Supersymmetric Lattice Models

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Benasque – 17-19 February 2020

Lecture 1/3
The name of the game

\[ N=2 \text{ lattice supersymmetry} \]

\[ \{Q, Q^+\} = H \]

this is supersymmetric (susy) quantum mechanics
  – space time susy can emerge in scaling limit

We propose and investigate cases susy models for
  (spinless) fermions on a lattice or (graph) and
connect to susy QFT in a continuum limit
Thanks to susy friends and collaborators ...

Paul Fendley,
Jan de Boer, Bernard Nienhuis,
Hendrik van Eerten,
**Liza Huijse**, Jim Halverson,
Jiri Vala, Nial Moran, Dhagash Mehta,
Bela Bauer, Erez Berg, Matthias Troyer,
**Thessa Fokkema**, 
Jan de Gier, Gyorgy Feher, Sasha Garbali,
Tristan Kuen, Ruben La, Sergey Shadrin,
Jasper Dingerink,
Bart van Voorden, Jiri Minar
Hosho Katsura, Natalia Chepiga
Twelve Easy Pieces
Twelve Easy Pieces

Easy Piece
+ 15 minutes
+ single topic
+ up to 10 slides
+ key idea on board
Twelve Easy Pieces

1. Teaser
2. $N=2$ susy – Witten index
3. $M_1$ model in 1D: Witten index, spectra, CFT connection
4. $M_1$ model: scaling form of 1-pt functions from CFT
5. $M_k$ models: Witten index, CFT
6. $M_k$ models off criticality $\rightarrow$ massive (integrable) QFT
7. $M_1$ model on square ladder: CFT, 1-pt functions, $\langle \sigma \sigma \sigma \sigma \rangle$
8. PH symmetric model and coupled fermion chains
9. Superfrustration on ladder: zig-zag, Nicolai, $Z_2$ Nicolai
10. Superfrustration on 2D grids
11. Back to 1D $M_1$ model: kink dynamics
12. Towards realization with Rydberg-dressed cold atoms
1. Teaser

2. N=2 susy – Witten index

3. $M_1$ model in 1D: Witten index, spectra, CFT connection

4. $M_1$ model: scaling form of 1-pt functions from CFT
QM with $N=2$ supersymmetry

$Q^2 = 0, \quad (Q^\dagger)^2 = 0$

$[Q, H] = 0, \quad H = \{Q, Q^\dagger\}, \quad [Q^\dagger, H] = 0$

[not to be confused with graded Lie algebra symmetries such as ‘supersymmetric $tJ$-model’]
Supersymmetric lattice models

susy QM for lattice fermions

\[ \{ c_i, c_j^\dagger \} = \delta_{ij}, \quad i, j \in \Lambda \]

supercharges expressed in fermion operators

- Hamiltonians with kinetic (hopping) terms and strong interactions
Nicolai model

Supercharge

\[ Q^{\text{Nic}} = \sum_{i} c_{2i-1}^\dagger c_{2i} c_{2i+1} \]

Nicolai 1976
M$_1$ model

configurations:
lattice fermions with nearest neighbor exclusion

supercharge
takes out particle where possible

\[ Q^{M_1} = \sum_i c_i P_i, \quad P_i = \prod_{<ij>} (1 - c_j^\dagger c_j) \]

Fendley-KjS-de Boer 2003
features of susy lattice models

- integrability
- critical behaviour $\mapsto$ supersymmetric CFT
- off-critical $\mapsto$ kink picture of integrable susy QFT
- superfrustration $\mapsto$ proliferation of susy ground states
- dynamics, I $\mapsto$ adiabatic driving in susy gs manifold
- dynamics, II $\mapsto$ out-of-equilibrium transport, MBL?
Susy groundstates in $M_1$ model

Susy groundstates ($M_1$ model):

- 1D closed chain, $L=3l$: $G = 2$

- 2D square lattice, periodic BC, $N \times N$ sites
  $G = 1, 1, 4, 7, 9, 14, 1, 7, 40, 9, 1, 166, \ldots$

- 2D triangular lattice, periodic BC, $N \times N$ sites
  $G_{6 \times 6} \geq 102$, $G_{8 \times 8} \geq 881$, $G_{10 \times 10} \geq 950592$, etc.
**M₁ model, 2D triangular lattice**

Witten index for $N \times M$ sites with periodic BC

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van Eerten 2005
M₁ model, 2D triangular lattice

Witten index for $N \times M$ sites with periodic BC

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`superfrustration’

van Eerten 2005
1. Teaser

2. \textbf{N=2 susy – Witten index}

3. \(M_1\) model in 1D: Witten index, spectra, CFT connection

4. \(M_1\) model: scaling form of 1-pt functions from CFT
Basic structure of susy spectra

- $E \geq 0$ for all states
- $E > 0$ states are paired into **doublets**
  \[
  \{|\psi\rangle, Q^\dagger |\psi\rangle\}, \quad Q |\psi\rangle = 0
  \]
- $E = 0$ iff a state is a **singlet** under supersymmetry
  \[
  Q |\psi_{gs}\rangle = 0, \quad Q^\dagger |\psi_{gs}\rangle = 0
  \]
Fermion number and Witten index

Supercharges change fermion number $F$ by $\pm 1$

$$[F, Q] = -Q, \quad [F, Q^\dagger] = Q^\dagger$$

Witten index

$$W = \text{Tr} \left[ (-1)^F \right]$$

- $W$ easily evaluated by computing trace over all states
- $E>0$ doublets cancel in $W$, only $E=0$ singlets contribute
- $W \neq 0$ implies existence of at least $|W|$ $E=0$ singlets
$W = \text{Tr} \left[ (-1)^F \right]$

$F = 0$: 1 state
$F = 1$: 6 states
$F = 2$: 9 states
$F = 3$: 2 states

$\Rightarrow W = 1 - 6 + 9 - 2 = 2$
$M_1$ model, 6 site chain, $W=2$
Finding groundstates: cohomology

+ supercharges define complex

\[ \mathcal{H} = \bigoplus C_n \quad C_0 \xrightarrow{Q} C_1 \xrightarrow{Q} C_2 \xrightarrow{Q} C_3 \ldots \]

+ can show that susy groundstates are 1-1 with cohomology classes

→ advanced math techniques available
configurations:
lattice fermions with nearest neighbor exclusion

supercharge
takes out particle where possible

\[ Q^{M_1} = \sum_i c_i P_i, \quad P_i = \prod_{\langle ij \rangle} (1 - c_j^\dagger c_j) \]

Fendley-KjS-de Boer 2003
Example of tetrahedron:
+ Witten index : $W=1-4=-3$
+ back-of-the envelope:
  + one $E=4$ susy doublet
  + three $E=0$ susy singlets
Challenge:
For each of the other platonic solid graphs, find the number of $E=0$ susy groundstates and, for each of them, the number of fermions

Reward
A chocolate bar for the first correct result @
c.j.m.schoutens@uva.nl
1. Teaser

2. N=2 susy – Witten index

3. $M_1$ model in 1D: Witten index, spectra, CFT connection

4. $M_1$ model: scaling form of 1-pt functions from CFT
M$_{1}$ model on 1D lattice

configurations
lattice fermions with nearest neighbor exclusion

supercharge and Hamiltonian

\[
Q^{M_{1}} = \sum_{i} (1 - n_{i-1})c_{i}(1 - n_{i+1}), \quad n_{i} = c_{i}^{\dagger}c_{i}
\]

\[
H^{M_{1}} = \sum_{i} \left[ (1 - n_{i-1})c_{i}^{\dagger}c_{i+1}(1 - n_{i+2}) + \text{h.c.} \right] + \sum_{i} n_{i-1}n_{i+1} - 2F + L
\]

n.n. exclusion
hopping
n.n.n. repulsion
Susy lattice model $M_1$, $D=1$

Closed chain
$W=2$ for $L=3l$ sites $\rightarrow$ find 2 groundstates at filling $f/L=1/3$

Open chain
$W=\pm 1$ for $L=3l$, $3l-1$; $W=0$ for $L=3l+1$

Analysis: Bethe Ansatz, mapping to XXZ @ $\Delta=-1/2$
Numerics: ED, finite size spectra

Find that $M_1$ model is critical $\rightarrow$ CFT
Susy lattice model $M_1$, $D=1$

CFT for open chain spectra

what CFT? $ \rightarrow$ $k=1$ minimal model of $N=2$ superconformal field theory

$c=1$, write using single scalar field

Waterson, 1986
Susy lattice model $M_1$, open chain

Ramond-sector affine $U(1)$ modules built on charge $m$ vertex operator $V_m$

$$m = 3f - L - \frac{1}{2}$$

Huijse, 2010

$L=21$

$f=6$  $f=7$  $f=8$  

CFT spectrum
Susy lattice model $M_1$, closed chain

$L=27, f=9$
Susy lattice model $M_1$, closed chain

$L=27, f=9$
Susy lattice model $M_1$, closed chain

$L = 27$, $f = 9$
Susy lattice model $M_1$, closed chain

$L = 27, f = 9$
Susy lattice model $M_1$, closed chain

$L=27$, $f=9$
Susy lattice model $M_1$, closed chain

- twist bc
- trace out $(E,P)$
- SCFT data extracted via parabolic fits to $(E,P)$

### numerics

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### SCFT ($c=1$)

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$L=27, f=9$
Susy lattice model $M_1$, closed chain, $S_E$

(a) $L = 22$ and $f = 7$.

(b) $L = 23$ and $f = 8$.

$$S(l_A) = \frac{c}{3} \ln \left( \frac{L}{\pi} \sin \left( \frac{l_A \pi}{L} \right) \right) + b, \quad c \approx 1.05$$
+ to break criticality, one can stagger the amplitudes in the supercharge with position dependent factors $\lambda_i$
+ for $\lambda_i$ periodic with period 3 the staggered model is still integrable, eg

$$\ldots \lambda 1 1 \lambda 1 1 \lambda 1 1 \ldots$$

+ ground states at $\lambda$ close to 0 are (generalized) product states $\to$ kinks as leading low energy excitations
1. Teaser

2. N=2 susy – Witten index

3. $M_1$ model in 1D: Witten index, spectra, CFT connection

4. $M_1$ model: scaling form of 1-pt functions from CFT
Figure 5.2: Example of the $Z_3$ behaviour for the one-point functions $\langle n_k \rangle$ for a chain of $L = 30$. There are three branches, one for each value of $k \mod 3$.  

Dingerink, MSc thesis 2019
Susy lattice model $M_1$, open chain

(a) Finite size scaling for $L \mod 3 = 0$, with a scaling of $\nu = 1/3$.

numerics: MPS

Dingerink,
MSc thesis 2019
Susy lattice model $M_1$, open chain

Step I
express density operator in CFT fields

\[
\begin{align*}
n_k &= \frac{1}{3} + \frac{1}{3} \left( A_1 V_1 + A_{-1} V_{-1} \right), \\
n_{k+1} &= \frac{1}{3} + \frac{1}{3} \left( e^{2\pi i / 3} A_1 V_1 + e^{-2\pi i / 3} A_{-1} V_{-1} \right), \\
n_{k-1} &= \frac{1}{3} + \frac{1}{3} \left( e^{-2\pi i / 3} A_1 V_1 + e^{2\pi i / 3} A_{-1} V_{-1} \right).
\end{align*}
\]

Huijse, 2010
Step II

determine density operator 1-pt functions in CFT

\[
\langle V_1(x) \rangle_{\text{strip}} = \lim_{z_1 \to \infty, z_4 \to 0} \left( \frac{i\pi}{\tilde{L}} \right)^{1/3} (-z_2 z_3)^{1/6} \langle 0 | V_{1/2}(z_1) z_1^{1/12} V_1(z_2) V_{-1}(z_3) V_{-1/2}(z_4) | 0 \rangle \\
= (-1)^{1/6} \left( \frac{i\pi}{\tilde{L}} \right)^{1/3} z_3^{1/3} (z_2 - z_3)^{-1/3} \\
= (-1)^{1/6} \left( \frac{\pi}{2\tilde{L}} \right)^{1/3} \frac{e^{-i\pi x/(3\tilde{L})}}{\sin^{1/3}(\pi x/\tilde{L})} \\
= \left( \frac{\pi}{2\tilde{L}} \right)^{1/3} \frac{e^{-i\pi (x-\tilde{L}/2)/(3\tilde{L})}}{\sin^{1/3}(\pi x/\tilde{L})}. \]

Huijse, 2010
Susy lattice model $M_1$, open chain

Step III
wrap up

\[
\langle n_k \rangle = \frac{1}{3} + \frac{2A_1}{3} \left( \frac{\pi}{2\tilde{L}} \right)^{1/3} \cos \left[ \frac{\pi(x - \tilde{L}/2)}{(3\tilde{L})} \right] \frac{\sin^{1/3}(\pi x/\tilde{L})}{\sin^{1/3}(\pi x/\tilde{L})},
\]

\[
\langle n_{k+1} \rangle = \frac{1}{3} + \frac{2A_1}{3} \left( \frac{\pi}{2\tilde{L}} \right)^{1/3} \sin \left[ \frac{\pi(x - \tilde{L})}{(3\tilde{L})} \right] \frac{\sin^{1/3}(\pi x/\tilde{L})}{\sin^{1/3}(\pi x/\tilde{L})},
\]

\[
\langle n_{k-1} \rangle = \frac{1}{3} - \frac{2A_1}{3} \left( \frac{\pi}{2\tilde{L}} \right)^{1/3} \sin \left[ \frac{\pi x}{(3\tilde{L})} \right] \frac{\sin^{1/3}(\pi x/\tilde{L})}{\sin^{1/3}(\pi x/\tilde{L})}.
\]

Huijse, 2010
Susy lattice model $M_1$, open chain

Figure 5.4: The one-point functions $\langle n_k \rangle$ of chains of length $L \mod 3 = 0$ for $k \mod 3 = 0$ (left), $k \mod 3 = 1$ (middle) and $k \mod 3 = 2$ (right). For each of the three cases the corresponding equation (5.23) was fitted (solid line) with a fit parameter of $A_1 = 0.810$. The $x$ on the x-axis is related to lattice position $k$ by $x = k + 1$. 

Dingerink, 
MSc thesis 2019
Twelve Easy Pieces

1. Teaser
2. $N=2$ susy – Witten index
3. $M_1$ model in 1D: Witten index, spectra, CFT connection
4. $M_1$ model: scaling form of 1-pt functions from CFT
5. $M_k$ models: Witten index, CFT
6. $M_k$ models off criticality $\rightarrow$ massive (integrable) QFT
7. $M_1$ model on square ladder: CFT, 1-pt functions, $\langle \sigma \sigma \sigma \sigma \rangle$
8. PH symmetric model and coupled fermion chains
9. Superfrustration on ladder: zig-zag, Nicolai, $Z_2$ Nicolai
10. Superfrustration on 2D grids
11. Back to 1D $M_1$ model: kink dynamics
12. Towards realization with Rydberg-dressed cold atoms