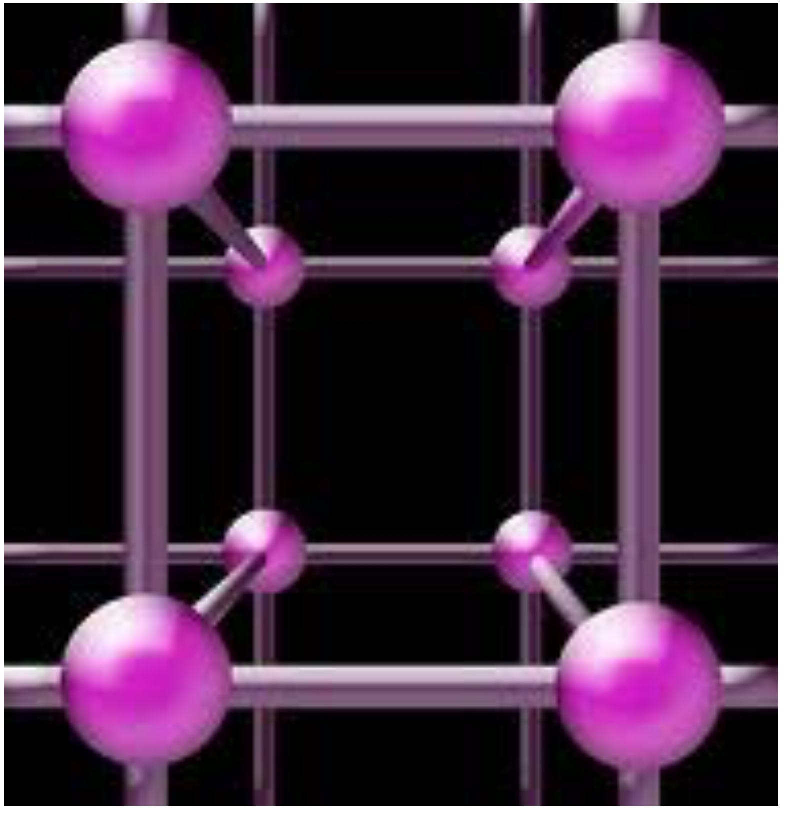
# Plan

Part I: Functional Formulation of QFT, renormalization, Wilson RG

Part II: Lattice Formulation of scalar, fermion and gauge QFT

Part III: Lattice QCD: numerical methods and applications



# Introduction to Lattice Field Theory

P. Hernández (IFIC, UVEG-CSIC)

# Plan

Part I: Functional Formulation of QFT, renormalization, Wilson RG

Part II: Lattice Formulation of scalar, fermion and gauge QFT

Part III: Lattice QCD: numerical methods and applications

# Bibliography

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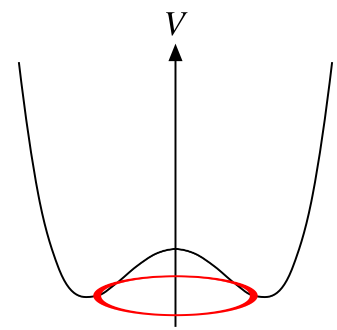
# Introduction

The Standard Model of particle physics has been tested sub % to be the theory describing microscopic particles and their interactions

$$\mathcal{L}_{SM} = \mathcal{L}_{gauge} + \mathcal{L}_{matter} + \mathcal{L}_{SSB}$$




Gauge principle

$$SU(3) \times SU(2) \times U(1)_Y$$

Lepton<-> quark (anomaly cancellation)

| $(\mathbf{1},\mathbf{2})_{-\frac{1}{2}}$ | $(\mathbf{3},\mathbf{2})_{-\frac{1}{6}}$ | $(\mathbf{1},\mathbf{1})_{-1}$ | $(\mathbf{3},\mathbf{1})_{-\frac{2}{3}}$ | $(\mathbf{3},\mathbf{1})_{-\frac{1}{3}}$ |
|---|---|---|---|---|
| $\begin{pmatrix}\nu_e\\ e\end{pmatrix}_L$ | $\begin{pmatrix}u^i\\ d^i\end{pmatrix}_L$ | $e_R$ | $u^i_R$ | $d^i_R$ |
| $\begin{pmatrix}\nu_\mu\\ \mu\end{pmatrix}_L$ | $\begin{pmatrix}c^i\\ s^i\end{pmatrix}_L$ | $\mu_R$ | $c^i_R$ | $s^i_R$ |
| $\begin{pmatrix}\nu_\tau\\ \tau\end{pmatrix}_L$ | $\begin{pmatrix}t^i\\ b^i\end{pmatrix}_L$ | $\tau_R$ | $t^i_R$ | $b^i_R$ |

Family

Parity Violation

# Introduction

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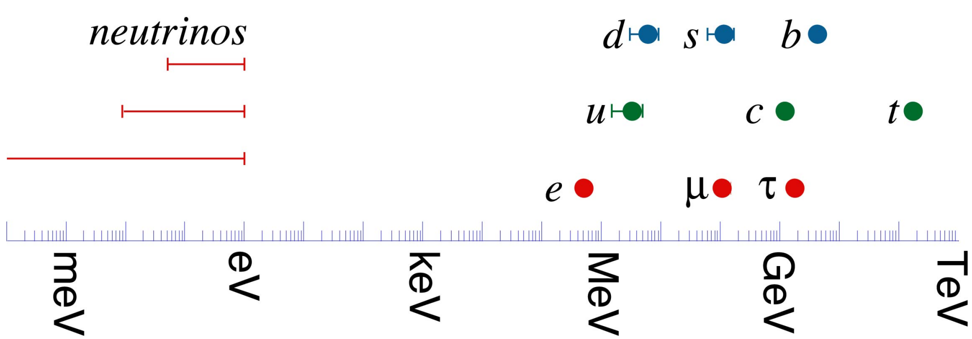
Gauge principle

$$SU(3) \times SU(2) \times U(1)_Y$$

Flavour Puzzle

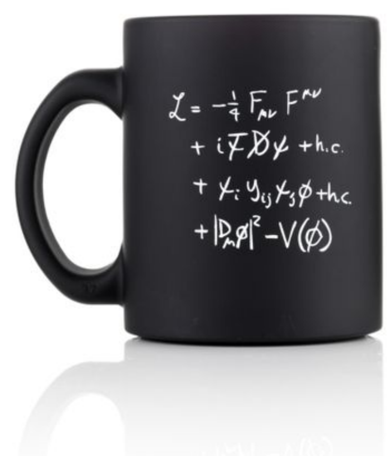


# Introduction

The Standard Model of particle physics has been tested sub % to be the theory describing microscopic particles and their interactions

$$\mathcal{L}_{SM} = \mathcal{L}_{gauge} + \mathcal{L}_{matter} + \mathcal{L}_{SSB}$$

$$\begin{aligned}
\mathcal{L}_{gauge} &= -\frac{1}{4g^2_{U(1)}} B_{\mu\nu} B_{\mu\nu} - \frac{1}{4g^2_{SU(2)}} W_{\mu\nu} W_{\mu\nu} - \frac{1}{4g^2_{SU(3)}} G_{\mu\nu} G_{\mu\nu} \\
\mathcal{L}_{matter} &= \sum_a \bar{\Psi}^a i \not{D} \Psi^a \\
\mathcal{L}_{SSB} &= \sum_{ab} \bar{\Psi}^a Y_{ab} \Phi \Psi^b + h.c. + \mathcal{L}(\Phi)
\end{aligned}$$



# Introduction

The Standard Model interactions imply a # of free parameters and accidental symmetries:

| Sector | Free Param. | Discrete Sym. | Flavour Sym. |
| --- | --- | --- | --- |
| Gauge | 3 | $C, P, T$ | |
| Gauge+matter | 3 | $T, \not{C}, \not{P}$ | $\prod_{\text{multiplet}} U(N_f)$ |
| Gauge+matter+SSB | 22-24 | $\not{C}, \not{P}, \not{T}$ | $U(1)_{B-L}$ or none |

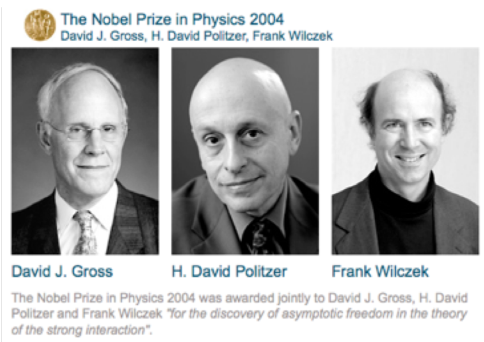
+ A non-accidental "symmetry": strong CP

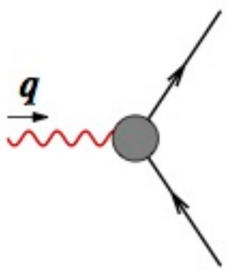
$$\mathcal{L}_{\rm SM} \supset \bar{\theta}\frac{\alpha_s}{8\pi}G\tilde{G} \qquad\qquad \bar{\theta} \leq 10^{-10}$$

Most of what we know is derived from perturbation theory and is not enough!

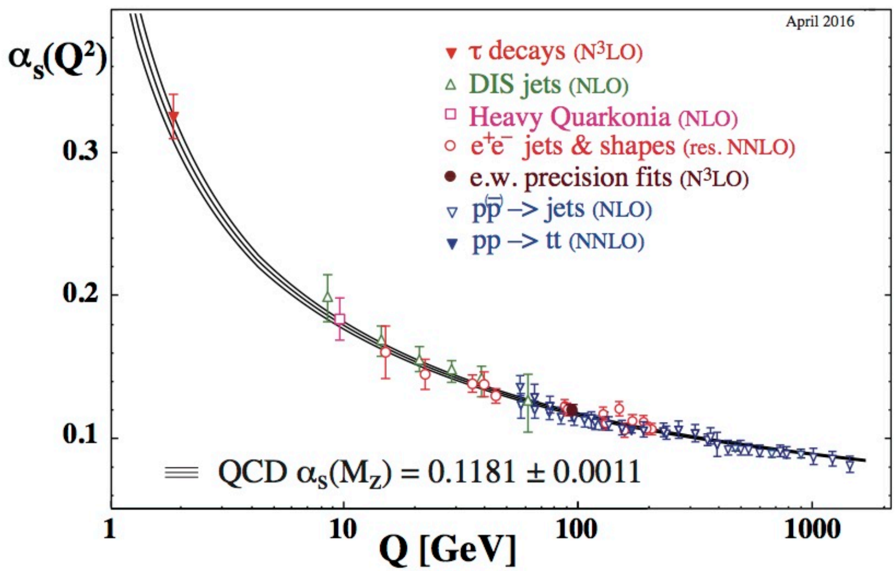
# The need to go beyond perturbation theory

SU(3) interactions weak at large energies become strong at low energies




$\bar{g}(q) =$ 


Asymptotic freedom




Growth of the coupling at low energies: Confinement

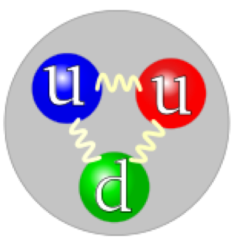
Generation of a mass gap

Chiral symmetry breaking

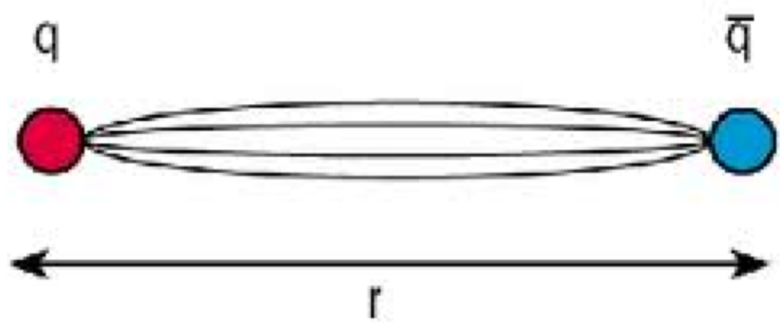
# Confinement

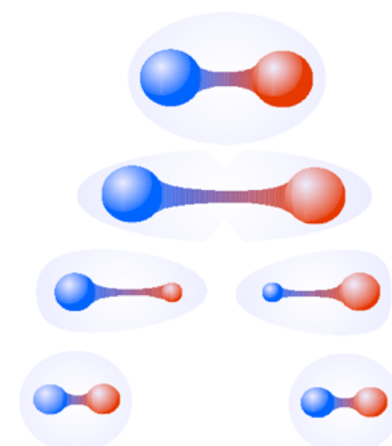
We do not observe asymptotic states with net color charge, only hadrons which are color singlets

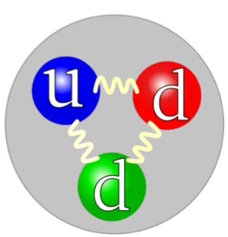
Static potential (potential between infinitely heavy quarks) grows with r:




Proton






Neutron

$$V(r) \sim \sigma r$$

string tension

# Mass gap

Light hadron masses (except pions) are dominated by the strong binding energy

$$\frac{m_{\text{proton}}}{2m_u + m_d} \sim 100$$

The mass of ordinary matter is mostly color binding energy!

One of the 6 Millennium Prize Problems still to be solved (1M$ prize!)

Yang–Mills Existence and Mass Gap. Prove that for any compact simple gauge group G, a non-trivial quantum Yang–Mills theory exists on $R_4$ and has a mass gap $\Delta > 0$. Existence includes establishing axiomatic properties at least as strong as those cited in [45, 35].

# Spontaneous Chiral Symmetry Breaking

The lightest pseudoscalar mesons are significantly lighter than the mass gap because they are Nambu-Goldstone bosons of chiral symmetry breaking

In the limit $m_u = m_d = 0$, there is a chiral global symmetry in QCD

$$\begin{pmatrix} u \\ d \end{pmatrix}_L \to U_L \begin{pmatrix} u \\ d \end{pmatrix}_L \qquad \begin{pmatrix} u \\ d \end{pmatrix}_R \to U_R \begin{pmatrix} u \\ d \end{pmatrix}_R$$

Due to the strong interactions a quark condensate forms:

$$\langle \bar{u}u \rangle = \langle \bar{d}d \rangle = \Sigma \ \delta_{ij}$$

Spontaneous symmetry breaking:

$$SU(2)_L \times SU(2)_R \to SU(2)_V$$

Three goldstone bosons: $\pi^{\pm}, \pi^0$

# Chiral Symmetry Breaking

Goldstone theorem:

$$\langle 0|A^a_\mu(x)|\pi^a(p)\rangle = ip_\mu F_\pi e^{-ipx}, \quad A^a_\mu = \bar{Q}\gamma_\mu\gamma_5 T^a Q$$

$$\langle \partial_\mu A^a_\mu(x) P^a(y)\rangle = \langle \bar{Q}(x)\{M, T^a\}\gamma_5 Q(x)\ P^a(y)\rangle - \delta(x-y)\langle \delta^a P^a(y)\rangle$$

$$\boxed{M_\pi^2 = (m_u + m_d)|\langle 0|(\bar{u}u + \bar{d}d)|0\rangle|\frac{1}{F_\pi^2}}$$

[Gell-Mann, Oakes, Renner]

Light pseudoscalar mesons are very sensitive to light quark masses, the latter can be extracted from the former

# Anomalous Chiral Symmetry Breaking

For $m_u = m_d = 0$ the symmetry group at the classical level also contains

$$U(1)_L \times U(1)_R \to U(1)_V$$

However: there is no fourth goldstone boson: $\eta'$

$$m_{\eta'} \sim m_{\text{proton}}$$

[t'Hooft; Witten; Veneziano]

$U(1)_A$ broken by the anomaly:

$$\partial_\mu A^\mu = \frac{g^2}{32\pi^2} G^a_{\mu\nu} \tilde{G}^a_{\mu\nu}$$

The term on the right-hand side has no effect in perturbation theory, but configurations exist that make this non vanishing beyond perturbation theory

# Asymptotic freedom versus Landau Poles/Triviality

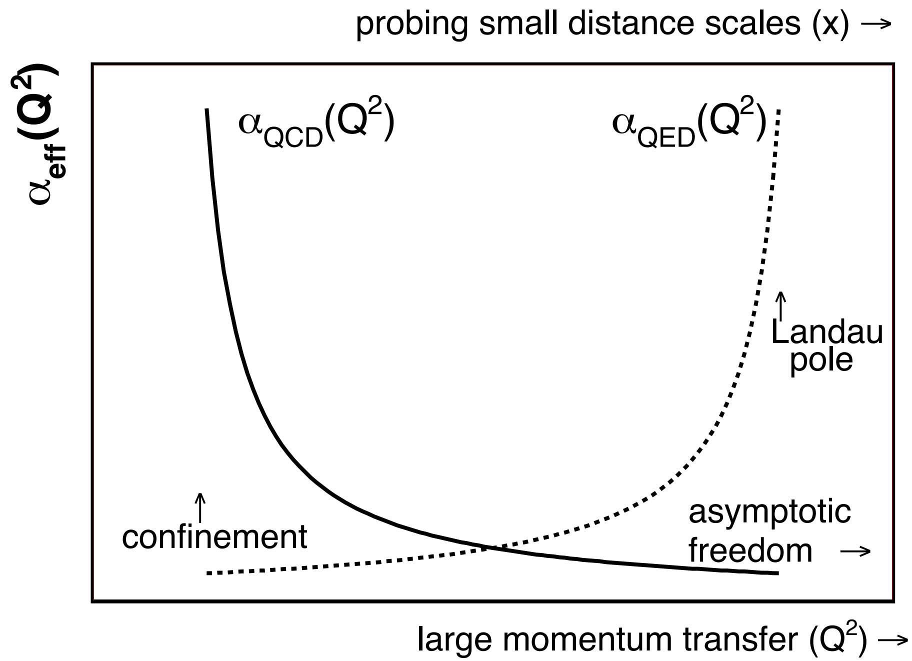
$$\beta(\bar{q}) = \frac{\partial \bar{g}(q)}{\partial \ln q} = \beta_0 \bar{g}^3 + \mathcal{O}(\bar{g}^5)$$

$\beta_0 < 0$ Asymptotic Freedom

$\beta_0 > 0$ Triviality and Landau Pole

$$[\alpha(q)]_{\mathrm{QCD}} \equiv \frac{\bar{g}(q)^2}{4\pi} \sim_{q\to\infty} \frac{c}{\ln(\frac{q}{\Lambda})}$$

$$(Q_{\mathrm{Landau\ Pole}})_{QED} = m_e \exp\left(\frac{1}{\beta_0 \bar{g}(m_e)^2}\right)$$



# The need to go beyond perturbation theory

The Standard Model at arbitrary high energy ?

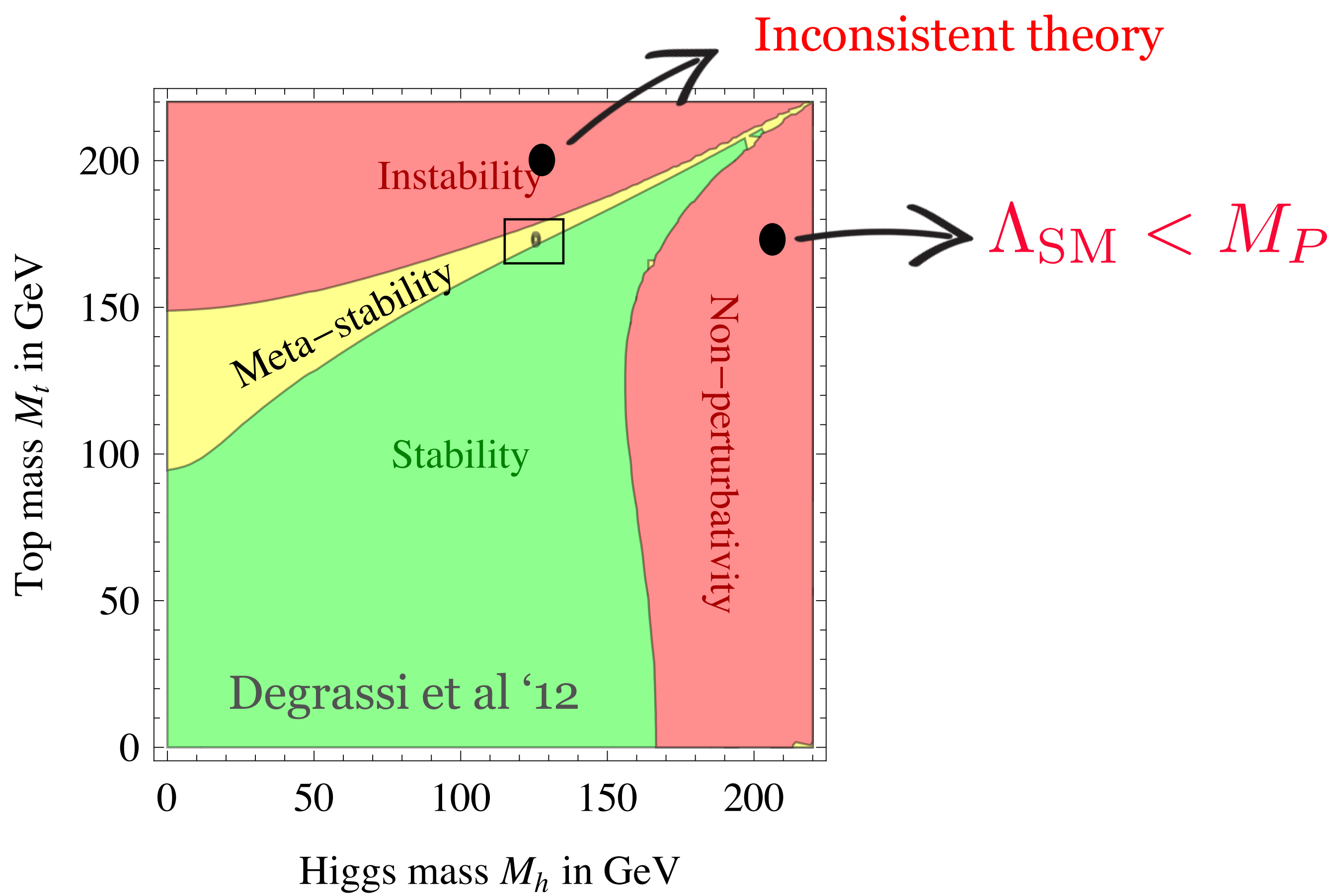
$$V(\phi) = -\mu^2 \phi^\dagger \phi + \lambda (\phi^\dagger \phi)^2 \qquad U_Y(1)$$

1. Landau poles (perturbative) <-> triviality (non-perturbative)

$$\Lambda \to \infty : \lambda_R(\mu) = 0, g_{1\,R}(\mu) = 0$$

2. Stability of the higgs potential <-> $\lambda_R(\mu) > 0$

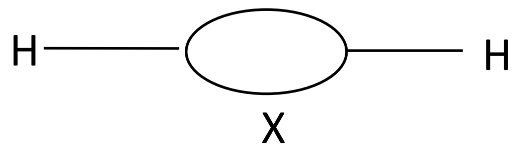
Intriguing correlation between SM pararameters: $m_t$, $m_h$, $\alpha_s$ !




The Standard Model is borderline OK up to the Planck scale but not beyond

# The (B)SM puzzles

- If there is new physics, there is a hierarchy problem




$$\delta M_H^2 \propto M_X^2 \log' s$$

Solutions involve strong interactions: SUSY breaking, Technicolor, etc

- Flavoured new physics (e.g. FCNC) more strongly constrained than unflavoured ones:

$$\frac{\Lambda}{\sqrt{c}} \geq [10^2 - 10^5]\ \mathrm{TeV}$$

Flavoured new physics in the quark sector involve strong interactions

- Strong CP problem…if CP is broken why not by QCD ? $\theta F^a_{\mu\nu} \tilde{F}^{a\mu\nu}$

  Term irrelevant in the perturbation theory

# Beyond PTh: Lattice Quantum Field Theory

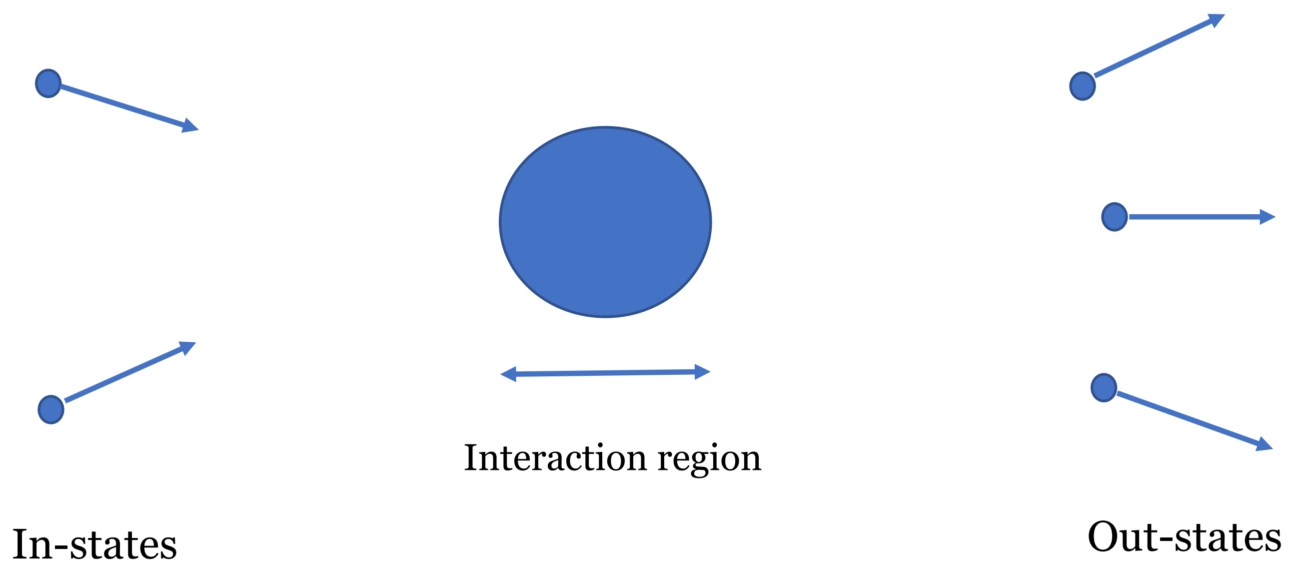
Basic idea due to K. Wilson: convert the path-integral formulation of a QFT into a statistical system by discretizing space-time

QCD and asymptotically free renormalizable theories are benchmarks

# Lattice Quantum Field Theory

## Lehman-Symanzik-Zimmerman Reduction Formula

Cross sections, decay widths <-> Field Correlation functions




$$\prod_{i=1}^{n}\int d^4x_i e^{ip_i\cdot x_i}\prod_{j=1}^{k}\int d^4y_j e^{-iq_j\cdot y_i}\langle 0|T\left(\hat{\phi}(x_1)...\hat{\phi}(x_n)\hat{\phi}(y_1)....\hat{\phi}(y_k)\right)|0\rangle$$

$$\simeq_{p_i^0\to E_{\mathbf{p}_i},q_j^0\to E_{\mathbf{q}_j}}\prod_{i=1}^{n}\left(\frac{i\sqrt{Z}}{p_i^2-m^2+i\epsilon}\right)\prod_{j=1}^{k}\left(\frac{i\sqrt{Z}}{q_j^2-m^2+i\epsilon}\right)\langle\mathbf{p}_1,....,\mathbf{p}_n,out|\mathbf{q}_1,...,\mathbf{q}_k;in\rangle,$$

# Lattice Quantum Field Theory

From Minkowski to Euclidean via a Wick rotation

$$W_n(t_1, \mathbf{x_1}; ...., t_n, \mathbf{x}_n) \quad = \quad \langle 0|\hat{\phi}(t_1, \mathbf{x}_1)...\hat{\phi}(t_n, \mathbf{x}_n)|0\rangle, \quad t_1 \geq t_2..... \geq t_n$$

$$S_n(x_1, ..., x_n) = W_n(-ix_1^0, \mathbf{x}_1; ... - ix_n^0, \mathbf{x}_n),$$

From quantum to classical variables: path integral representation

$$S_n = \frac{\int_{PBC} \mathcal{D}\phi \; e^{-S[\phi]} \phi(\mathbf{x}_1, t_1)....\phi(\mathbf{x}_n, t_n)}{\int_{PBC} \mathcal{D}\phi \; e^{-S[\phi]}} \equiv \langle \phi(x_1)....\phi(x_n) \rangle$$

# Lattice Quantum Field Theory

Characterization of asymptotic states (Z, m):

KL Spectral decomposition of the propagator in energy and momentum eigenstates |α>

$$\langle\phi(x)\phi(0)\rangle = \sum_\alpha \int \frac{d^3p}{(2\pi)^3 2E_{\mathbf{p}}(\alpha)} |Z_\alpha|^2 e^{-E_{\mathbf{p}}(\alpha)x_0} e^{i\mathbf{p}\cdot\mathbf{x}}$$

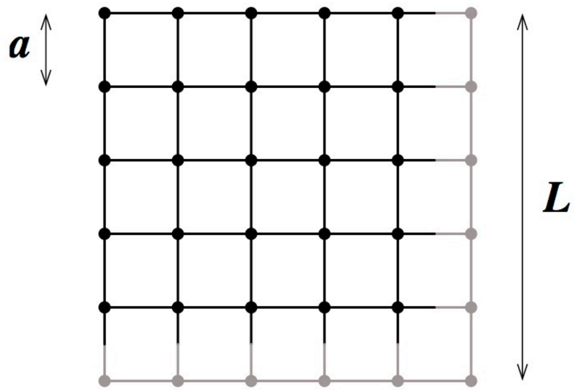
$$Z_\alpha \equiv \langle 0|\phi(0)|\alpha(\mathbf{0})\rangle \qquad E_\alpha^2(\mathbf{p}) = m_\alpha^2 + \mathbf{p}^2$$

Dominated by the lowest energy states: one particle states

$$\lim_{x_0\to\infty} \int d^3x \ \langle\phi(x)\phi(0)\rangle \propto e^{-m_\alpha x_0}$$

# Lattice Quantum Field Theory

From functional integrals to multidimensional ordinary integrals via discretization of space-time




$x = a\,(n_0, n_1, n_2, n_3)\,, \quad n_\mu \in \mathbb{Z}$

$a \equiv$ lattice spacing

$L/a \equiv$ lattice size

$L \leq \infty$, possibly $T \neq L$

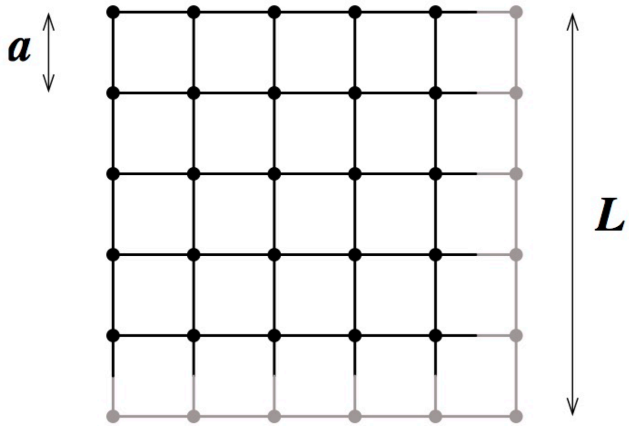
$$\phi(x) \to \phi(an) \qquad \int dx_i \to a \sum_{n_i \in Z} \qquad \int d^4x \to a^4 \sum_x \equiv a^4 \sum_{n \in Z^4}$$

$$\tilde{F}(p) = a^4 \sum_n e^{-ipna} F(na) \qquad p \in \left[-\frac{\pi}{a}, \frac{\pi}{a}\right]$$

$$\int_{-\pi/a}^{\pi/a} \frac{d^4p}{(2\pi)^4} e^{ipna} \tilde{F}(p) = F(na) \qquad \frac{\pi}{a} \leftrightarrow \text{momentum cutoff}$$

# Lattice Quantum Field Theory

From functional integrals to multidimensional ordinary integrals via discretization of space-time




$x = a\,(n_0, n_1, n_2, n_3)\,, \qquad n_\mu \in \mathbb{Z}$

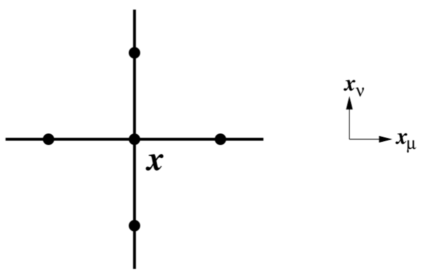
$a \equiv$ lattice spacing

$L/a \equiv$ lattice size

$L \leq \infty$, possibly $T \neq L$

$$\partial_\mu \phi(x) \rightarrow \;\; \hat{\partial}_\mu \phi(x) \equiv \frac{1}{a}\left(\phi(x + a\hat{\mu}) - \phi(x)\right)$$

$$\hat{\partial}^*_\mu \phi(x) \equiv \frac{1}{a}\left(\phi(x) - \phi(x - a\hat{\mu})\right)$$




Any lattice derivative involves high dimension operators

$$\hat{\partial}_\mu \phi = \partial_\mu \phi + \frac{1}{2} a \partial_\mu \partial_\mu \phi + \mathcal{O}(a^2) \qquad\qquad \frac{1}{2}\left(\hat{\partial}_\mu + \hat{\partial}^*_\mu\right)\phi = \partial_\mu \phi + \mathcal{O}(a^2)$$

# Lattice Quantum Field Theory

Perturbative renormalizability: generic 1PI diagram in scalar $\lambda\phi^4$

$$\Gamma^{(N)}(p_1,\ldots,p_N) \sim \int^{\Lambda} \prod_{l=1}^{L} d^4 q_l \prod_{i=1}^{I} \frac{1}{k_i(q_l,p_j)^2+m^2} \quad \propto \Lambda^{\omega}$$

$$\omega = \text{superficial degree of divergence} = 4L - 2I$$

Using:

$$4V = N + 2I$$

$$L = I - V + 1$$

$$\boxed{\omega = 4 - N}$$

Only N=2, 4 can be divergent

$$\Gamma^{(2)} = A\,\partial_\mu\phi\partial_\mu\phi + B\phi^2$$

$$\Gamma^{(4)} = C\phi^4$$

Divergences in A, B, C can be reabsorbed in Z, m, $\lambda$ (proof to all others difficult!)

# Lattice Quantum Field Theory

d>4 interactions

$$V^{(1)}[\phi] = g_V \left(\partial\right)^{N_\partial} \left(\phi\right)^{N_\phi} \qquad [g_V] = 4 - N_\phi - N_\partial$$

$$\omega = 4 - N - [g_V]V$$

As more vertices of this type are included in a diagram higher N is necessary to absorb the divergence

| | | |
|---|---|---|
| Perturbative renormalizability: | $[g_V] > 0$ | Superrenormalizable |
| | $[g_V] = 0$ | Renormalizable |
| | $[g_V] < 0$ | Non renormalizable |

The lattice formulation is not renormalizable in this sense...

# Wilsonian Renormalization

Consider a QFT with a fundamental cutoff $\int_{-\Lambda}^{\Lambda} dp$

$$S_\Lambda[\phi] = \int_x \frac{1}{2}\partial_\mu\phi\partial_\mu\phi + \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4 + \frac{\lambda'}{\Lambda^2}\phi^6 + \frac{c_2}{\Lambda^2}\phi\partial^4\phi + \ldots$$

$\omega$ counts the powers of $\Lambda$

*Exercise: show that*

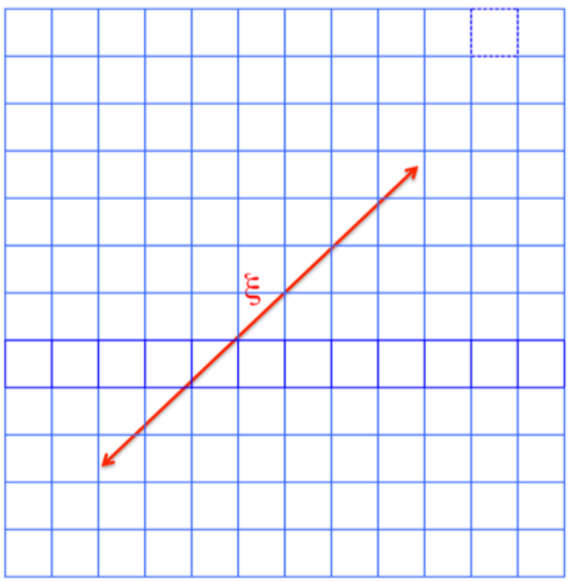
$$\boxed{\omega = 4 - N}$$

There is nothing special about a theory that is perturbatively renormalizable!

# Wilsonian Renormalization

Renormalizability is an emergent phenomenon in a theory with a fundamental cutoff (such as a lattice QFT $\Lambda = a^{-1}$ )

Renormalizability <-> existence of a continuum limit (criticality)




$$\langle\phi(x)\phi(0)\rangle \propto e^{-x/\xi}$$

$$\xi = m^{-1} \gg a$$

Continuum limit: a->0, keeping ξ fixed

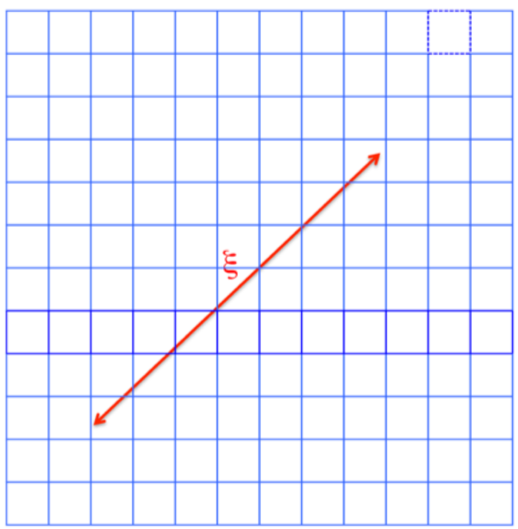
$$\frac{\xi}{a} \to \infty \;\; \Leftrightarrow \;\; ma \to 0$$

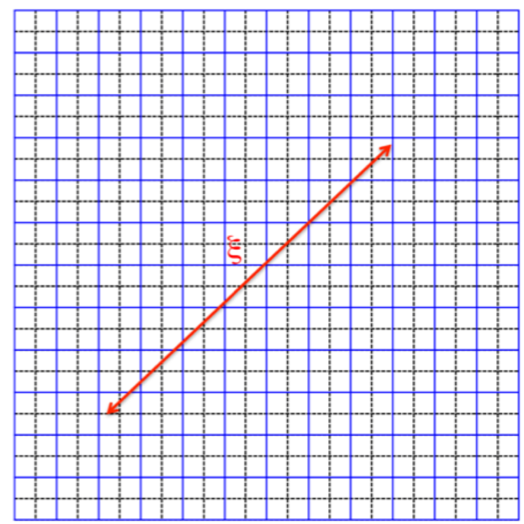
Empirical fact: many systems near critical points behave similarly (universality class)

# Wilsonian Renormalization

$$a \geq a_1 \geq a_2 ... \geq a_n = (1-\epsilon)^n a, \quad \epsilon \ll 1$$

At each step we integrate the modes $[a_{n-1}^{-1}, a_n^{-1}]$




$$S^{(n)}(a) = \sum_{i,x} g_i^{(n)} O_i(x) \qquad S^{(n+1)}(a) = \sum_{i,x} g_i^{(n+1)} O_i(x)$$

Renormalization group transformation $\quad g_i^{(n+1)} = R_i^{(n)}(g^{(n)})$

# Wilsonian Renormalization

At each step we integrate the modes $[a_{n-1}^{-1}, a_n^{-1}]$ to match to a theory with the same cutoff

$$
\begin{aligned}
S(a_1) &\rightarrow \quad S^{(1)}(a) = \sum_\alpha g_\alpha^{(1)}(a) \sum_x O_\alpha(\phi(x), a) \\
S(a_2) &\rightarrow \quad S^{(1)}(a_1) \rightarrow S^{(2)}(a) = \sum_\alpha g_\alpha^{(2)}(a) \sum_x O_\alpha(\phi(x), a) \\
\ldots. \\
S(a_n) &\rightarrow \quad \ldots. \sum_\alpha g_\alpha^{(n)}(a) \sum_x O_\alpha(\phi(x), a)
\end{aligned}
$$

Fixed Point of RG $\qquad g_i^* = R_i(g^*)$

$$m_\alpha(g^*) = \text{fixed} \rightarrow m_\alpha(g*)a \rightarrow 0$$

# Wilsonian Renormalization

Renormalizability <-> Universality $\Longrightarrow$ Fixed Point of RG

$$g_i^{(n+1)} = R_i^{(n)}(g^{(n)})$$

Near a FP:

$$g_\alpha^{(n+1)} - g_\alpha^* = \left.\frac{\partial R_\alpha}{\partial g_\beta}\right|_{g^*} (g_\beta^{(n)} - g_\beta^*),$$

$$\Delta g_\alpha^{(n+1)} = M_{\alpha\beta} \Delta g_\beta^{(n)}, \qquad M_{\alpha\beta} \equiv \left.\frac{\partial R_\alpha}{\partial g_\beta}\right|_{g^*}$$

Different situations depending on eigenvalues of M

| | | |
|---|---|---|
| $\lambda > 1$ | $\Delta g_\alpha^{(n)}$ increases as $n \to \infty$ | $\alpha$ is a relevant direction |
| $\lambda = 1$ | $\Delta g_\alpha^{(n)}$ stays the same as $n \to \infty$ | $\alpha$ is a marginal direction |
| $\lambda < 1$ | $\Delta g_\alpha^{(n)}$ decreases as $n \to \infty$ | $\alpha$ is an irrelevant direction |

# of relevant directions is usually small: universality & renormalizability

*Exercise:* *convince yourself that a free massless scalar is a fixed point (gaussian fixed point)*

*Start with a generic lattice action quadratic in the fields but otherwise arbitrary*

$$S(a) = \int_{BZ(a)} \frac{d^4p}{(2\pi)^4} \frac{1}{2} \phi(-p) \left( p^2 + m_0^2 \frac{1}{a^2} + g_1 a^2 p^4 + \dots \right) \phi(p)$$

*(i) Construct the effective action* $S^{(1)}(a)$ $\quad a_1 = (1-\epsilon)a$

*(ii) Construct the matrix M from this one step and find the relevant, irrelevant and marginal directions*

# Wilsonian Renormalization

Critical Regions $\leftrightarrow$ Effective QFT

Fixed Points $\leftrightarrow$ Renormalizable QFT: continuum limit

Universality $\leftrightarrow$ small number of relevant directions

Any local action with the same degrees of freedom and symmetries lead to the same continuum limit (via the tuning of a small set of relevant parameters)

Asymptotic freedom ensures the existence of FP in perturbation theory

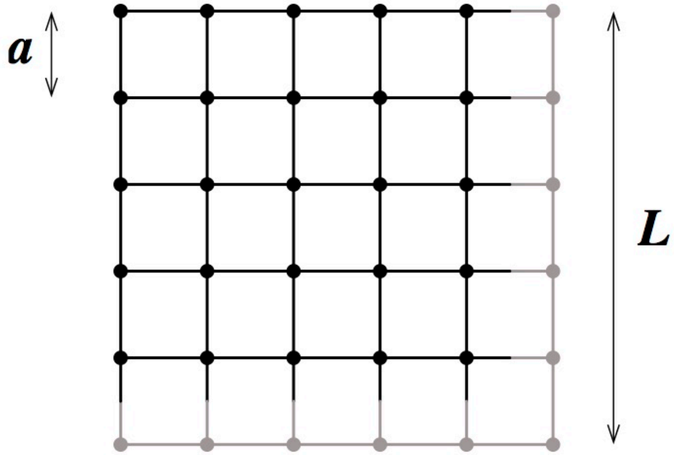
QCD has the (marginally) relevant couplings: $g_0$, $m_u$ ,$m_d$ ,$m_s$ ,$m_c$ ,...

$$g_0 \to 0 \,,\; m_q\, a \to 0$$

Caveat: a discretization that breaks any of the symmetries in general requires more relevant couplings

# Lattice Scalar Fields

From functional integrals to multidimensional ordinary integrals via discretization of space-time




$x = a\,(n_0, n_1, n_2, n_3)\,, \qquad n_\mu \in \mathbb{Z}$

$a \equiv$ lattice spacing

$L/a \equiv$ lattice size

$L \leq \infty$, possibly $T \neq L$

$$\mathcal{Z} = \int \mathcal{D}\phi\; e^{-S[\phi]}, \quad \mathcal{D}\phi \to \prod_x d\phi(x)$$

$$\partial_\mu \phi(x) \to \;\; \hat{\partial}_\mu \phi(x) \equiv \frac{1}{a}\left(\phi(x + a\hat{\mu}) - \phi(x)\right)$$

$$\hat{\partial}^*_\mu \phi(x) \equiv \frac{1}{a}\left(\phi(x) - \phi(x - a\hat{\mu})\right)$$

$$S[\phi] \;\; \to \;\; a^4 \sum_x \left\{ \frac{1}{2}\hat{\partial}_\mu\phi(x)\hat{\partial}_\mu\phi(x) + \frac{1}{2}m_0^2\phi(x)^2 + \frac{\lambda}{4!}\phi(x)^4 \right\}$$

# Lattice Scalar Fields

As in the continuum, the limit $\lambda = 0$ is solvable

$$S^{(0)}[\phi] \quad = \quad a^4 \sum_x \left\{ \frac{1}{2} \hat{\partial}_\mu \phi \hat{\partial}_\mu \phi + \frac{m_0^2}{2} \phi^2 \right\} = \frac{a^4}{2} \sum_{x,y} \phi(x) K_{xy} \phi(y),$$

$$K_{xy} \equiv -\frac{1}{a^2} \sum_{\hat{\mu}=0}^{3} \left( \delta_{x+a\hat{\mu} y} + \delta_{x-a\hat{\mu} y} - 2\delta_{xy} \right) + m_0^2 \delta_{xy}$$

$$Z^{(0)}[J] = e^{\frac{a^4}{2} \sum_{x,y} J_x (K^{-1})_{xy} J_y} \det \left( a^4 K \right)^{-1}$$

$$\boxed{\langle \phi(x) \phi(y) \rangle = a^{-4} K^{-1}_{xy} = \int \frac{d^4 p}{(2\pi)^4} \frac{e^{ip\cdot(x-y)}}{\hat{p}^2 + m_0^2}}$$

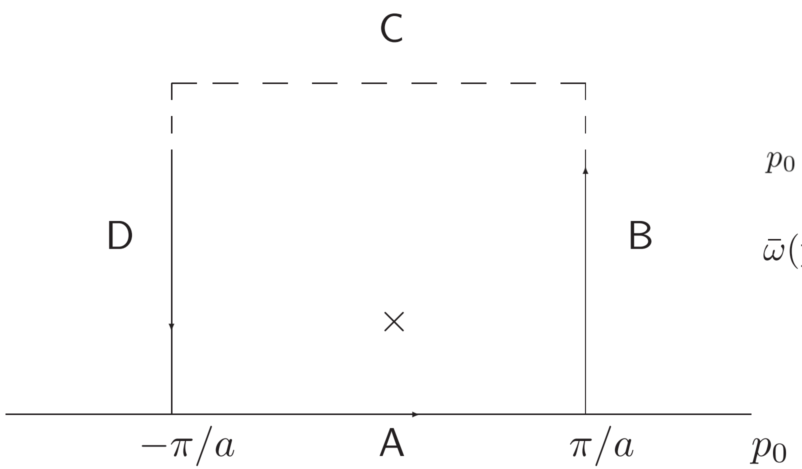
$$\hat{p}_\mu \equiv \frac{2}{a} \sin\left( \frac{p_\mu a}{2} \right) \qquad \hat{p}^2 \equiv \sum_\mu \hat{p}_\mu^2.$$

# Lattice Scalar Fields

Particle interpretation ? Continuum limit ?

*Exercise:*

$$\langle\phi(x)\phi(0)\rangle = \int_A (\dots) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\bar{\omega}(\mathbf{p})}e^{-\omega(\mathbf{p})x_0}e^{i\mathbf{p}\cdot\mathbf{x}}$$




One pole:

$$p_0 = \pm i\omega(\mathbf{p})\left(mod\ \frac{2\pi}{a}\right) \qquad \cosh\omega(\mathbf{p})a = 1+\frac{a^2}{2}\left(m_0^2+\frac{4}{a^2}\sum_{i=1}^{3}\sin^2\frac{p_i a}{2}\right)$$

$$\bar{\omega}(\mathbf{p}) \equiv \frac{1}{a}\sinh\left(\omega(\mathbf{p})a\right)$$

One particle state of mass $m_o$

$$\int_A(\dots)+\cancel{\int_B(\dots)}+\cancelto{0}{\int_C(\dots)}+\cancel{\int_D(\dots)} = 2\pi i\sum_{poles}\text{Residues.}$$

$$\lim_{a\to 0}\ \omega(\mathbf{p}) = \lim_{a\to 0}\ \bar{\omega}(\mathbf{p}) = \sqrt{m_0^2+\mathbf{p}^2}+\mathcal{O}(a^2)$$

$$Z = \sqrt{\frac{\omega(\mathbf{p})}{\bar{\omega}(\mathbf{p})}} \to 1+\mathcal{O}(a^2)$$

# Lattice Fermion Fields

Euclidean fermions :

$$S_{\text{cont}} = \int \mathrm{d}^4x\, \bar{\psi}(x)\,[\gamma_\mu \partial_\mu + m]\,\psi(x)\,; \qquad \{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}\,, \quad \gamma_\mu^\dagger = \gamma_\mu$$

One particle states (KL representation):

$$\langle 0|\psi(x)\bar{\psi}(0)|0\rangle_F\big|_{x_0>0} = \sum_\alpha \int \frac{d^3p}{(2\pi)^3}\,|Z_\alpha|^2\, \frac{i\gamma_\mu p_\mu - m_\alpha}{2ip_0}\bigg|_{p_0=iE_{\mathbf{p}}(\alpha)} e^{-E_{\mathbf{p}}(\alpha)x_0} e^{i\mathbf{p}\mathbf{x}}$$

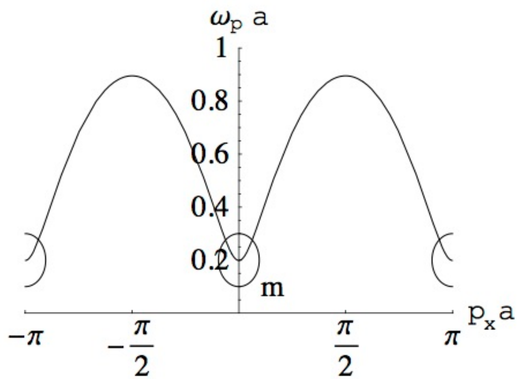
Chiral symmetry m -> 0: $\quad \psi(x) \to e^{i\alpha\gamma_5}\psi(x)$

# Lattice Fermion Fields

Naïve discretization:

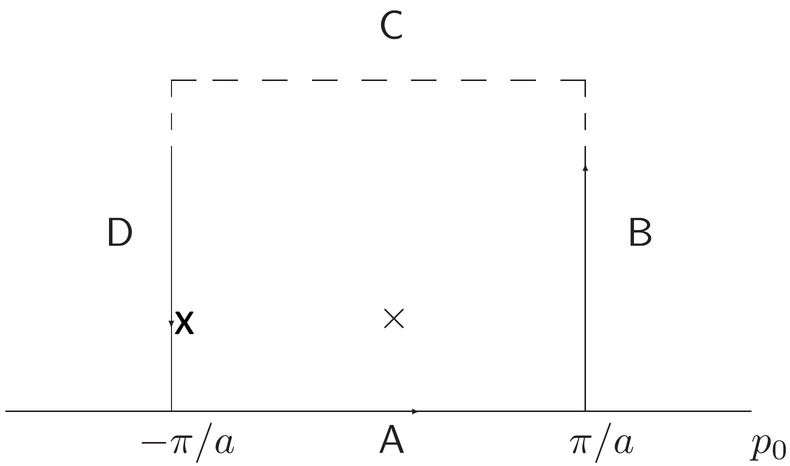
$$S_{\text{latt}} = a^4 \sum_x \bar{\psi}(x) \left\{ \tfrac{1}{2} \left[ \gamma_\mu (\partial_\mu^* + \partial_\mu) \qquad\qquad \right] + m \right\} \psi(x)$$

$$\langle \psi_\alpha(x) \bar{\psi}_\beta(y) \rangle_F = \int_{BZ} \frac{d^4 p}{(2\pi)^4} \frac{e^{ip(x-y)}}{\sum_\mu i\gamma_\mu \frac{\sin(p_\mu a)}{a} + m}.$$




KL representation: one particle states ?

# Lattice Fermion Fields




Two poles:

$$e^{ip_0 a} = \pm e^{-\omega_{\mathbf{p}} a} \equiv \pm\left(\sqrt{1+M_{\mathbf{p}}^2} - M_{\mathbf{p}}\right)$$

$$M_{\mathbf{p}}^2 \equiv m^2 a^2 + \sum_{k=1}^{3} \sin(p_k a)^2$$

$$\begin{aligned}\langle\psi_\alpha(x)\bar{\psi}_\beta(0)\rangle_F \quad &= \int \frac{d^3p}{(2\pi)^3}\frac{e^{i\mathbf{px}}e^{-\omega_{\mathbf{p}}x_0}}{\sinh(2\ \omega_{\mathbf{p}}a)}\Bigg[\left(\gamma_0 \sinh\omega_{\mathbf{p}}a - i\sum_k \gamma_k \sin p_k a + ma\right) \\ &+ \quad (-1)^{x_0/a}\left(-\gamma_0 \sinh\omega_{\mathbf{p}}a - i\sum_k \gamma_k \sin p_k a + ma\right)\Bigg].\end{aligned}$$

- Two poles with same energy $\omega_{\mathbf{p}}$ with different residues
- Minimum of the energy @ $p_k = \bar{p}_k \equiv n_k \frac{\pi}{a} \quad n_k = 0, 1$

$$\lim_{a\to 0} \ \omega_{\mathbf{p}}\big|_{p_k = n_k\pi/a} = m$$

# Lattice Fermion Fields

$$p_j = \bar{p}_j^{(i)} + k_j, \quad k_j a \ll 1 \quad j = 1, ..., 2^3$$

$$\bar{p}_\mu^{(\alpha)} = (n_0^{(\alpha)}, n_1^{(\alpha)}, n_2^{(\alpha)}, n_3^{(\alpha)})\frac{\pi}{a}, \qquad n_\mu^{(\alpha)} = 0, 1$$

$$\langle \psi_\alpha(x)\bar{\psi}_\beta(0)\rangle_F = \sum_{\alpha=1}^{16} e^{i\bar{p}^{(\alpha)}x} \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\mathbf{kx}} e^{-\omega_\mathbf{p} t}}{2k_0} S_\alpha \left[ \left( \gamma_0 k_0 - i \sum_k \gamma_k k_k + m \right) \right] S_\alpha^{-1}$$

$$S_\alpha \equiv \prod_\mu (i\gamma_\mu\gamma_5)^{n_\mu^{(\alpha)}}$$

Doubling Problem: $2^d$ massive fields in the continuum limit and not one !

# Lattice Fermion Fields

Deep connection between doubling problem and chirality

Naive chiral fermion:

$$(1-\gamma_5)\ \sum_{\alpha=1}^{16} e^{i\bar{p}^{(\alpha)}x} \int \frac{d^3k}{(2\pi)^3}\frac{e^{i\mathbf{k}\mathbf{x}}e^{-\omega_{\mathbf{p}}t}}{2k_0} S_\alpha \left[\left(\gamma_0 k_0 - i\sum_k \gamma_k k_k + m\right)\right] S_\alpha^{-1}\ (1+\gamma_5)$$

$$(1-\gamma_5)S_\alpha = S_\alpha(1-(-1)^{\sum_\mu n_\mu^{(\alpha)}}\gamma_5)$$

8 right-movers + 8 left-movers: vector-like theory !

# Lattice Fermion Fields

Deep connection between doubling problem and chirality

Nielsen-Ninomiya no-go theorem: doubling problema is common to any fermion action that satisfies

- Invariant under space-time translations
- Quadratic in fermions & hermitian
- Local (smooth kernel in Fourier space)
- Chirally symmetric

# Lattice Fermion Fields

Wilson discretization:

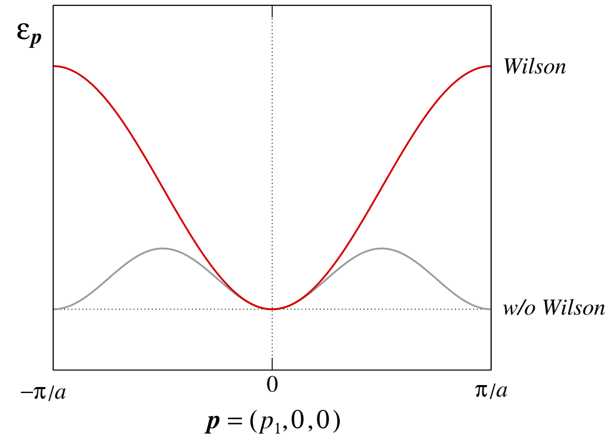
$$S_{\text{latt}} = a^4 \sum_x \bar{\psi}(x) \left\{ \tfrac{1}{2} \left[ \gamma_\mu (\partial^*_\mu + \partial_\mu) - a\partial^*_\mu \partial_\mu \right. + m \right\} \psi(x)$$

Vanishes in the naïve continuum limit, but breaks chiral symmetry!

$$\langle \psi_\alpha(x) \bar{\psi}_\beta(y) \rangle_F = \int_{BZ} \frac{d^4p}{(2\pi)^4} \frac{e^{ip(x-y)}}{\sum_\mu i\gamma_\mu \frac{\sin(p_\mu a)}{a} + m + \frac{r}{a} \sum_\mu (1 - \cos p_\mu a)}$$

- Only one pole in the $p_0$
- Minimum energy unique

$$\omega^{(\alpha)}_{\mathbf{p}} = \frac{1}{a} \log \left( 1 + ma + 2 \sum_k n^{(\alpha)}_k \right)$$



# Lattice Fermion Fields

The breaking of chiral symmetry brings many complications:

- bare quark masses become relevant (instead of marginally relevant)
- cutoff effects are O(a) (instead of O(a^2))
- operator mixing and renormalization much more complicated
- in the context of chiral gauge theories: breaking of gauge symmetry

Alternatively discretizations to tame chiral symmetry breaking

- ➢ Staggered fermions (Kogut-Susskind)
- ➢ Twisted mass Wilson
- ➢ Ginsparg-Wilson (overlap fermions, domain-wall,...)

# Lattice Gauge Theory

SU(N) gauge theory in the continuum: e.g. a charged fermion + gauge connection

$$\psi^i(x) \qquad i = 1, .., N \qquad\qquad A_\mu(x) = A_\mu^a(x) T^a \qquad a = 1, .., N^2 - 1$$

Gauge Symmetry:

$$\psi(x) \to \psi'(x) = \Omega(x)\psi(x), \quad \Omega(x) \in SU(N)$$

$$A_\mu(x) \to A'_\mu(x) = \Omega(x) A_\mu(x) \Omega(x)^\dagger + i\Omega(x)\partial_\mu \Omega(x)^\dagger$$

Action: start with the free fermion action ($A_\mu$=0) and do a gauge transformation

$$\bar{\psi}\gamma_\mu \partial_\mu \psi \to \bar{\psi}' \gamma_\mu D_\mu \psi'$$

$$D_\mu \psi' \equiv \left(\partial_\mu + \Omega\ \partial_\mu \Omega^+\right) \psi' = (\partial_\mu - iA'_\mu)\psi'$$

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# Lattice Gauge Theory

Gauge action in terms of the field strength

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) - i[A_\mu(x), A_\nu(x)]$$

$$\boxed{S_{\mathrm{cont}}[\phi, A_\mu] = -\frac{1}{2g^2}\mathrm{Tr}\left[F_{\mu\nu}F^{\mu\nu}\right] + \bar{\psi}\gamma_\mu D_\mu \psi}$$

How do we discretize this, maintaining gauge symmetry ?

# Lattice Gauge Theory

Free fermion in two different gauges to get the lattice covariant derivative:

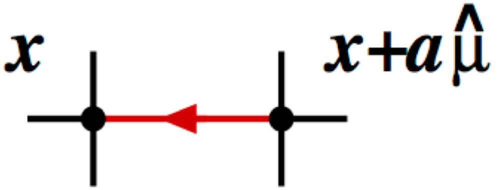
$$\bar{\psi}\gamma_\mu\hat{\partial}_\mu\psi \to \bar{\psi}'\gamma_\mu\hat{\nabla}_\mu\psi'$$

$$\hat{\nabla}_\mu\psi(x) = \Omega(x)^\dagger\Omega(x+a\hat{\mu})\psi(x+a\hat{\mu}) - \psi(x) \equiv U_\mu(x)\psi(x+a\hat{\mu}) - \psi(x)$$

The basic variable on the lattice is the link variable

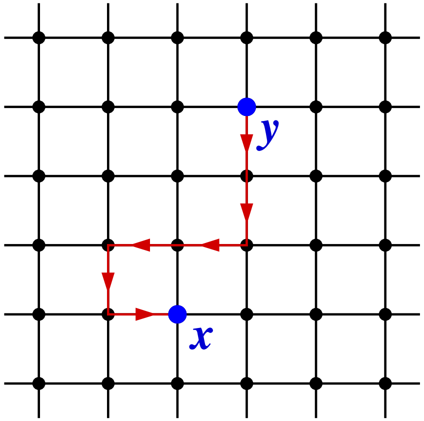
$$U_\mu(x) \in SU(N)$$

It is a parallel transporter




$$U_\mu(x) \to \Omega(x)U_\mu(x)\Omega(x+a\hat{\mu})^\dagger$$

# Lattice Gauge Fields




We still need the pure gauge action…

Wilson lines: path ordered product of links (y-> x)

$$P(x, y; \text{path}) \to \Omega(x) P(x, y; \text{path}) \Omega(y)^\dagger$$

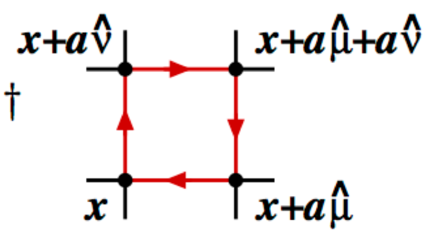
Wilson loops: products of link variables forming a closed loop

$$\mathrm{W} = \mathrm{Tr}[\mathrm{P(x,x;path)}]$$

Gauge invariant!

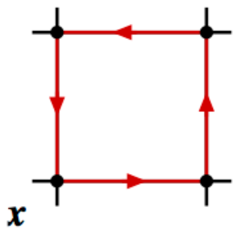
Plaquette: smallest Wilson loop

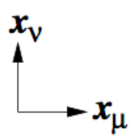
$$U_{\mu\nu}(x) \equiv \quad U_\mu(x) U_\nu(x + a\hat{\mu}) U_\mu(x + a\hat{\nu})^\dagger U_\nu(x)^\dagger$$



# Lattice Gauge Fields

Discretized Gauge Action ? Local, real, lattice rotation and gauge invariant




$$S_W[U] = \frac{2}{g_0^2} \sum_x \sum_{\mu<\nu} \mathrm{Tr}\left[1 - \frac{1}{2}(U_{\mu\nu}(x) + U^\dagger_{\mu\nu}(x))\right]$$

Continuum Limit ?

$$U_\mu(x) = \mathrm{P}\exp\left\{ia\int_0^1 \mathrm{d}t\, A_\mu(x + (1-t)a\hat{\mu})\right\}$$
$$= \mathbf{1} + iaA_\mu(x) + \mathcal{O}(a^2)$$

$$\lim_{a\to 0} S_W[U] = \frac{1}{2g_0^2} a^4 \sum_x \mathrm{Tr}[F_{\mu\nu}(x)F^{\mu\nu}(x)] + \mathcal{O}(a^6)$$

# Strong Coupling Expansion: mass gap

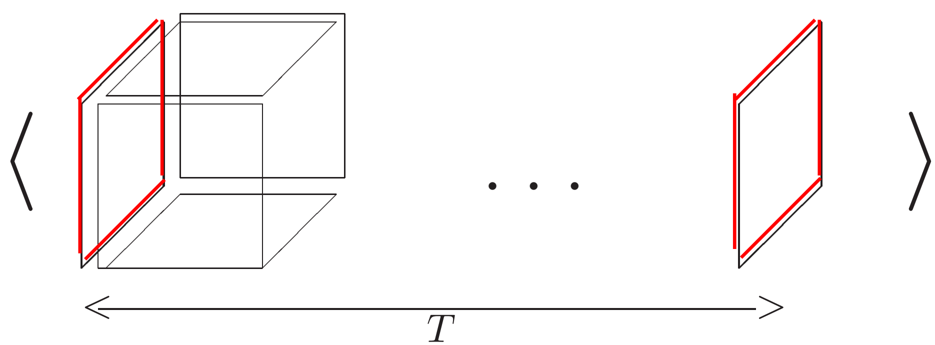
An Taylor expansion in $\beta \equiv \dfrac{2N}{g_0^2}$

- each power of β brings down a plaquette
- $\int dU \; U_{\alpha\beta} = 0$ each link must appear more than once

Glueball spectrum: the large time behaviour of any local operator with the right quantum numbers

$$\lim_{x_0 \to \infty} a^3 \sum_{\mathbf{x}} \langle \mathrm{Tr}[U_{ij}(x)] \mathrm{Tr}[U_{ij}(0)] \rangle \propto e^{-m_{\text{glueball}} x_0}$$

$$\int dU \; U_{\alpha\beta} U^{\dagger}_{\gamma\delta} = \frac{1}{N} \delta_{\alpha\delta} \delta_{\beta\gamma}$$



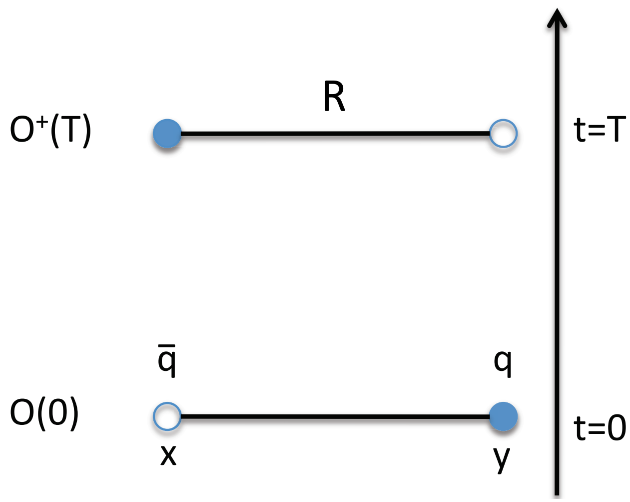

$$\propto \left(\frac{\beta}{2N^2}\right)^{4T/a} = \exp\left(-\frac{4}{a} \log\left(\frac{2N^2}{\beta}\right) T\right)$$

$$m \sim \frac{4}{a} \log\left(\frac{2N^2}{\beta}\right)$$

Mass gap, but no continuum limit...

# Strong Coupling Expansion: confinement

Static potential: potential energy between two infinitely heavy quark/antiquark separated a distance R

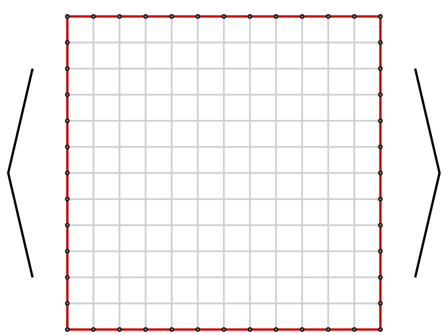



$$\mathcal{O}(t) = \phi^\dagger(\mathbf{y},t)U(\mathbf{y},t;\mathbf{x},t)\phi(\mathbf{x},t)$$

$$C_{q\bar{q}}(T) \equiv \langle \mathcal{O}^\dagger(T)\mathcal{O}(0)\rangle_{\phi,U}$$

$$C_{q\bar{q}}(T) \sim \exp(-E(R)T), \quad E(R) = E_0 + V(R)$$

$$C_{qq}(T) \propto$$



$$\lim_{\beta\to 0} V(R) = \frac{R}{a^2}\log\left(\frac{2N^2}{\beta}\right) + \ldots = \sigma R + \ldots$$

String tension but no continuum limit!

$$\lim_{\beta\to 0} \sigma = \frac{1}{a^2}\log\left(\frac{2N^2}{\beta}\right)$$

# Lattice QCD

$$\mathcal{Z} = \int D[U] e^{-S_W[U]} \int D[\psi] D[\bar{\psi}] \;\; e^{-S_{WF}[U,\psi,\bar{\psi}]}$$

$$S_W[U] = \frac{2}{g_0^2} \sum_x \sum_{\mu<\nu} \mathrm{Tr}\left[1 - \frac{1}{2}(U_{\mu\nu}(x) + U_{\mu\nu}^\dagger(x))\right]$$

$$S_{WF}[U,\psi,\bar{\psi}] = a^4 \sum_{x,a} \bar{\psi}_a(x)(D_W + M_a)\psi_a$$

$$D_W = \frac{1}{2}\left[\left(\nabla_\mu + \nabla_\mu^*\right)\gamma_\mu - a\nabla_\mu \nabla_\mu^*\right]$$

$$D[U] = \prod_{x,\mu} dU_\mu(x), \quad D[\psi] = \prod_x d\psi(x)$$

Haar measure

Grassman variables

Integration over fermion variables can be done analytically:

$$\mathcal{Z}_F \equiv \int D[\psi] D[\bar{\psi}] \;\; e^{-S_{WF}[U,\psi,\bar{\psi}]} = \prod_a \det(D_W + M_a)$$

# Lattice QCD

$$\mathcal{Z} = \int D[U] \quad \prod_q \det(D_W + m_q)\; e^{-S_W[U]}$$

- Integrals over link variables are compact: no need to fix the gauge
- Integrand is positive definite: Monte Carlo methods can be used

$$\langle \phi_1(x_1)\cdots\phi_n(x_n)\rangle = \frac{1}{\mathcal{Z}}\int D[U]\langle \phi_1(x_1)\cdots\phi_n(x_n)\rangle_{\mathrm{F}} \times$$

$$\times \prod_{q=1}^{N_f} \det[D_{\mathrm{w}}(U) + m_q] e^{-S_{\mathrm{G}}[U]}$$

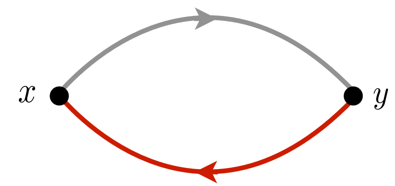
$\langle \psi_1(x_1)\ldots\psi_n(x_n)\bar{\psi}_1(y_1)\ldots\bar{\psi}_n(y_n)\rangle_F$ = Tr[ Product of quark propagators ]

$$\langle \psi(x)\bar{\psi}(y)\rangle_{\mathrm{F}} = S(x,y;U) \qquad\qquad (D_{\mathrm{w}} + M)\, S(x,y;U) = a^{-4}\delta_{xy}$$

# Lattice QCD

*Example:* pion propagator

$$\langle(\bar{u}\gamma_5 d)(x)(\bar{d}\gamma_5 u)(y)\rangle_{\mathrm{F}} = -\mathrm{tr}\,\{\gamma_5\, S(x,y;U)_d\,\gamma_5\, S(y,x;U)_u\}$$




$$\langle\pi(x)\pi(y)\rangle = -\mathcal{Z}^{-1}\int D[U]\;tr[\gamma_5 S_d(x,y;U)\gamma_5 S_u(y,x;U)]\prod_q \det(D_W+m_q)\;e^{-S_W[U]}$$

$$\lim_{x_0\to\infty} a^3\sum_{\mathbf{x}}\langle\pi(x)\pi(0)\rangle \propto e^{-m_\pi x_0}$$

# Lattice QCD: continuum limit ?

Is there a fixed point: $m_{\text{phys}} a \to 0$

How to approach this continuum limit: how many couplings do we have to tune ?

Asymptotic freedom ensures the existence of FP in perturbation theory

$$\beta(g_0) \equiv -a \left.\frac{\partial g_0}{\partial a}\right|_{g_R \text{ fixed}} = -\beta_0 g_0^3 - \beta_1 g_0^5 + \ldots \quad \beta_0 = \frac{N_c}{16\pi^2}\frac{11}{3} > 0$$

$$\Rightarrow \quad g_0^2 \underset{a\to 0}{\sim} \frac{1}{b_0 \ln(a\mu)} + \ldots$$

$$g_0 = 0 \;\; \text{UV fixed point}$$

QCD has the relevant couplings: $g_0$, $m_u$ ,$m_d$ ,$m_s$ ,$m_c$ ,...

# Continuum Limit

$N_f$=2+1 three parameters: $g_0$, $m_u$=$m_d$, $m_s$

e.g. we measure three quantities and predict everything else

$$a^{\text{phys}} \to M_p a / M_p^{\text{phys}}, \;\; m_u a = m_d a \to M_\pi a / a^{\text{phys}} = M_\pi^{\text{phys}},$$
$$m_s a \to M_K a / a^{\text{phys}} = M_K^{\text{phys}}$$

$a \to a' < a$ (L/a, increase) increasing $\beta \propto \dfrac{1}{g_0^2}$ sot that

so that physics remains constant

$$M_p a' = M_p a \left( \frac{a'}{a} \right) = M_p a \left( \frac{L}{a} \frac{a'}{L} \right)$$

# Plan

Part I: Functional Formulation of QFT, renormalization, Wilson RG

Part II: Lattice Formulation of scalar, fermion and gauge QFT

Part III: Lattice QCD: numerical methods and applications

# Numerical Aspects of Lattice QCD

Well defined problem for finite a and volume but:

$$N_f = 2+1+1\,, \quad (L/a)^3 \times (T/a) = 64^3 \times 128$$

$$\Rightarrow \quad D_{\mathrm{w}} = (1.6 \times 10^9)^2 \quad \text{complex matrix}$$

$$\int \underbrace{\prod_{x,\mu} dU_\mu(x)}_{>10^8 \ \mathrm{SU(N)\ integrals}}$$

Monte Carlo integration mandatory

$$I = \int_0^1 \mathrm{d}x_0 \int_0^1 \mathrm{d}x_1 \cdots \int_0^1 \mathrm{d}x_{K-1}\, P(\mathbf{x})\, f(\mathbf{x})$$

Generate N random K-vectors $\left\{\mathbf{x}^{[i]}\right\}$ distributed according to P(x) (normalized)

$$I(N) = \frac{1}{N} \sum_{i=1,..N} f[\mathbf{x}^{[i]}]$$

Convergence guarantied by central limit theorem:

$$\lim_{N\to\infty} I(N) = I + \mathcal{O}(1/\sqrt{N})$$

# Numerical Aspects of Lattice QCD

$$\mathbf{x} \to U_\mu(x), \qquad P(\mathbf{x}) \to \frac{e^{-S[U]}}{\mathcal{Z}}$$

Markov Chains: a procedure to get the required samples $\{\mathbf{x}^{[i]}\}$

Stocastic process to get one configuration from the previous one via a

Transition Probability $T(\mathbf{x} \to \mathbf{x}')$

With the following properties guarantied to get the right distribution (asymptotically):

1) $T(\mathbf{x} \to \mathbf{x}') \geq 0 \quad \sum_{\mathbf{x}'} T(\mathbf{x} \to \mathbf{x}') = 1$

2) $\sum_{\mathbf{x}} P(\mathbf{x}) T(\mathbf{x} \to \mathbf{x}') = P(\mathbf{x}')$

3) ergodicity

# Numerical Aspects of Lattice QCD

What $T(\mathbf{x} \to \mathbf{x}')$ ?

Metropolis-Hastings algorithm

$$T(\mathbf{x} \to \mathbf{x}') = \begin{cases} \min(1, P(\mathbf{x}')/P(\mathbf{x})) & \mathbf{x}' \neq \mathbf{x} \\ 1 - \sum_{\mathbf{x}'} \min(1, P(\mathbf{x}')/P(\mathbf{x})) & \mathbf{x}' = \mathbf{x} \end{cases}$$

Not very efficient when the domain is much larger than the region where P(x) is significant: small acceptance rate...

# Numerical Aspects of Lattice QCD

Hybrid Monte Carlo (HMC) algorithm [Duane et al '87]

Molecular dynamics: $\mathbf{x} \to \mathbf{x}, \mathbf{p}$ $\Delta S = \frac{\mathbf{p}^2}{2}$

1) Starting with some gaussian random momenta, Hamilton equation is solved (approximately) with Hamiltonian and new state in the chain is the solution

$$\left.\begin{aligned} H(x,p) &= S(x) + \Delta S(p) \\ \mathbf{x}(0) = \mathbf{x}&, \mathbf{p}(0) = \mathbf{p}, \end{aligned}\right\} \quad \mathbf{x}' = \mathbf{x}(\tau)$$

2) Ergodicity is achieved by the change in positions from random gaussian momentum updates

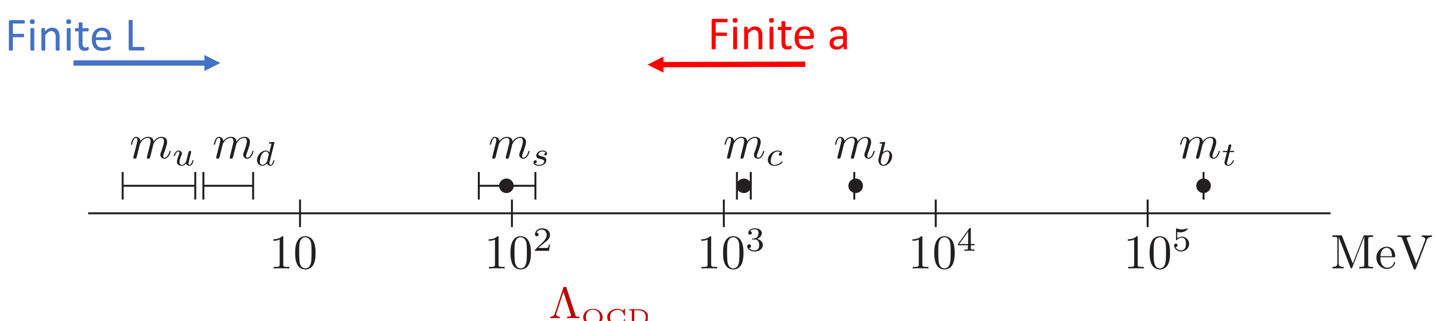
3) A MH accept-reject step because the solution to Hamilton eqs. is not exact

# Numerical Aspects of Lattice QCD

Systematic errors:

- Continuum limit: $Ma \ll 1 \qquad a \to 0$
- Infinite volume limit: $ML \gg 1$

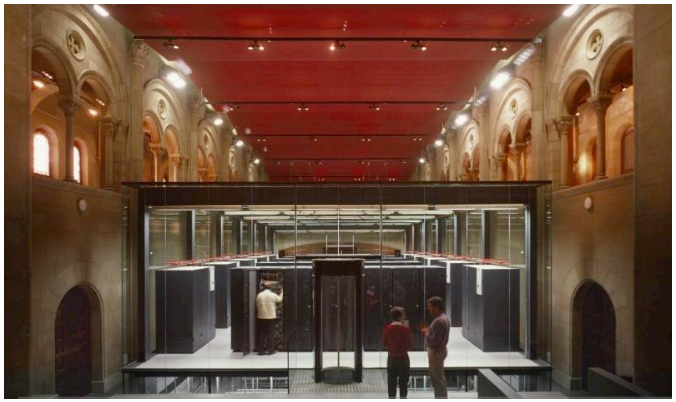
Challenge: multiscale problem

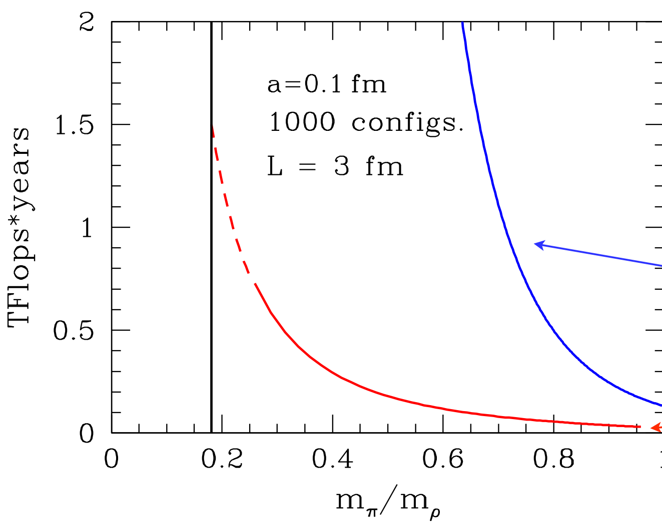



$$L^{-1} \ll \mu \ll a^{-1}$$

# Numerical Aspects of Lattice QCD

Need for HPC and smart algorithms!



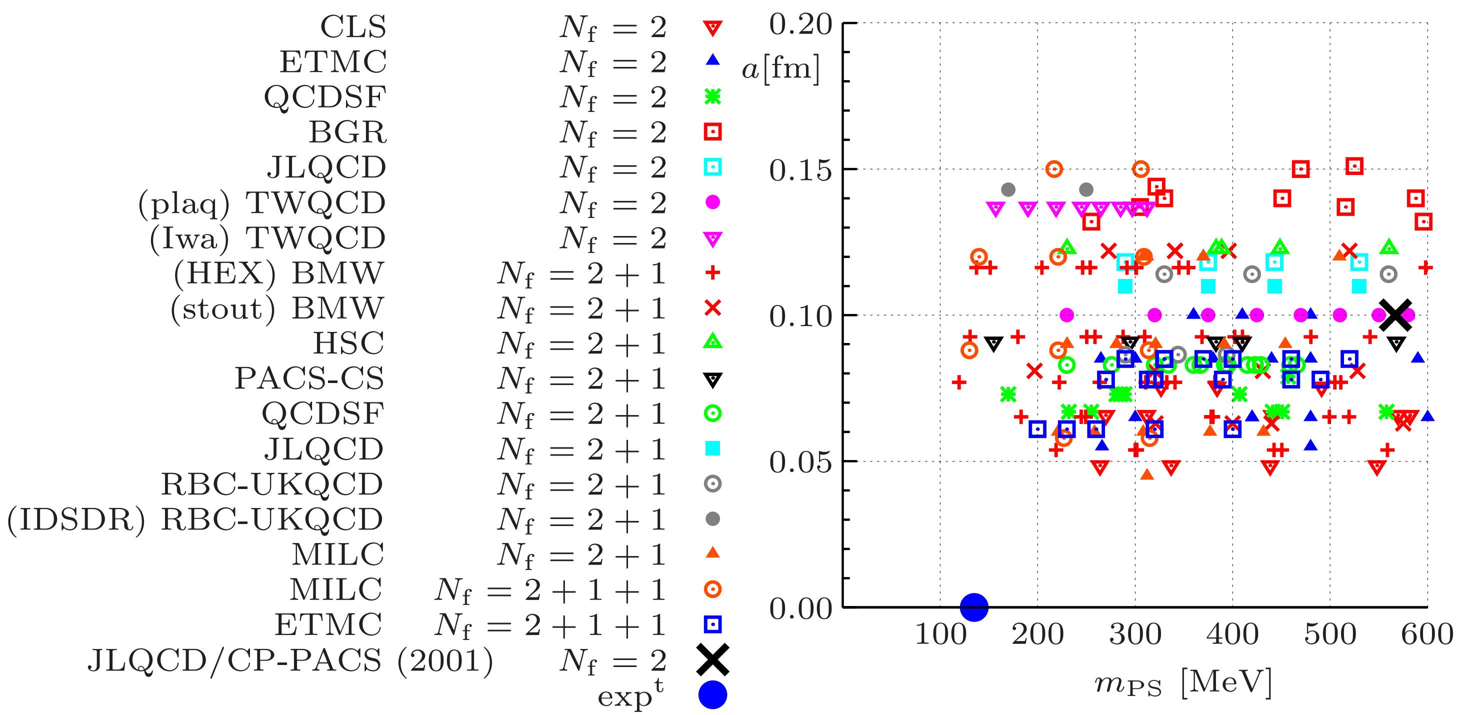




$$5\left[\frac{20\ \mathrm{MeV}}{m}\right]^3\left[\frac{L}{3\ \mathrm{fm}}\right]^5\left[\frac{0.1\ \mathrm{fm}}{a}\right]^7$$ [Ukawa 2001]

$$0.05\left[\frac{20\ \mathrm{MeV}}{m}\right]^1\left[\frac{L}{3\ \mathrm{fm}}\right]^5\left[\frac{0.1\ \mathrm{fm}}{a}\right]^6$$ [Giusti 2006]

# Numerical Aspects of Lattice QCD

Slowly getting there…

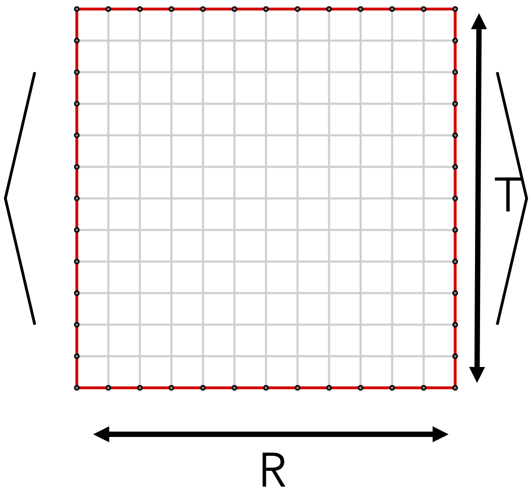


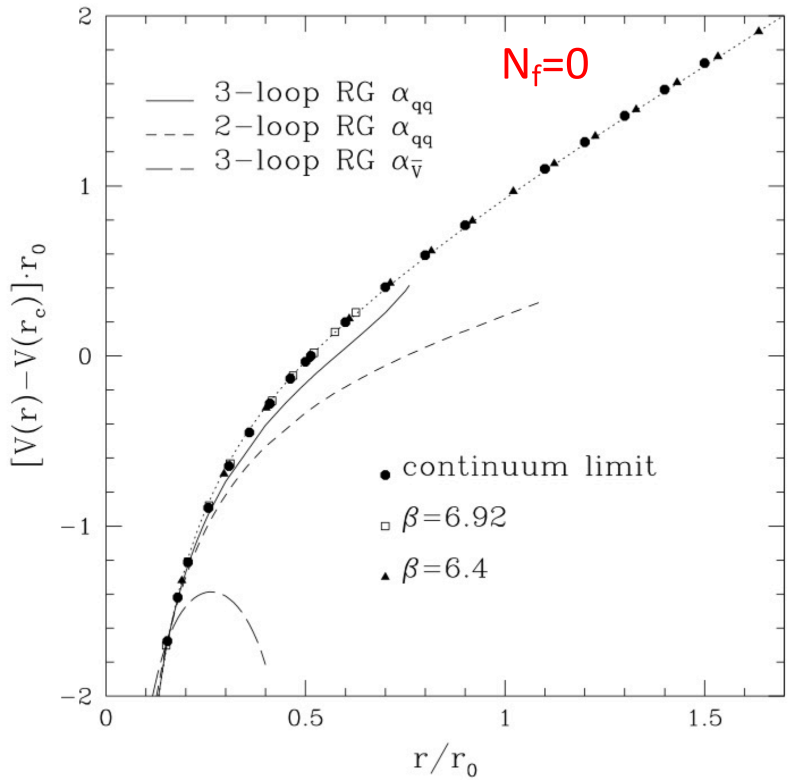

[G. Herdoiza]

# Confinement

$$\mathcal{O}(t) = \phi^\dagger(\mathbf{y},t)U(\mathbf{y},t;\mathbf{x},t)\phi(\mathbf{x},t) \qquad C_{q\bar{q}}(T) \equiv \langle \mathcal{O}^\dagger(T)\mathcal{O}(0)\rangle_{\phi,U}$$

$$C_{q\bar{q}}(T) \sim \exp(-E(R)T), \quad E(R) = E_0 + V(R)$$



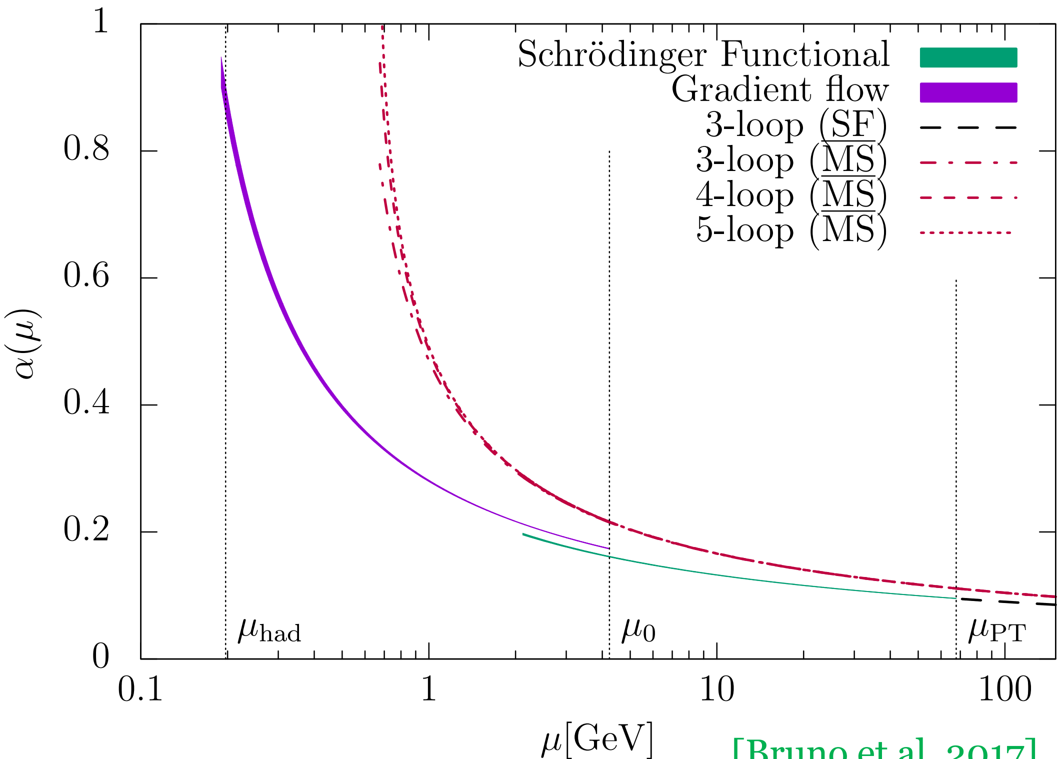




$\sigma \approx (0.4\ \mathrm{GeV})^2$

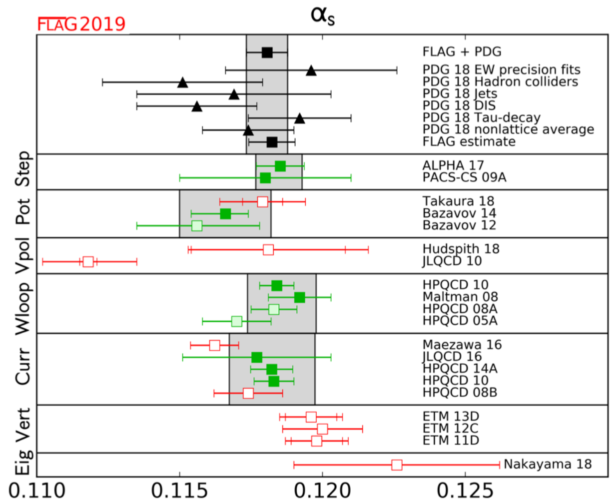
[Necco, Sommer 2001]

# Running coupling

Define a coupling at finite box size: g(L)




[Bruno et al, 2017]



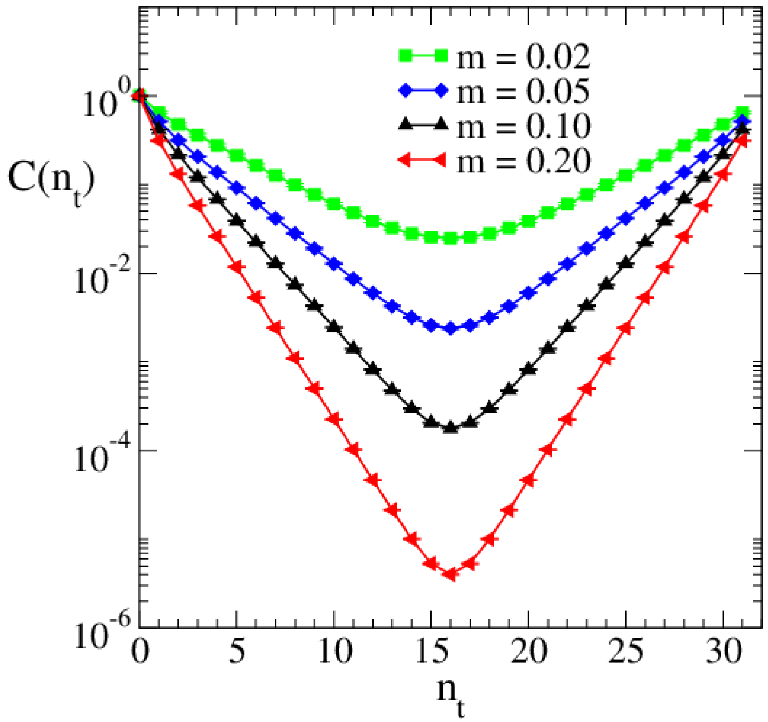

$[\alpha_s(M_Z)]_{PDG18} = 0.1174(16)$ → $[\alpha_s(M_Z)]_{MS} = 0.11852(84)$

# Hadron spectrum

Choose an operator with the right quantum numbers O(x):

$$M^a(x) \equiv \bar{\psi}_{\alpha i c}(x)\Gamma_{\alpha\beta}T^a_{ij}\psi_{\beta j c}(x), \qquad B^{abc}_{\alpha\beta\gamma} = \psi(x)_\alpha \equiv \epsilon_{c_1 c_2 c_3}\psi_{\alpha a c_1}\psi_{\beta b c_2}\psi_{\gamma c c_3}$$

$$\lim_{x_0\to\infty}\int d^3x\ \langle O(x)O(0)\rangle \propto e^{-M_{\rm lightest}x_0}$$



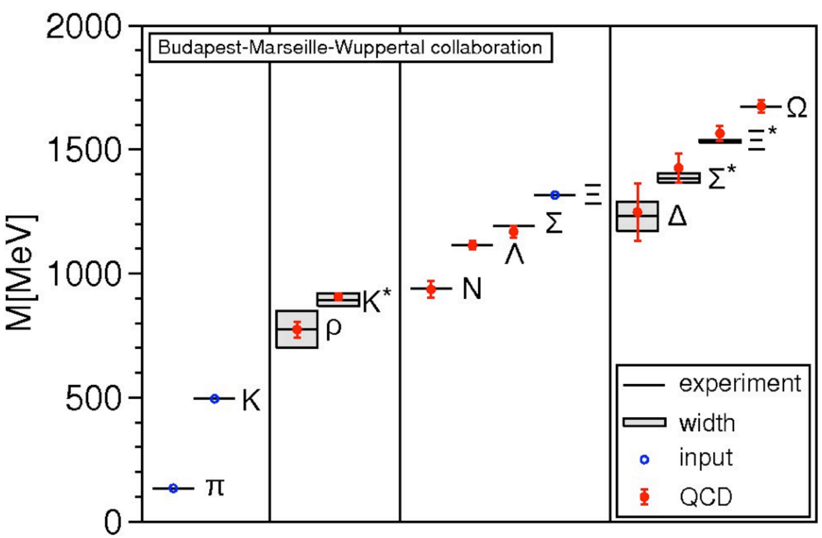

Pion propagator

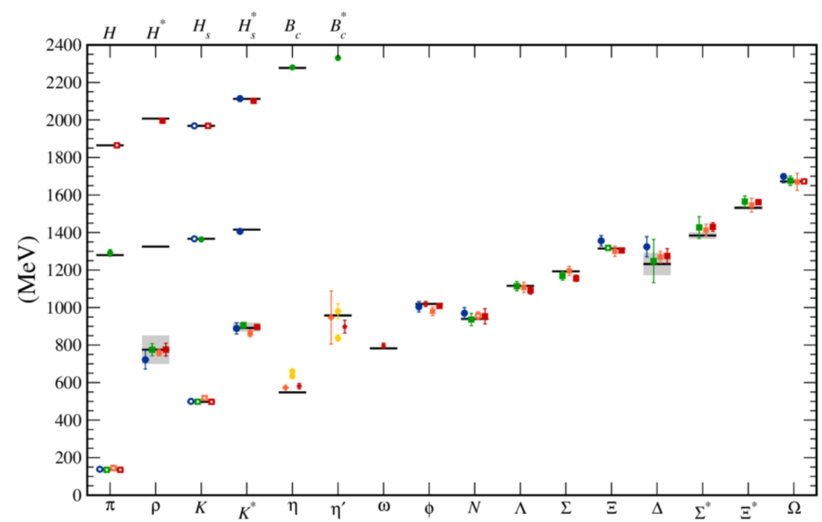
# Hadron spectrum

Choose an operator with the right quantum numbers O(x):

$$M^a(x) \equiv \bar{\psi}_{\alpha i c}(x)\Gamma_{\alpha\beta}T^a_{ij}\psi_{\beta j c}(x), \qquad B^{abc}_{\alpha\beta\gamma} = \psi(x)_\alpha \equiv \epsilon_{c_1c_2c_3}\psi_{\alpha a c_1}\psi_{\beta b c_2}\psi_{\gamma c c_3}$$

$$\lim_{x_0\to\infty}\int d^3x\, \langle O(x)O(0)\rangle \propto e^{-M_{\text{lightest}}x_0}$$




[BMW Collaboration 2008]

FLAG 2019

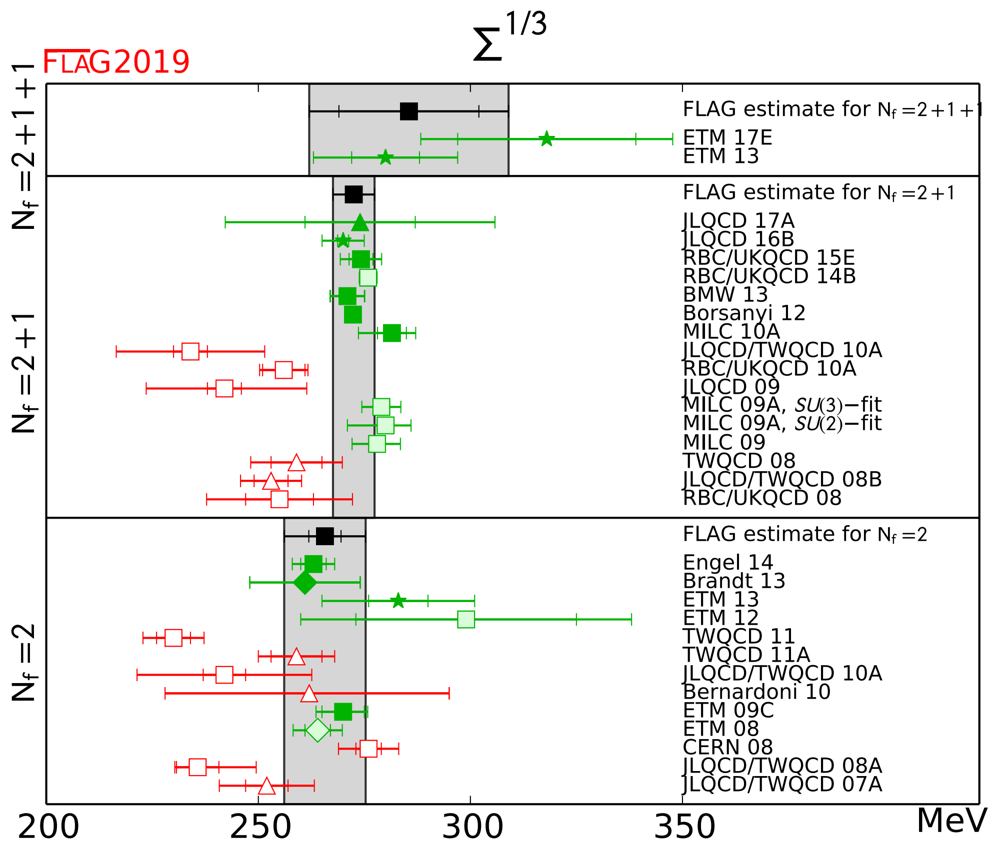
Flavour Lattice Averaging Group

Averages of quanities of phenomenological interest:

- Quark masses
- Vud and Vus
- Low-energy constants
- Kaon mixing
- D-meson decay constants and form factors
- B-meson decay constants, mixing parameters, and form factors
- The strong coupling $\alpha_s$
- Nucleon matrix elements

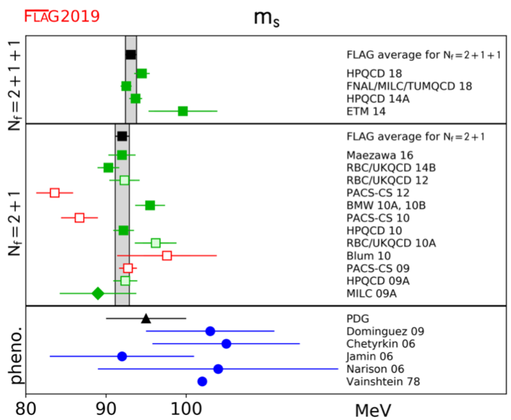
# Chiral Symmetry Breaking

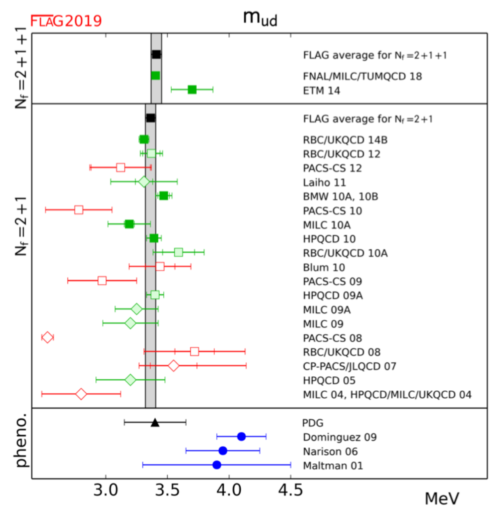
Spontaneous Chiral Symmetry Breaking takes place in QCD via a quark condensate:



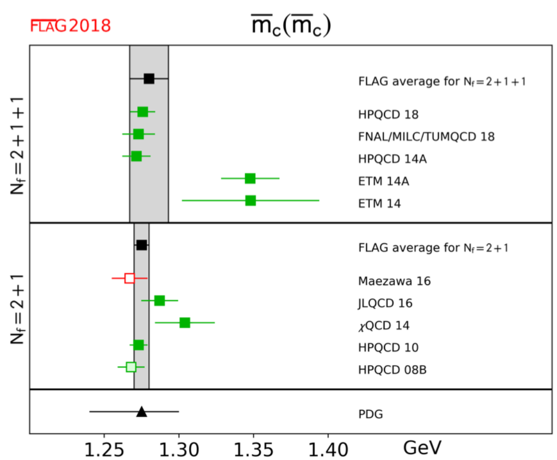
# Fundamental Parameters in the SM

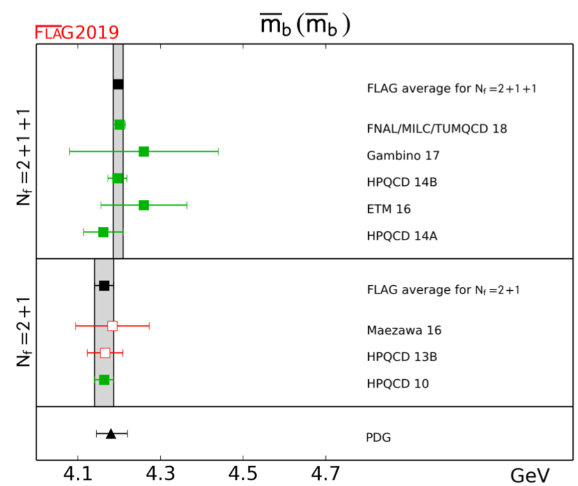
## Light quark masses from pion and kaon masses




## Heavy quark masses from D mesons and B mesons



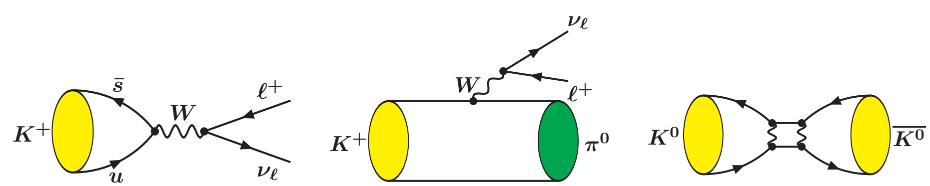



# Fundamental Parameters in the SM

CKM mixing matrix from leptonic and semileptonic decays

Rate = $|V_{CKM}|^2$ x Wilson coefficients x Form factors

$$
\left(\begin{array}{c|c|c}
\mathbf{V_{ud}} & \mathbf{V_{us}} & \mathbf{V_{ub}} \\
\pi \to \ell\nu & K \to \ell\nu & B \to \pi\ell\nu \\
 & K \to \pi\ell\nu & \\
\hline
\mathbf{V_{cd}} & \mathbf{V_{cs}} & \mathbf{V_{cb}} \\
D \to \ell\nu & D_s \to \ell\nu & B \to D\ell\nu \\
D \to \pi\ell\nu & D \to K\ell\nu & B \to D^*\ell\nu \\
\hline
\mathbf{V_{td}} & \mathbf{V_{ts}} & \mathbf{V_{tb}} \\
B_d \leftrightarrow \overline{B}_d & B_s \leftrightarrow \overline{B}_s & \\
K_0 \leftrightarrow \overline{K}_0 & K_0 \leftrightarrow \overline{K}_0 &
\end{array}\right)
$$



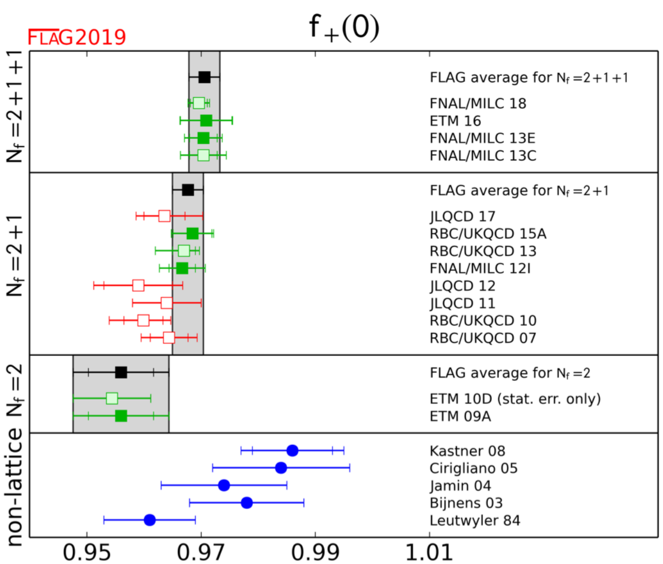

$K_{\ell 2} \to f_K$ $K_{\ell 3} \to f_+(0)$ $K^0$-$\overline{K^0}$ mixing $\to B_K$

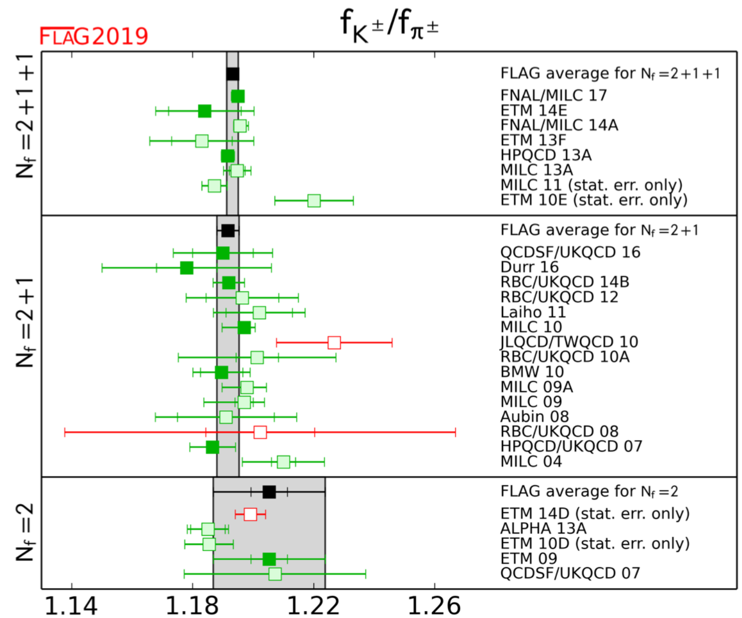
$$S_{\rm w} \approx \int \mathrm{d}^4x\, \mathcal{H}_{\rm w}(x), \qquad \mathcal{H}_{\rm w}(x) = \frac{g_{\rm w}^2}{4M_W^2}(V_{us})^* V_{ud} \sum_n k_n \mathcal{Q}_n(x)$$

4-fermion operators

# Fundamental Parameters in the SM

Leptonic and semileptonic form factors:



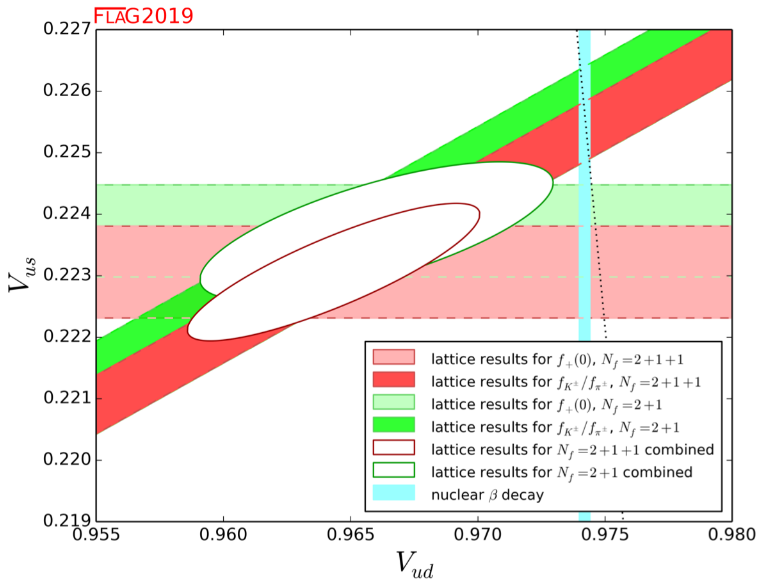




$$K_{\ell 3} \quad \Rightarrow \quad |V_{us}| f_+(0) = 0.2165(4) \quad \Rightarrow \quad |V_{us}| = 0.2231(7)$$

$$K_{\mu 2}/\pi_{\mu 2} \quad \Rightarrow \quad \left|\frac{V_{us}}{V_{ud}}\right| \frac{f_{K^\pm}}{f_{\pi^\pm}} = 0.2760(4) \quad \Rightarrow \quad \left|\frac{V_{us}}{V_{ud}}\right| = 0.2313(7)$$

# Precision tests of the SM

## CKM unitarity:

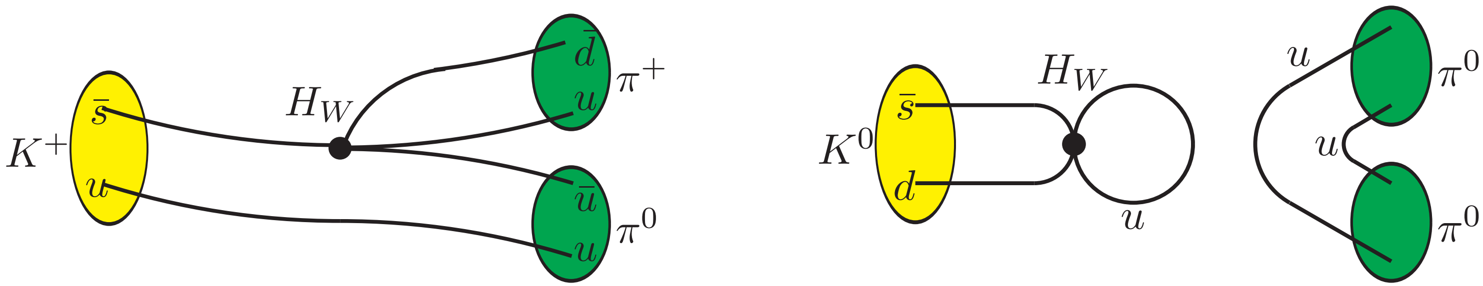



$$|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = \ 0.9998(5)$$

# Precision tests of the SM

Still very uncertain quantities…hard for lattice QCD

Ex: K -> ππ



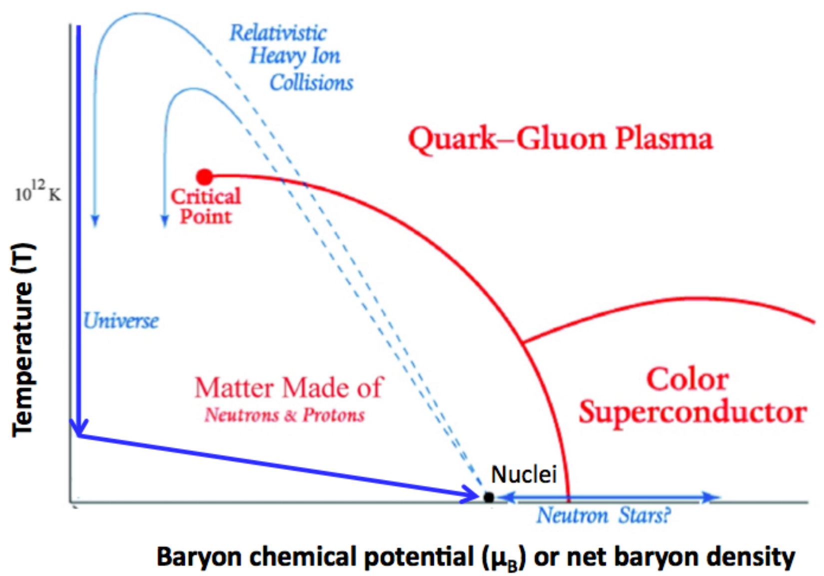
The Δ=1/2 rule: one of the most misterious hierarchies in QCD:

$$\frac{\Gamma(K_S^0 \to \pi\pi)}{\Gamma(K^+ \to \pi\pi)} \approx 330 \qquad T\left(K^0 \to \pi\pi|_{I=\alpha}\right) = A_\alpha e^{i\delta_\alpha} \qquad \frac{A_0}{A_2} = 22.1$$

# QCD @ finite T and density

Asymptotic freedom predicts that the theory should approach a perturbative regime as T -> $\infty$ relevant for the Early Universe, heavy ion collisions

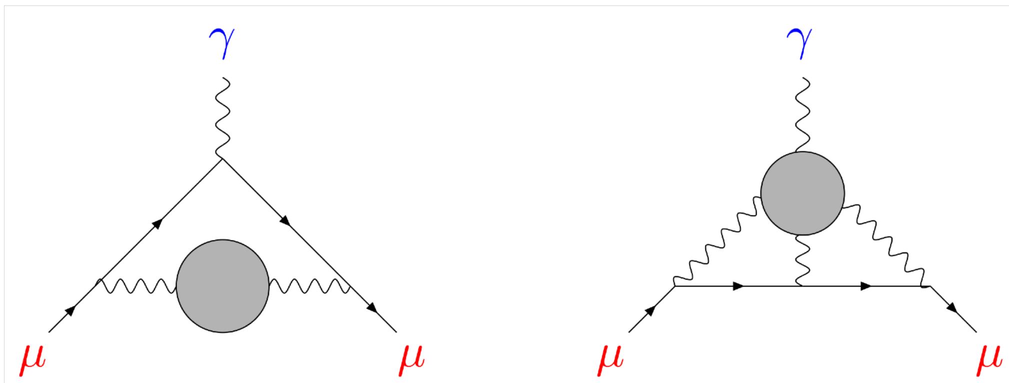
Quark-Gluon plasma: deconfined phase, chiral symmetry breaking restored




Perturbation theory has proved not good enough for the regimes accesible to Experiment. Finite T straightforward on the lattice, finite $\rho$ has sign problem

# $(g\text{-}2)_\mu$ anomaly

$$a_\mu^{\mathrm{SM}} = a_\mu^{\mathrm{QED}} + a_\mu^{\mathrm{EW}} + a_\mu^{\mathrm{had}},$$




HVP HLBL

$$a_\mu^{\mathrm{E821}} - a_\mu^{\mathrm{SM}} = 27.4\,\underbrace{(2.7)}_{\mathrm{HVP}}\,\underbrace{(2.6)}_{\mathrm{HLbL}}\,\underbrace{(0.1)}_{\mathrm{other}}\,\underbrace{(6.3)}_{\mathrm{E821}}\times 10^{-10}$$

# Beyond SM: Alternative to SM Higgs ?

Old Technicolor paradigm: condensate of techniquarks plays the role of the Higgs

$$\langle \bar{Q}Q \rangle \neq 0 : SU(2)_L \times SU(2)_R \to SU(2)_V$$

Three GB: $W^{+-}$,Z

Generically FCNC ($\Lambda_{TC}$> 5TeV): but now there is a light Higgs!

In QCD $\sigma$ particle:

$m_\sigma = O(\Lambda_{QCD})$

$m_\sigma/\Gamma_\sigma = O(1)$

The SM H particle:

$m_H << O(\Lambda_{TC})$

$m_H/\Gamma_H \sim O(30)$

# Beyond SM: Alternative to SM Higgs ?

Modern Technicolor paradigms

- Dilatonic Higgs: TC with approximate conformal symmetry: $N_f$ large enough

  Higgs -> Pseudo-Goldstone boson of this symmetry

  Examples: SU(2) $N_f$=8 fund; SU(2) $N_f$=1,2 adj; SU(3) $N_f$=2 sextet

- Composite Higgs: TC breaking pattern leads to ($W^{+-}$, Z, H) goldstone bosons

  Higgs potential from EW corrections

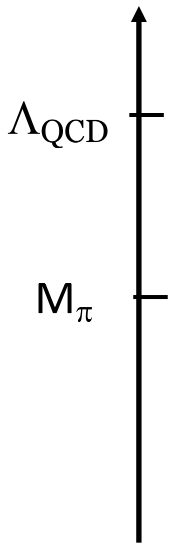
Whether these models are viable alternatives to the SM will rely ultimately on lattice methods…

# Conclusions

- Lattice QFT is a first-principles non-perturbative method to solve asymptotically free QFTs such as QCD
- Lattice QCD has demonstrated quark confinement, a mass gap, spontaneous chiral symmetry breaking
- It has provided precise determination of hadron masses and form factors needed to infer quark masses and mixings from experiment
- Present and future precision tests of the flavour sector of the SM rely on lattice input
- Still more progress is needed: heavy quarks, multi-hadron states, finite density, chiral gauge theories…
- Open problems in particle physics might require non-perturbative physics BSM (eg. composite higgs models)

# Chiral Symmetry Breaking

Chiral symmetry dictates the dynamics of pions and kaons




Effective theory of pion dynamics: Chiral Perturbation Theory

[Weinberg; Gasser and Leutwyler]

$$\mathcal{L}_\chi = \mathcal{L}_\chi^{(2)} + \mathcal{L}_\chi^{(4)} + \ldots$$

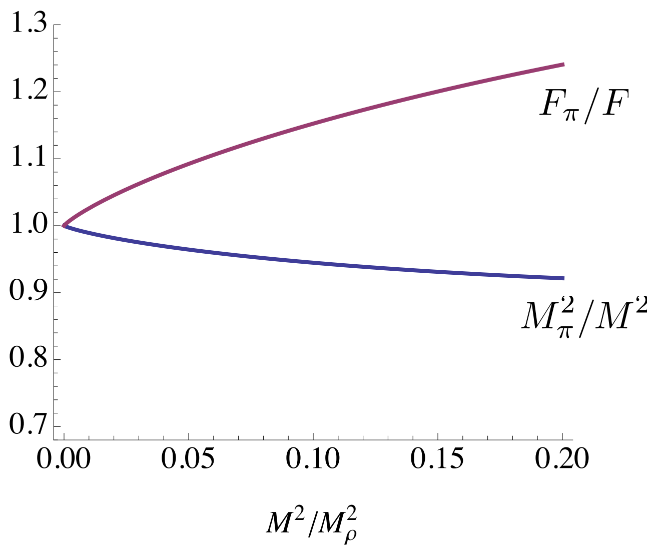
$$\mathcal{L}_\chi^{(2)} = \frac{F^2}{4}\mathrm{Tr}\left[\partial_\mu U^\dagger \partial_\mu U\right] - \frac{\Sigma}{2}\mathrm{Tr}\left[e^{i\theta/N_\mathrm{f}} MU + \mathrm{h.c.}\right]$$

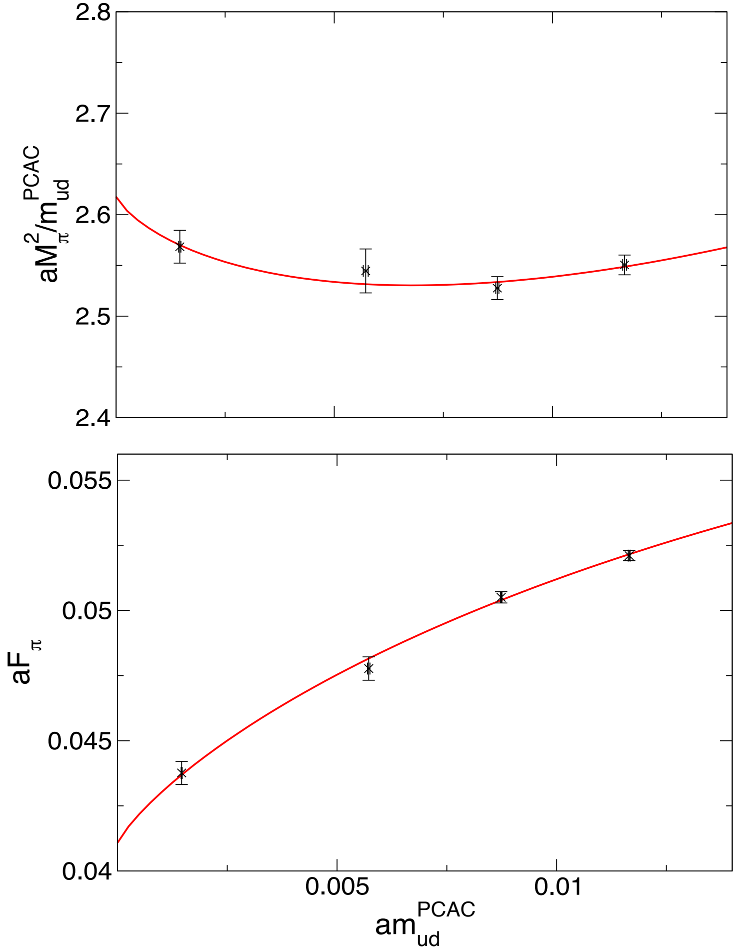
$$\mathcal{L}_\chi^{(4)} = \sum_i \mathcal{C}_i \mathcal{O}_i$$

expansion in $\frac{p^2}{\Lambda_\chi^2}$

Low-energy couplings can be obtained from lattice QCD

# Chiral Symmetry Breaking




[BMW col.]