$\Pi(p^2)_{1-\text{loop}} =$$

+



$$= -\frac{im^2\lambda}{32\pi^2}\left[\frac{\Lambda^2}{m^2} - \log\left(\frac{\Lambda^2}{m^2}\right)\right] + \frac{im^2\lambda}{32\pi^2}\left[\frac{\Lambda^2}{m^2} - \log\left(\frac{\Lambda^2}{\mu^2}\right)\right]$$

$$= -\frac{im^2\lambda}{32\pi^2}\log\left(\frac{m^2}{\mu^2}\right)$$

which is **momentum independent**. Thus, the **physical mass** is given in terms of the **renormalized parameters** $\boldsymbol{m}$ and $\boldsymbol{\lambda}$ by

$$m^2_{\text{phys}} = m^2\left[1 + \frac{\lambda}{32\pi^2}\log\left(\frac{m^2}{\mu^2}\right)\right]$$

which **does not depend** on the (unphysical) momentum **cutoff**.

$$\boxed{m_{\rm phys}^2 - m^2 - i\Pi(m_{\rm phys}^2) = 0}$$

From our loop **calculation**,

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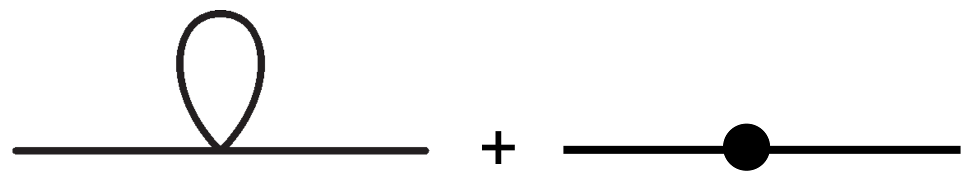


$$= -\frac{im^2\lambda}{32\pi^2}\left[\frac{\Lambda^2}{m^2} - \log\left(\frac{\Lambda^2}{m^2}\right)\right] + \frac{im^2\lambda}{32\pi^2}\left[\frac{\Lambda^2}{m^2} - \log\left(\frac{\Lambda^2}{\mu^2}\right)\right]$$

$$= -\frac{im^2\lambda}{32\pi^2}\log\left(\frac{m^2}{\mu^2}\right)$$

at one loop: $\frac{d\Pi}{dp^2} \equiv 0$

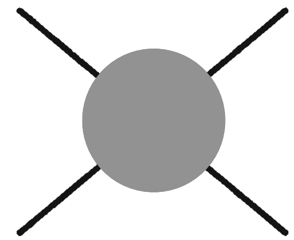
which is **momentum independent**. Thus, the **physical mass** is given in terms of the **renormalized parameters** $\boldsymbol{m}$ and $\boldsymbol{\lambda}$ by

$$m_{\rm phys}^2 = m^2\left[1 + \frac{\lambda}{32\pi^2}\log\left(\frac{m^2}{\mu^2}\right)\right]$$

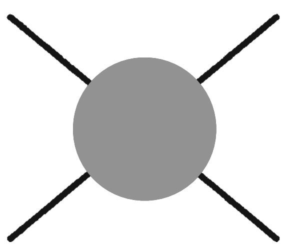
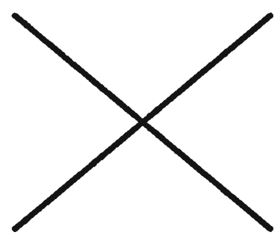
which **does not depend** on the (unphysical) momentum **cutoff**.

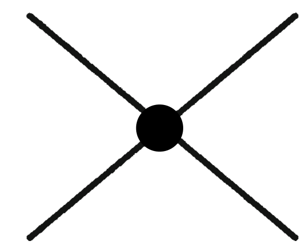
Next we look at the **coupling constant**.

We can define the **physical coupling constant**, for example, as

$$-i\lambda_{\text{phys}} \equiv$$

$$\Bigg|_{\substack{s=4m^2\\ t=u=0}}$$

From our calculation


$$=$$

$$+$$

$$+\ \text{crossed}\ +$$


$$= -i\lambda + \frac{i\lambda^2}{32\pi^2}\int_0^1 dx\left\{\log\left[\frac{\Lambda^2}{m^2-x(1-x)s}\right] + \log\left[\frac{\Lambda^2}{m^2-x(1-x)t}\right]\right.$$

$$\left. + \log\left[\frac{\Lambda^2}{m^2-x(1-x)u}\right]\right\} - \frac{3i\lambda^2}{32\pi^2}\log\left(\frac{\Lambda^2}{\mu^2}\right)$$

$$\begin{pmatrix} s=4m^2 \\ t=u=0\end{pmatrix} = -i\lambda + \frac{i\lambda^2}{32\pi^2}\int_0^1 dx\left\{\log\left[\frac{\mu^2}{m^2(1-2x)^2}\right] + 2\log\left(\frac{\mu^2}{m^2}\right)\right\}$$

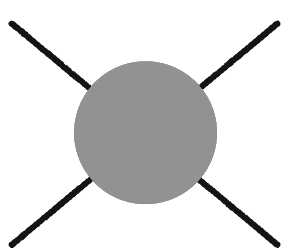
$$-i\lambda_{\rm phys} = -i\lambda + \frac{i\lambda^2}{32\pi^2}\int_0^1 dx\left\{\log\left[\frac{\mu^2}{m^2(1-2x)^2}\right] + 2\log\left(\frac{\mu^2}{m^2}\right)\right\}$$

$$= -i\lambda + \frac{3i\lambda^2}{32\pi^2}\int_0^1 dx\left[\log\left(\frac{\mu^2}{m^2}\right) - \frac{1}{3}\log(1-2x)^2\right]$$

$$\boxed{\int_0^1 dx\,\log(1-2x)^2 = -2}$$

$$\lambda_{\rm phys} = \lambda - \frac{\lambda^2}{16\pi^2}\left[1 + \frac{3}{2}\log\left(\frac{\mu^2}{m^2}\right)\right]$$

Other definitions of the physical coupling lead to **different results**. For example:

$-i\lambda_{\rm phys} \equiv$  $\Big|_{s=t=u=\frac{4}{3}m^2}$

$$\lambda_{\rm phys} = \lambda - \frac{3\lambda^2}{32\pi^2}\left[2\left(1-\sqrt{2}\arctan\frac{1}{\sqrt{2}}\right) + \log\left(\frac{\mu^2}{m^2}\right)\right]$$

$$m_{\text{phys}}^2 = m^2 \left[1 + \frac{\lambda}{32\pi^2} \log\left(\frac{m^2}{\mu^2}\right)\right] \qquad \lambda_{\text{phys}} = \lambda - \frac{\lambda^2}{16\pi^2}\left[1 + \frac{3}{2}\log\left(\frac{\mu^2}{m^2}\right)\right]$$

But **physical** quantities **cannot depend** on the **fiducial subtraction scale** μ. The **explicit dependence** is **compensated** by the one of the **renormalized parameters**.

Let us begin with the **coupling**

$$\mu \frac{d\lambda_{\text{phys}}}{d\mu} = 0$$

$$\Downarrow$$

$$\left(\mu \frac{d\lambda}{d\mu}\right) - \frac{\lambda}{8\pi^2}\left(\mu\frac{d\lambda}{d\mu}\right)\left[1 + \frac{3}{2}\log\left(\frac{\mu^2}{m^2}\right)\right] - \frac{3\lambda^2}{16\pi^2} = 0$$

At **leading order** in $\boldsymbol{\lambda}$

$$\mu\frac{d\lambda}{d\mu} - \frac{3\lambda^2}{16\pi^2} = 0 \quad \Longrightarrow \quad \beta(\lambda) \equiv \mu\frac{d\lambda}{d\mu} = \frac{3\lambda^2}{16\pi^2}$$

This defines the **beta function**.

$$m^2_{\rm phys} = m^2\left[1+\frac{\lambda}{32\pi^2}\log\left(\frac{m^2}{\mu^2}\right)\right] \qquad\qquad \lambda_{\rm phys} = \lambda - \frac{\lambda^2}{16\pi^2}\left[1+\frac{3}{2}\log\left(\frac{\mu^2}{m^2}\right)\right]$$

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At **leading order** in **λ**

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This defines the **beta function**.

$$m^2_{\text{phys}} = m^2\left[1+\frac{\lambda}{32\pi^2}\log\left(\frac{m^2}{\mu^2}\right)\right] \qquad \lambda_{\text{phys}} = \lambda - \frac{\lambda^2}{16\pi^2}\left[1+\frac{3}{2}\log\left(\frac{\mu^2}{m^2}\right)\right]$$

Next we deal with the **physical mass**

$$\mu\frac{dm^2_{\text{phys}}}{d\mu} = 0$$

$$\Downarrow$$

$$\left(\mu\frac{dm^2}{d\mu}\right)\left[1+\frac{\lambda}{32\pi^2}\log\left(\frac{m^2}{\mu^2}\right)\right] + m^2\left[\frac{1}{32\pi^2}\left(\mu\frac{d\lambda}{d\mu}\right)\log\left(\frac{m^2}{\mu^2}\right)+\frac{\lambda}{32\pi^2 m^2}\left(\mu\frac{dm^2}{d\mu}\right)-\frac{\lambda}{16\pi^2}\right] = 0$$

Dropping **subleading** terms in **λ**

$$\mu\frac{dm^2}{d\mu} - \frac{\lambda m^2}{16\pi^2} = 0 \qquad \Longrightarrow \qquad \gamma_{m^2}(\lambda) \equiv \frac{\mu}{m^2}\frac{dm^2}{d\mu} = \frac{\lambda}{16\pi^2}$$

with is the **Callan-Symanzik gamma function**.

$$m^2_{\text{phys}} = m^2\left[1+\frac{\lambda}{32\pi^2}\log\left(\frac{m^2}{\mu^2}\right)\right] \qquad \lambda_{\text{phys}} = \lambda - \frac{\lambda^2}{16\pi^2}\left[1+\frac{3}{2}\log\left(\frac{\mu^2}{m^2}\right)\right]$$

Next we deal with the **physical mass**

$$\mu\frac{dm^2_{\text{phys}}}{d\mu} = 0$$

$$\boxed{\mu\frac{d\lambda}{d\mu} = \frac{3\lambda^2}{16\pi^2}}$$

$$\left(\mu\frac{dm^2}{d\mu}\right)\left[1+\cancel{\frac{\lambda}{32\pi^2}\log\left(\frac{m^2}{\mu^2}\right)}\right] + m^2\left[\cancel{\frac{1}{32\pi^2}\left(\mu\frac{d\lambda}{d\mu}\right)\log\left(\frac{m^2}{\mu^2}\right)} + \cancel{\frac{\lambda}{32\pi^2 m^2}\left(\mu\frac{dm^2}{d\mu}\right)} - \frac{\lambda}{16\pi^2}\right] = 0$$

Dropping **subleading** terms in **λ**

$$\mu\frac{dm^2}{d\mu} - \frac{\lambda m^2}{16\pi^2} = 0 \quad \Longrightarrow \quad \gamma_{m^2}(\lambda) \equiv \frac{\mu}{m^2}\frac{dm^2}{d\mu} = \frac{\lambda}{16\pi^2}$$

with is the **Callan-Symanzik gamma function**.

$$m^2_{\text{phys}} = m^2\left[1+\frac{\lambda}{32\pi^2}\log\left(\frac{m^2}{\mu^2}\right)\right] \qquad \lambda_{\text{phys}} = \lambda - \frac{\lambda^2}{16\pi^2}\left[1+\frac{3}{2}\log\left(\frac{\mu^2}{m^2}\right)\right]$$

N

There is a **further relevant function** to be defined

$$\gamma(\lambda) \equiv \frac{1}{2}\mu\frac{d}{d\mu}\log Z$$

but at **one loop** for the ϕ^4 theory

$$\gamma(\lambda) = 0$$

(no field renormalization)

$$0$$

$$\boxed{\mu\frac{d\lambda}{d\mu} = \frac{3\lambda^2}{16\pi^2}}$$

$$\cancel{\text{og}\left(\frac{m^2}{\mu^2}\right)} + \cancel{\frac{\lambda}{32\pi^2 m^2}\left(\mu\frac{dm^2}{d\mu}\right)} - \frac{\lambda}{16\pi^2}\bigg] = 0$$

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with is the **Callan-Symanzik gamma function**.

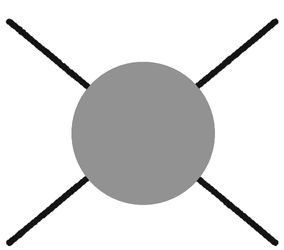
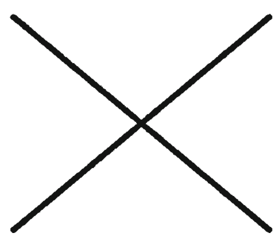
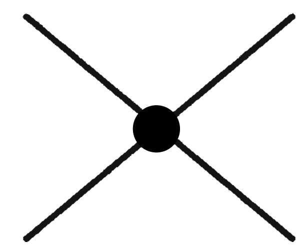
We can now compute the **four-point amplitude** in terms of our **physical** quantities:

$$m^2_{\text{phys}} = m^2\left[1+\frac{\lambda}{32\pi^2}\log\left(\frac{m^2}{\mu^2}\right)\right] \qquad \lambda_{\text{phys}} = \lambda - \frac{\lambda^2}{16\pi^2}\left[1+\frac{3}{2}\log\left(\frac{\mu^2}{m^2}\right)\right]$$

Inverting them at **this order**, we have

$$m^2 = m^2_{\text{phys}}\left[1-\frac{\lambda_{\text{phys}}}{32\pi^2}\log\left(\frac{m^2_{\text{phys}}}{\mu^2}\right)\right] \qquad \lambda = \lambda_{\text{phys}} + \frac{\lambda^2_{\text{phys}}}{16\pi^2}\left[1+\frac{3}{2}\log\left(\frac{\mu^2}{m^2_{\text{phys}}}\right)\right]$$

while for the **amplitude** we have found

 $=$  $+$  $+$ crossed $+$ 

$$= -i\lambda + \frac{i\lambda^2}{32\pi^2}\int_0^1 dx\left\{\log\left[\frac{\mu^2}{m^2-x(1-x)s}\right] + \log\left[\frac{\mu^2}{m^2-x(1-x)t}\right]\right.$$

$$\left. + \log\left[\frac{\mu^2}{m^2-x(1-x)u}\right]\right\}$$

$$
i\mathcal{M} = -i\lambda + \frac{i\lambda^2}{32\pi^2}\int_0^1 dx\left\{\log\left[\frac{\mu^2}{m^2 - x(1-x)s}\right] + \log\left[\frac{\mu^2}{m^2 - x(1-x)t}\right] + \log\left[\frac{\mu^2}{m^2 - x(1-x)u}\right]\right\}
$$

$$
m^2 = m^2_{\rm phys}\left[1 - \frac{\lambda_{\rm phys}}{32\pi^2}\log\left(\frac{m^2_{\rm phys}}{\mu^2}\right)\right]
$$

$$
\lambda = \lambda_{\rm phys} + \frac{\lambda^2_{\rm phys}}{16\pi^2}\left[1 + \frac{3}{2}\log\left(\frac{\mu^2}{m^2_{\rm phys}}\right)\right]
$$

At order $\boldsymbol{\lambda^2}$ the corrections to the **mass** are **irrelevant**, thus

$$
i\mathcal{M}(s,t,u) = -i\lambda_{\rm phys} + \frac{i\lambda^2_{\rm phys}}{32\pi^2}\int_0^1 dx\left\{\log\left[\frac{m^2_{\rm phys}}{m^2_{\rm phys} - x(1-x)s}\right] + \log\left[\frac{m^2_{\rm phys}}{m^2_{\rm phys} - x(1-x)t}\right]\right.
$$

$$
\left. + \log\left[\frac{m^2_{\rm phys}}{m^2_{\rm phys} - x(1-x)u}\right] - 2\right\}
$$

The result is **independent of $\boldsymbol{\mu}$** and satisfies the **renormalization condition**

$$
i\mathcal{M}(4m^2_{\rm phys}, 0, 0) = -i\lambda_{\rm phys}
$$

$$i\mathcal{M} = -i\lambda + \frac{i\lambda^2}{32\pi^2}\int_0^1 dx\left\{\log\left[\frac{\mu^2}{m^2 - x(1-x)s}\right] + \log\left[\frac{\mu^2}{m^2 - x(1-x)t}\right] + \log\left[\frac{\mu^2}{m^2 - x(1-x)u}\right]\right\}$$

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$$\left.+\overbrace{\log\left[\frac{m^2_{\rm phys}}{m^2_{\rm phys} - x(1-x)u}\right]}^{=0} - 2\right\}$$

$s = 4m^2_{\rm phys}$

$t = u = 0$

The result is **independent of $\boldsymbol{\mu}$** and satisfies the **renormalization condition**

$$i\mathcal{M}(4m^2_{\rm phys}, 0, 0) = -i\lambda_{\rm phys}$$

$$\int_0^1 dx\,\log(1-2x)^2 = -2$$

To summarize:

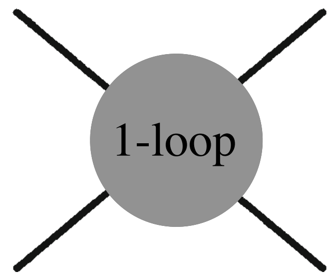
Bare quantities $\phi_0(\Lambda), m_0(\Lambda), \lambda_0(\Lambda)$ (unphysical, cutoff dependent)

Renormalized quantities ϕ, m, λ (unphysical, cutoff independent)

Physical quantities $m_{\rm phys}, \lambda_{\rm phys}$ (physical, operationally defined)

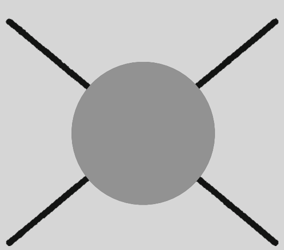
Scattering amplitudes can depend only on **physical quantities**!

Once the **one-loop correction** has been included, we can defined an **effective coupling constant** at a **characteristic momentum scale** q as

$$-i\lambda_{\text{eff}}(q^2) =$$


$$\Bigg|_{s\sim t\sim u\sim q^2} = -i\lambda + \frac{3i\lambda^2}{32\pi^2}\int_0^1 dx\,\log\left[\frac{\mu^2}{m^2 - x(1-x)q^2}\right]$$

$$= -i\lambda + \frac{3i\lambda^2}{32\pi^2}\left[\log\left(\frac{\mu^2}{m^2}\right) + 2 - \sqrt{1-\frac{4m^2}{q^2}}\,\log\left(\frac{\sqrt{1-\frac{4m^2}{q^2}}+1}{\sqrt{1-\frac{4m^2}{q^2}}-1}\right)\right]$$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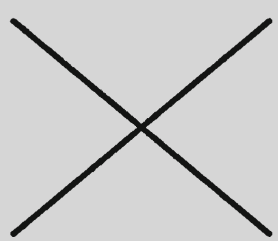
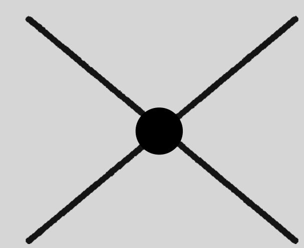

$$+$$

$$+\ \text{crossed}\ +$$


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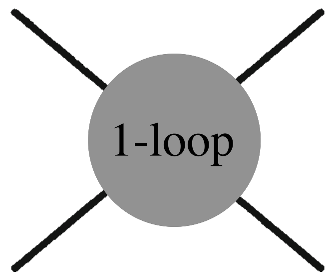

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For **large momenta** $q^2 \gg m^2$, this is given by

$$\lambda_{\text{eff}}(q^2) = \lambda\left[1 + \frac{3\lambda}{32\pi^2}\log\left(\frac{q^2}{\mu^2}\right)\right]$$

Noticing that $\lambda_{\text{eff}}(\mu^2) = \lambda$, this can be written as

$$\lambda_{\text{eff}}(q^2) = \lambda_{\text{eff}}(\mu^2)\left[1 + \frac{3\lambda_{\text{eff}}(\mu^2)}{32\pi^2}\log\left(\frac{q^2}{\mu^2}\right)\right] \qquad \longrightarrow \qquad \mu \equiv \text{reference scale}$$

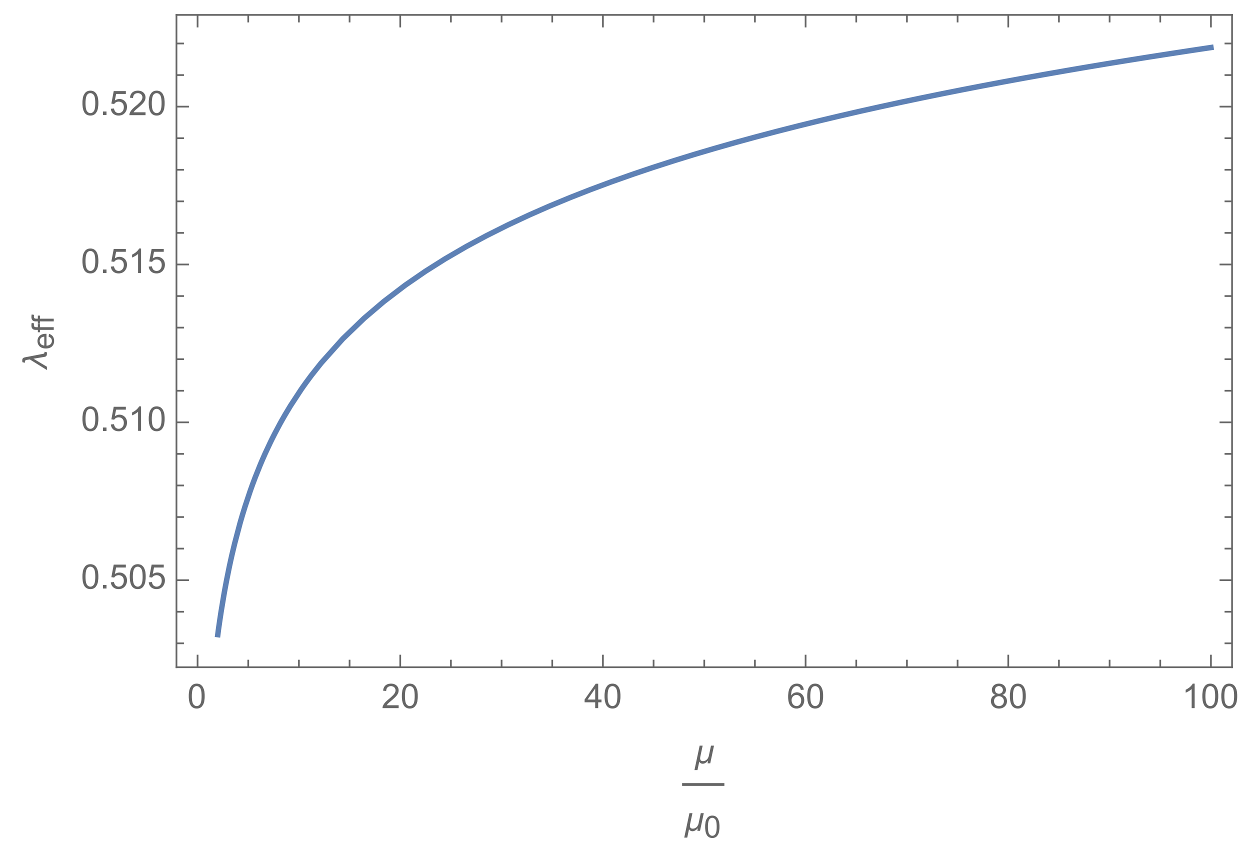
$$\boxed{\lambda_{\text{eff}}(\mu) = \lambda_{\text{eff}}(\mu_0)\left[1 + \frac{3\lambda_{\text{eff}}(\mu_0)}{32\pi^2}\log\left(\frac{\mu^2}{\mu_0^2}\right)\right]}$$

Beware!
small change
in notation

Quantum corrections make **couplings run with energy**.

This **running** is also governed by the **one loop beta function**

$$\mu\frac{d\lambda_{\text{eff}}}{d\mu} = \frac{3\lambda_{\text{eff}}^2}{16\pi^2}$$



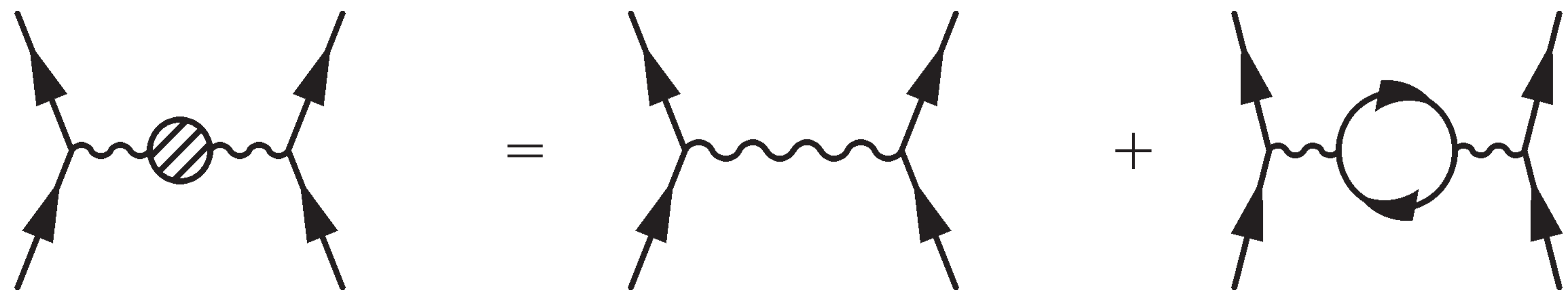
For the ϕ^4 theory, the effective coupling **grows** with energy $\longrightarrow$ $\boldsymbol{\beta(\lambda) > 0}$

Integrating the beta function equation we have

$$\lambda_{\text{eff}}(\mu) = \frac{\lambda_{\text{eff}}(\mu_0)}{1 - \frac{3\lambda_{\text{eff}}(\mu_0)}{16\pi^2}\log\left(\frac{\mu}{\mu_0}\right)} \xrightarrow{\text{blows up at}} \mu = \mu_0 e^{\frac{16\pi^2}{3\lambda_{\text{eff}}(\mu_0)}}$$

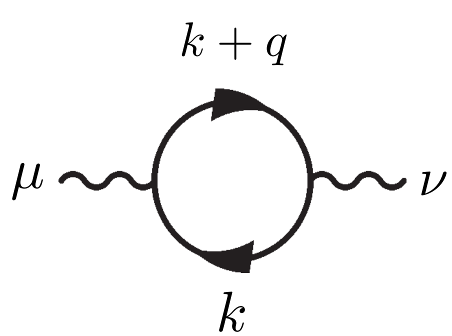
Landau pole

A similar calculation of the **effective coupling** can be carried out in **QED**:



$$= \eta_{\alpha\beta}(\overline{v}_e\gamma^\alpha u_e)\frac{e^2}{4\pi q^2}(\overline{v}_\mu\gamma^\beta u_\mu) + \eta_{\alpha\beta}(\overline{v}_e\gamma^\alpha u_e)\frac{e^2}{4\pi q^2}\Pi(q^2)(\overline{v}_\mu\gamma^\beta u_\mu)$$

where



$$\equiv \Pi^{\mu\nu}(q) = i^2(-ie)^2(-1)\int\frac{d^4k}{(2\pi)^4}\frac{\mathrm{Tr}\,[(\not{k}+m_f)\gamma^\mu(\not{k}+\not{q}+m_f)\gamma^\nu]}{(k^2-m_f^2+i\epsilon)[(k+q)^2-m_f^2+i\epsilon]}$$

Regulating the divergence using a **sharp cutoff** Λ, we have

$$\Pi_{\mu\nu}(q) = c\Lambda^2\eta_{\mu\nu} + \Pi(q^2)(q^2\eta_{\mu\nu} - q_\mu q_\nu)$$

regularization artefact

Breaks gauge invariance, cured by adding a **local counterterm**

$$\Delta\mathscr{L} \sim \Lambda^2 A_\mu A^\mu$$

Gauge invariant and logarithmically divergent

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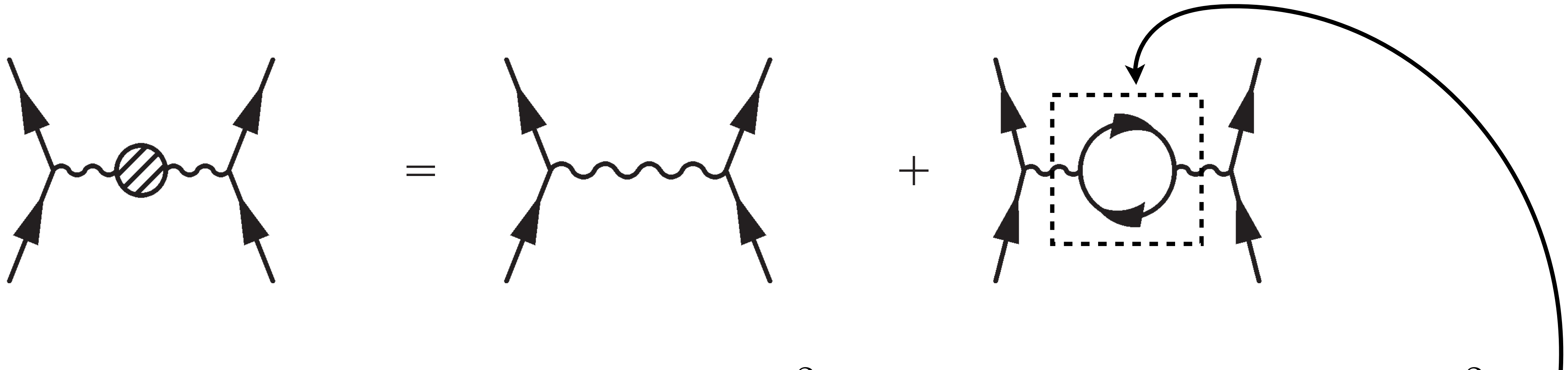

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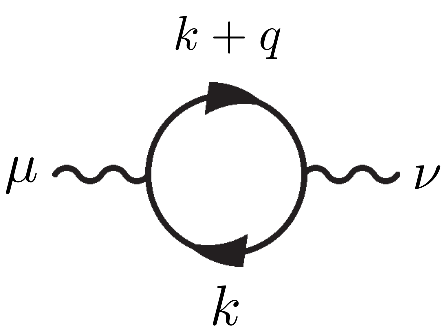

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Gauge invariant and logarithmically divergent

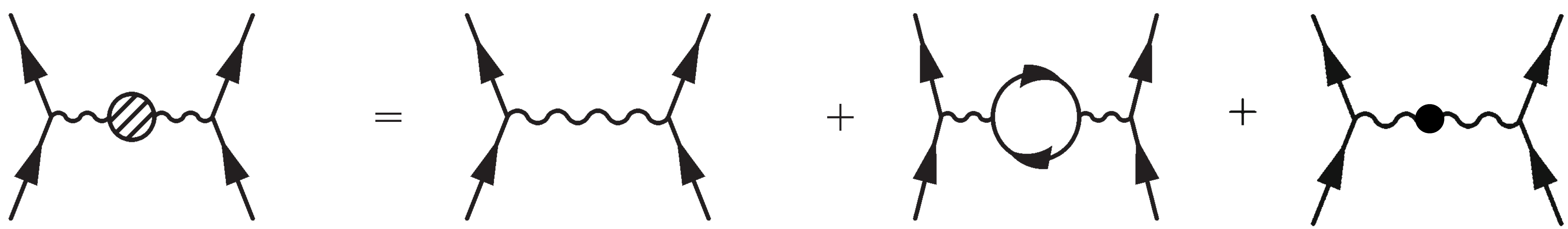
Forgetting about the **spurious quadratic divergence**, we have

$$\Pi_{\mu\nu}(q) = \left[\frac{e^2}{12\pi^2}\log\left(\frac{q^2}{\Lambda^2}\right) + \text{finite}\right](q^2\eta_{\mu\nu} - q_\mu q_\nu)$$

The logarithmic divergence can be cancelled by a **counterterm**

μ  ν $= -\dfrac{e^2}{12\pi^2}\log\left(\dfrac{\mu^2}{\Lambda^2}\right)(q^2\eta_{\mu\nu} - q_\mu q_\nu)$

The **total** contribution to the $e^-e^+ \to \mu^-\mu^+$ scattering is then



$$= \eta_{\alpha\beta}(\overline{v}_e\gamma^\alpha u_e)\left\{\frac{e^2}{4\pi q^2}\left[1 + \frac{e^2}{12\pi^2}\log\left(\frac{q^2}{\mu^2}\right)\right]\right\}(\overline{v}_\mu\gamma^\beta u_\mu)$$

$$\equiv \eta_{\alpha\beta}(\overline{v}_e\gamma^\alpha u_e)\left[\frac{e_{\text{eff}}(q^2)^2}{4\pi q^2}\right](\overline{v}_\mu\gamma^\beta u_\mu)$$

The **QED running effective charge** is then defined by

$$e_{\text{eff}}(q^2)^2 = e^2\left[1 + \frac{e^2}{12\pi^2}\log\left(\frac{q^2}{\mu^2}\right)\right]$$

$$e_{\text{eff}}(\mu)^2 = e_{\text{eff}}(\mu_0)^2\left[1 + \frac{e_{\text{eff}}(\mu_0)^2}{12\pi^2}\log\left(\frac{\mu^2}{\mu_0^2}\right)\right]$$

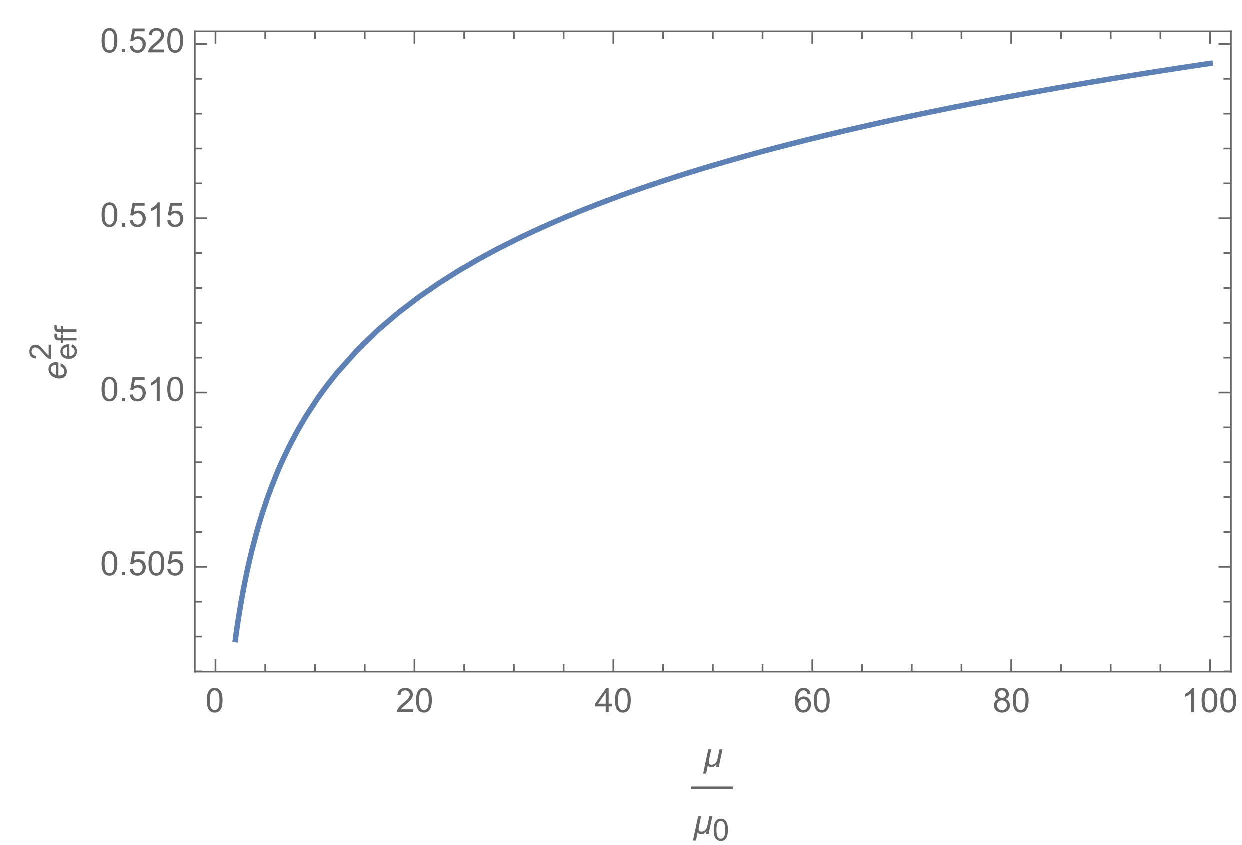
As in the ϕ^4 case, the **QED beta function** is **positive** and the coupling grows with energy

$$\beta(e)_{\text{QED}} = \frac{e^3}{12\pi^2} > 0$$

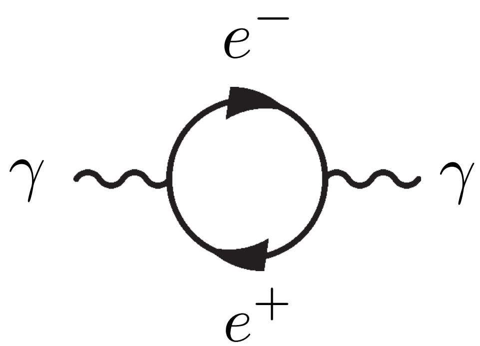
Again, there is a **Landau pole**, which for the **Standard Model** is located at

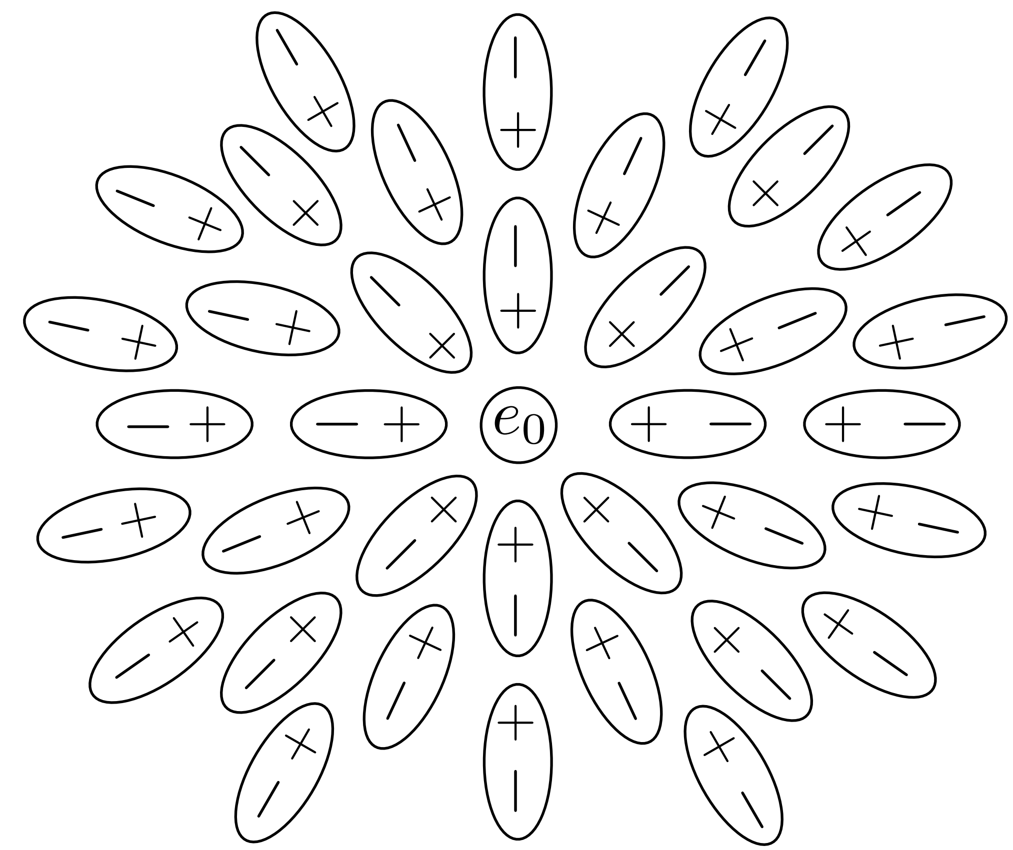
$$\mu_{\text{Landau}} \sim 10^{34}\ \text{GeV}$$

well **beyond** any other relevant energy scale.



Heuristically, the **running coupling** can be understood in terms of **screening**





As in a dielectric medium, the **polarization** of the vacuum screens the **bare** charge

$$e(\mu)^2 = e_0(\Lambda)^2 \left[1 + \frac{e_0(\Lambda)^2}{12\pi^2} \log\left(\frac{\mu^2}{\Lambda^2}\right)\right]$$

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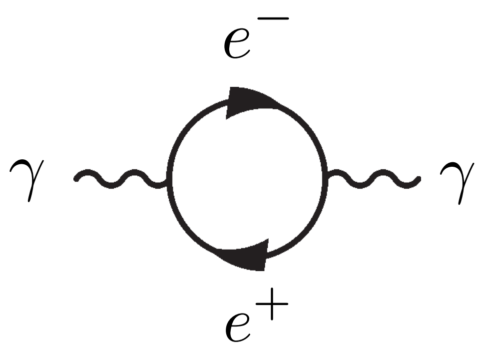

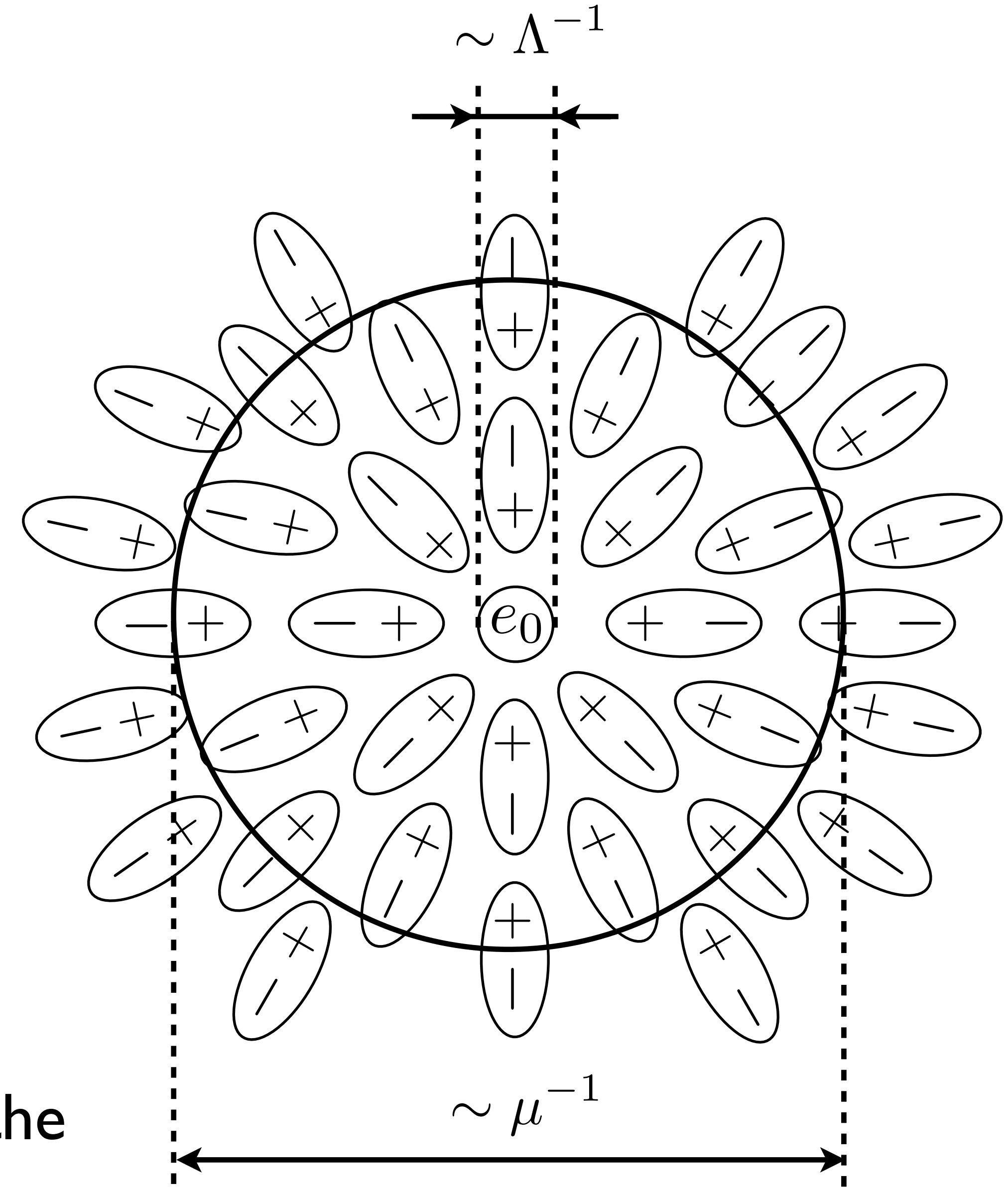

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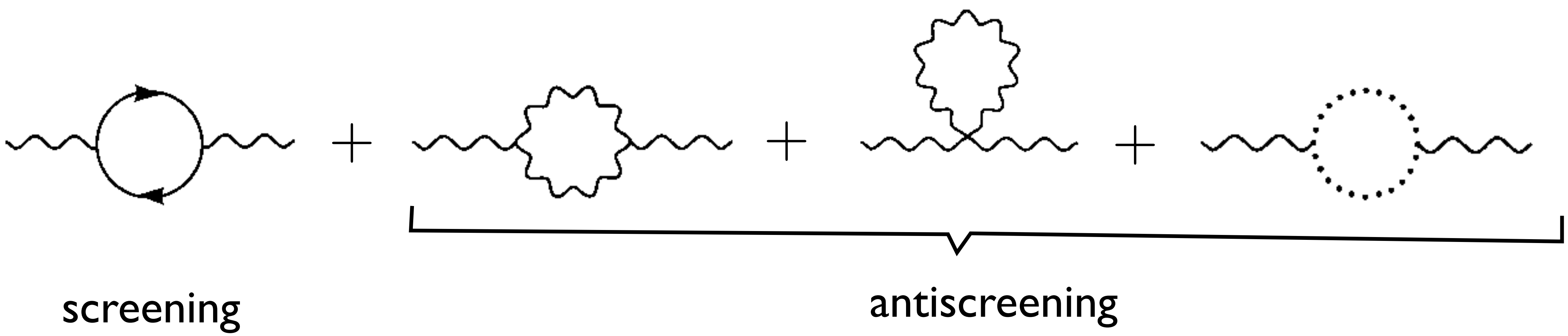
The behavior of the effective coupling is quite different for **non-Abelian gauge theories**

$$\mathscr{L} = -\frac{1}{4}F^A_{\mu\nu}F^{A\mu\nu} + i\sum_{k,\ell=1}^{N_c}\overline{\psi}_k\gamma^\mu(\delta_{k\ell}\partial_\mu - igA^B_\mu T^B_{k\ell})\psi_\ell$$

$$\boxed{F^A_{\mu\nu} = \partial_\mu A^A_\nu - \partial_\nu A^A_\mu + gf^{ABC}A^B_\mu A^C_\nu}$$

Now, both **fermions** and **gauge bosons** contribute to the gauge boson polarization tensor



For a $\mathbf{SU}(N_c)$ gauge theory, the **beta function** can be **negative**

$$\beta(g) = -\frac{g^3}{16\pi^2}\left(\frac{11}{3}N_c - \frac{2}{3}N_f\right)$$

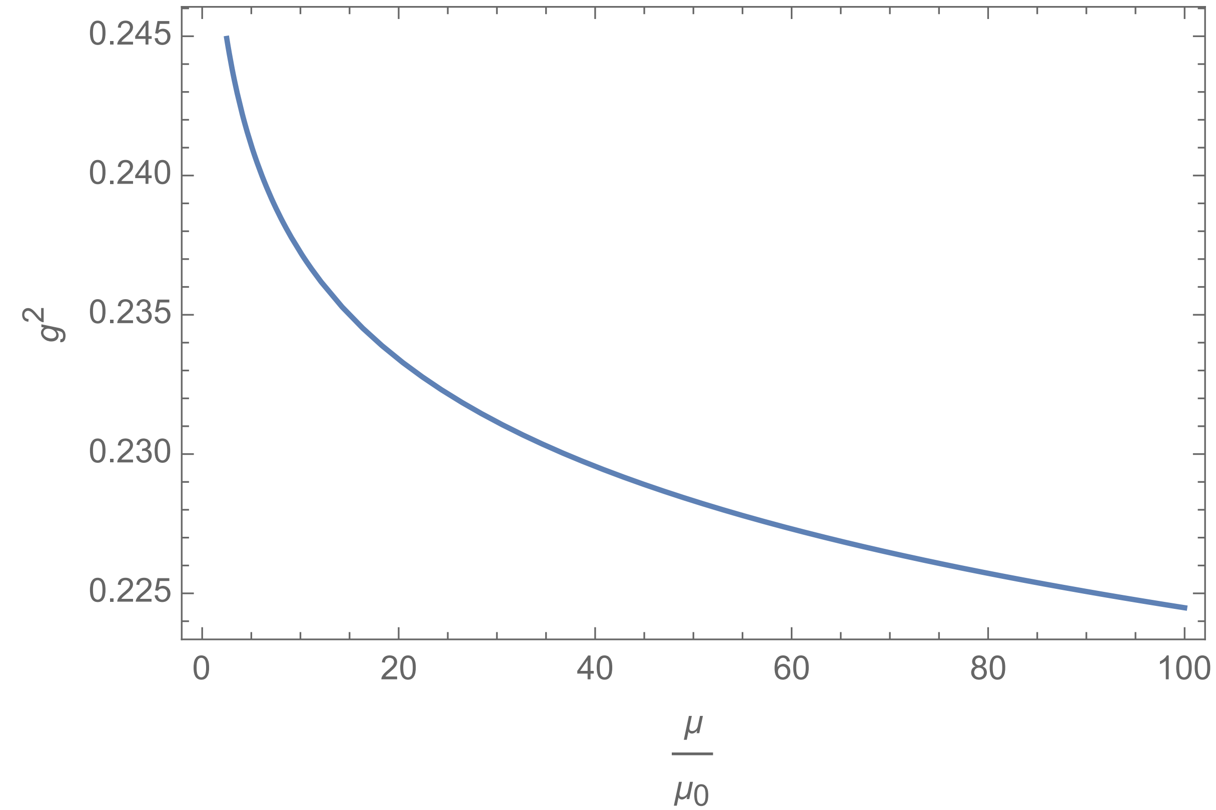
$N_c \equiv$ # of colors

$N_f \equiv$ # of flavors

For **QCD**, the beta function is **negative**

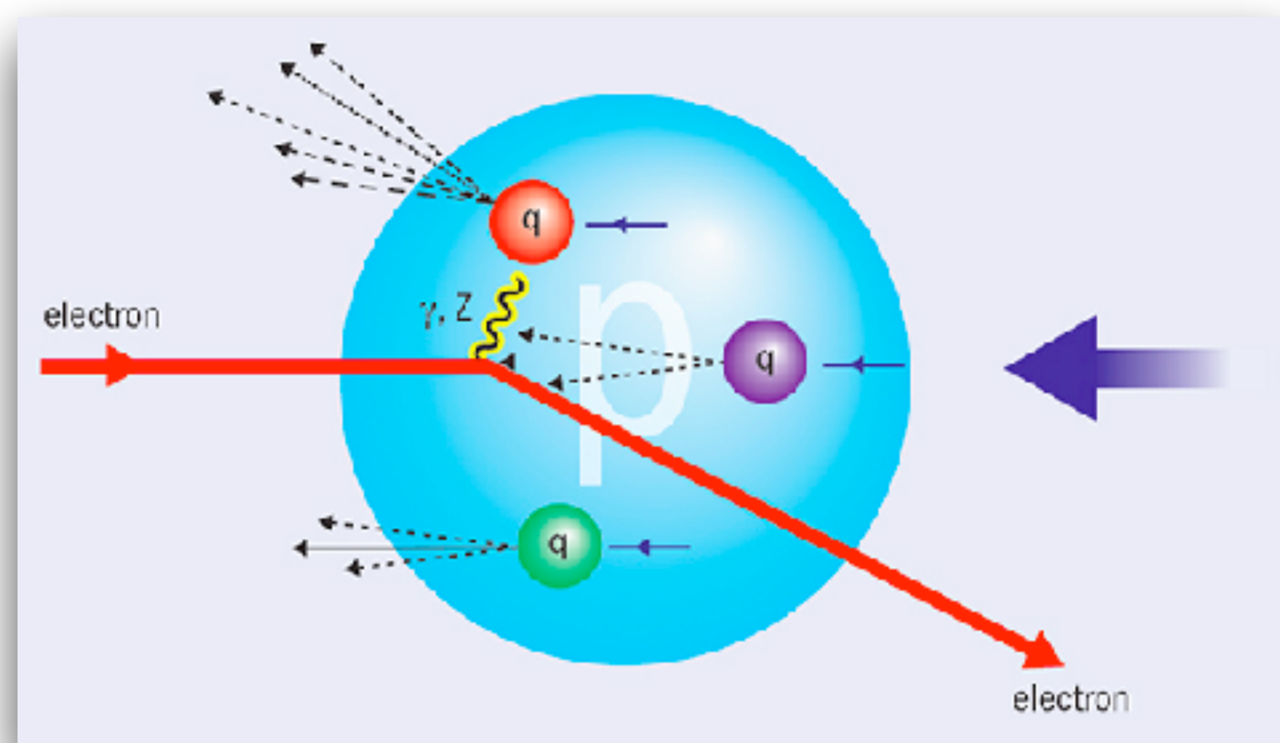
$$\left.\begin{array}{l} N_c = 3 \\ N_f = 6 \end{array}\right\} \quad \beta(g)_{\rm QCD} = -\frac{7g^3}{16\pi^2} < 0$$

$$g(\mu)^2 = g(\mu_0)^2\left[1 - \frac{7g(\mu_0)^2}{16\pi^2}\log\left(\frac{\mu^2}{\mu_0^2}\right)\right]$$



and the theory is **asymptotically free** at high energies.

This result explains the **quasifree behavior** of **partons** exhibited in **deep inelastic scattering**



David J. Gross
(b. 1941)

H. David Politzer
(b. 1949)

Frank Wilcek
(b. 1951)

What do ϕ^4 and **QED**, and **QCD** have in common?

Infinities are taken care of by renormalizing a **finite number** of **quantites**.

The **renormalized Lagrangian** contains a **finite** number of **operators**, e.g.

$$\mathscr{L}_{\rm ren} = Z_\psi(\Lambda)\overline{\psi}[i\gamma^\mu\partial_\mu - m_0(\Lambda)]\psi - \frac{1}{4}Z_A(\Lambda)F_{\mu\nu}F^{\mu\nu} - e_0(\Lambda)Z_\psi(\Lambda)\sqrt{Z_A(\Lambda)}A_\mu\overline{\psi}\gamma^\mu\psi$$

The theory is **renormalizable**

Rule of thumb: a theory is **renormalizable** if its bare Lagrangian **does not contain** higher-dimensional (**> 4**) operators.

All **coupling** constants have **non-negative energy dimensions**.

What do ϕ^4 and **QED**, and **QCD** have in common?

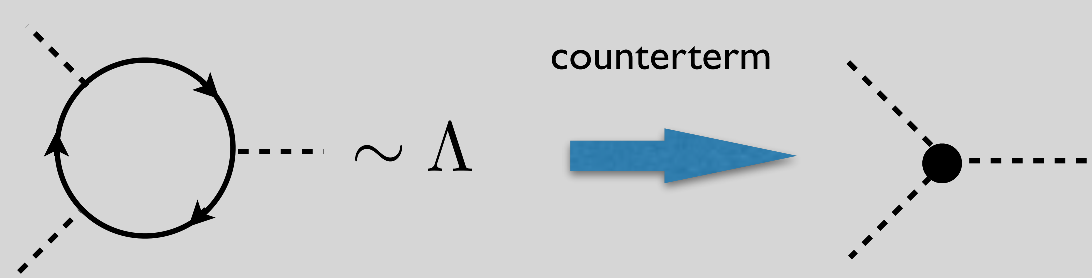
Infinities are taken care of by renormalizing a **finite number** of **quantites**.

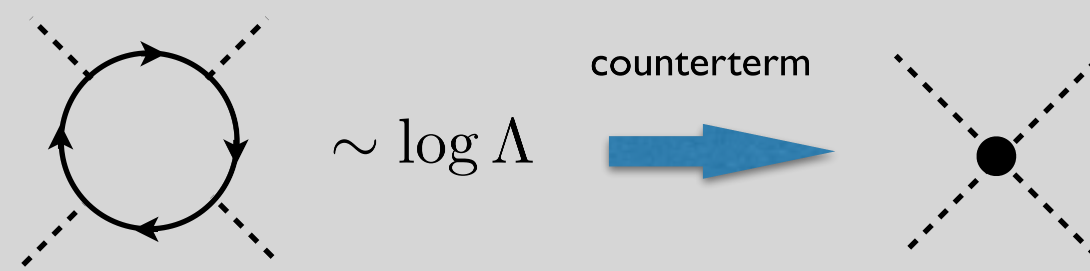
The **renormalized Lagrangian** contains a **finite** number of **operators**, e.g.

Sometimes, we need to **add operators** to renormalize the theory. In the **Yukawa** theory

$$\mathscr{L}_{\text{Yukawa}} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 + \overline{\psi}(i\partial\!\!\!/ - m)\psi + g\phi\overline{\psi}\psi$$

we have





and the renormalized Lagrangian contains **cubic** and **quartic couplings** for the scalar field

$$\mathscr{L}_{\text{ren}} = \frac{1}{2}Z_\phi(\Lambda)\partial_\mu\phi\partial^\mu\phi - Z_\phi(\Lambda)\frac{m_0(\Lambda)^2}{2}\phi^2 + Z_\psi(\Lambda)\overline{\psi}[i\partial\!\!\!/ - m_0(\Lambda)]\psi$$
$$+\sqrt{Z_\phi(\Lambda)}Z_\psi(\Lambda)g_0(\Lambda)\phi\overline{\psi}\psi - Z_\phi(\Lambda)^{\frac{3}{2}}\frac{\alpha_0(\Lambda)}{3!}\phi^3 - Z_\phi(\lambda)^2\frac{\lambda_0(\Lambda)}{4!}\phi^4$$

Take the case of a **scalar ϕ^n theory**.

For a diagram with E **external lines**, I **internal lines** and V **vertices**

total # of legs coming out of vertices

internal legs join two vertices

$$nV = E + 2I$$

external legs are attached to a single vertex

global conservation delta function

$$L = I - (V - 1) = I - V + 1$$

# of integrations

# of independent delta functions

On the other hand, the **superficial degree of divergence** of a diagram with E **external lines** is

$$D = 4L - 2I$$

This can be expressed in terms of E and V as

$L = I - V + 1$

$2I = nV - E$

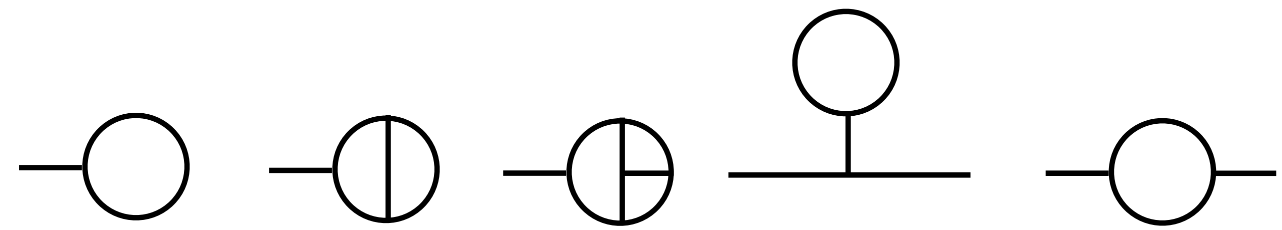
$$D = 4L - 2I = 2I - 4V + 4 = (n-4)V - E + 4$$

$$\boxed{D = (n-4)V - E + 4}$$

$$D = (n-4)V - E + 4$$

∗ $n = 3$

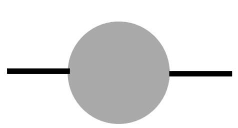
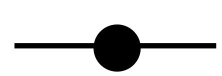
$D = 4 - E - V$ There is only a **finite** number of **superficially divergent diagrams**.

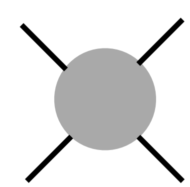
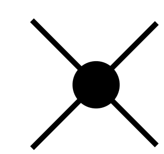


ϕ^3 theory is **superrenormalizable**

∗ $n = 4$

$D = 4 - E$ There are **infinitely many superficially divergent diagrams**. but **only** with **2** and **4** external legs

 $\sim \Lambda^2$ 

 $\sim \log\Lambda$ 

ϕ^4 theory is **renormalizable**

$$D = (n-4)V - E + 4$$

* **$n = 6$**

$$D = 2V - E + 4$$

There are **infinitely many divergent diagrams** with an **arbitrary number of external legs**

Thus, we need to add an **infinite number** of **counterterms** with **arbitrary** number of **external legs**.

The **renormalized Lagrangian** contains an **infinite number** of **operators**

$$\mathscr{L}_{\rm ren} = \frac{1}{2}Z_\phi(\Lambda)\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}Z_\phi(\Lambda)m_0(\Lambda)^2\phi^2 - \sum_{n=2}^{\infty}\frac{1}{(2n)!}Z_\phi(\Lambda)^n\lambda_{2n,0}(\Lambda)\phi^{2n}$$

In principle, to compute amplitudes we need to specify **infinitely many renormalizations conditions**!

ϕ^6 theory is **not renormalizable** (as well as ϕ^n for $n > 4$)

$$D = (n-4)V - E + 4$$

* $n = 6$

$$D = 2V - E + 4$$ ⟶ There are **infinitely many divergent diagrams** with an **arbitrary number of external legs**

Let us look at the **energy dimensions** of the **coupling constants**:

$$[\lambda_n] = 4 - n$$

* ϕ^3 **theory**: $[\lambda_3] = 1$ ⟶ superrenormalizable
* ϕ^4 **theory**: $[\lambda_4] = 0$ ⟶ renormalizable
* ϕ^n **theory** ($n > 4$): $[\lambda_n] = 4 - n < 0$ ⟶ non renormalizable

In

conditions!

ϕ^6 **theory** is **not renormalizable** (as well as ϕ^n for $n > 4$)

A **physical** (i.e., **Wilsonian**) view of **renormalization**.

Kenneth G. Wilson (1936-2013)

Let us **take** the **cutoff seriously** and start with our quantum field theory defined at the **scale Λ**

bare couplings

$E = \Lambda$

$$S[\phi_a, \Lambda] = \int d^4x \left\{ \mathscr{L}_0[\phi_a] + \sum_i g_i(\Lambda)\mathscr{O}_i[\phi_a] \right\}$$

bare fields

$$e^{iS[\phi'_a, \mu]} = \int\limits_{\mu<p<\Lambda} \prod_a \mathscr{D}\phi_a \, e^{iS[\phi_a, \Lambda]}$$

renormalized couplings

$E = \mu$

$$S[\phi'_a, \mu] = \int d^4x \left\{ \mathscr{L}_0[\phi'_a] + \sum_i g_i(\mu)\mathscr{O}_i[\phi'_a] \right\}$$

renormalized fields

Thank you