Supersymmetric Lattice Models

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Lecture 2/3
1. Teaser
2. N=2 susy – Witten index
3. $M_1$ model in 1D: Witten index, spectra, CFT connection
4. $M_1$ model: scaling form of 1-pt functions from CFT
5. $M_k$ models: Witten index, CFT
6. $M_k$ models off criticality $\rightarrow$ massive (integrable) QFT
7. $M_1$ model on square ladder: CFT, 1-pt functions, $\langle \sigma \sigma \sigma \sigma \rangle$
8. PH symmetric model and coupled fermion chains
9. Superfrustration on ladder: zig-zag, Nicolai, $Z_2$ Nicolai
10. Superfrustration on 2D grids
11. Back to 1D $M_1$ model: kink dynamics
12. Towards realization with Rydberg-dressed cold atoms
Twelve Easy Pieces

Easy Piece
+ 15 minutes
+ single topic
+ up to 10 slides
+ key idea on board
5. $M_k$ models: Witten index, CFT

6. $M_k$ models off criticality $\rightarrow$ massive (integrable) QFT

7. $M_l$ model on square ladder: CFT, 1-pt functions, $<\sigma\sigma\sigma\sigma>$

8. PH symmetric model and coupled fermion chains
**$M_k$ model in 1D**

configurations

lattice fermions **up to** $k$

nearest neighbors occupied

supercharge

\[
Q^{M_k} = \sum_i \sum_{a,b} \lambda_{[a,b],i} d_{[a,b],i}
\]

annihilates particle at position $b$ from string of length $a$
**M$_2$ model**

**configurations**

lattice fermions **up to 2** nearest neighbors occupied

**M$_2$[x]**:

amplitude $x$ if $Q$ annihilates particle w/o neighbor, else 1

$x=0$: maps to ferromagnetic integrable tJ model, $J=-2t$

$x=\sqrt{2}$: maps to integrable spin-1 XXZ model; critical

General $x$ not well studied
$M_k$ model, properties


Closed chain:

\[ W = k+1 \text{ for } L = l(k+2), \]
\[ \implies k+1 \text{ groundstates at filling } f/L = k/(k+2) \]

Open chain:

\[ W = (-1)^k \text{ for } L = k+1, k+2 \text{ modulo } k+2 \]
\[ 0 \text{ else} \]

Critical behavior:

$CFT_k$, $k^{th}$ minimal model of $N=2$ SCFT,

$U(1) \times Z_k$ parafermions
$M_2$ model, open chain

CFT modules:

$m = 2f - L - \frac{1}{2}$

$f$ even: $V_m$

$f$ odd: $\psi V_m$

$E = L_0 - \frac{1}{16}$

$L=25, \text{open BC}$
$\sigma$-type defects and BC

- locally forbid simultaneous occupancy of n.n. sites
- in the CFT these correspond to the injection of spin fields $\sigma$
M_2 model, σ/open boundary conditions

\[ f = 11 \quad f = 12 \quad f = 13 \quad f = 14 \]

CFT modules:
\[ \sigma V_m \text{ with } m = 2f - L \]

L=25, σ/open BC
$M_3$ model, various boundary conditions

$k=3$ minimal model of $N=2$ SCFT $\rightarrow \mathbb{Z}_3$ parafermions
5. $M_k$ models: Witten index, CFT

6. $M_k$ models off criticality $\rightarrow$ massive (integrable) QFT

7. $M_f$ model on square ladder: CFT, 1-pt functions, $\langle \sigma \sigma \sigma \sigma \rangle$

8. PH symmetric model and coupled fermion chains
+ to break criticality, one can stagger the amplitudes in the supercharge with position dependent factors $\lambda_i$
+ for $\lambda_i$ periodic with period 3 the staggered model is still integrable, eg

$$\ldots \lambda 1 1 \lambda 1 1 \lambda 1 1 \ldots$$
Hagendorf, Huijze and Fokkema found 1-parameter family of models $M_k[\lambda]$ with couplings $\lambda_{[a,b],j}$ such that

- all couplings repeat for $j$ modulo $(k+2)$
- $M_k[\lambda]$ model integrable by nested Bethe Ansatz for all $\lambda$
- $M_k[\lambda]$ is critical for $\lambda=1$

Hagendorf-Fokkema-Huijse 2014
Hagendorf-Huijse 2015
$M_k[\lambda]$, definitions

$M_2$: $x = \sqrt{2}$, 
staggering modulo 4

$\lambda_{[1,1],j}: \ldots \sqrt{2} \sqrt{2}\lambda \sqrt{2} \sqrt{2}\lambda \ldots$
$\lambda_{[2,1],j}: \ldots 1 \lambda 1 \lambda \ldots$
$\lambda_{[2,2],j}: \ldots 1 \lambda 1 \lambda \ldots$

$M_3$: staggering mod 5, $\lambda << 1$

$\lambda_{[1,1],j}: \ldots 1 \sqrt{2} \sqrt{2}\lambda \sqrt{2} 1 \ldots$
$\lambda_{[2,1],j}: \ldots 1 \sqrt{2} \lambda \sqrt{2} \lambda \ldots$
$\lambda_{[2,2],j}: \ldots \lambda \sqrt{2} \lambda 1 1 \ldots$
$\lambda_{[3,1],j}: \ldots 1 \lambda \lambda 1 \frac{\lambda}{\sqrt{2}} \ldots$
$\lambda_{[3,2],j}: \ldots \frac{\lambda}{\sqrt{2}} 1 \frac{\lambda^2}{\sqrt{2}} 1 \frac{\lambda}{\sqrt{2}} \ldots$
$\lambda_{[3,3],j}: \ldots \frac{\lambda}{\sqrt{2}} 1 \lambda \lambda 1 \ldots$
M₁ model with staggering

+ plotting the lowest energies of the model on $L=13$ sites, OBC, staggering

$$1 1 \lambda 1 1 \lambda 1 1 \lambda 1 1 \lambda 1$$

$\lambda = 0 \quad \lambda = 1$
M$_1$, model with staggering

$\lambda = 0$  \hspace{2cm} $\lambda = 1$

kink states
$|K_1>, |K_4>, \ldots |K_{13}>$
energy $E=1$

scaling limit for $\lambda \uparrow 1$:
supersymmetric $N=2$ QFT:
sine-Gordon theory at $\beta^2 = 4/3$
\[ M_k[\lambda<1] \text{ connects to } N=2 \text{ supersymmetric integrable massive QFT, with superpotentials of Chebyshev form} \]

\[ k=1: \text{sine-Gordon at } N=2 \text{ susy point } \beta^2=4/3 \]
\[ k=2: N=1 \text{ supersymmetric sine-Gordon at } N=2 \text{ susy point} \]

lattice model excitations at \( \lambda \ll 1 \) are kinks between \( W=k+1 \) possible \( E=0 \) states – they are in 1-1 correspondence with kinks in the \( N=2 \) integrable QFT
Integrable massive QFT

susy lattice model $M_3[\lambda]$
susy ground-states at $\lambda \ll 1$

massive $N=2$ integrable QFT
groundstates as extrema of $k=3$
Chebyshev superpotential

$$|1\rangle = \ldots 1110011100 \ldots, \quad |\frac{1}{2}\rangle = \ldots (\cdot 1 \ldots)(\cdot 1 \ldots) \ldots, \quad |0\rangle = \ldots 0101101011 \ldots, \quad |-\frac{1}{2}\rangle = \ldots 0(11\cdot)0(11\cdot) \ldots,$$
5. $M_k$ models: Witten index, CFT

6. $M_k$ models off criticality $\rightarrow$ massive (integrable) QFT

7. $M_1$ model on square ladder: CFT, 1-pt functions, $<\sigma\sigma\sigma\sigma>$

8. PH symmetric model and coupled fermion chains
$M_1$ model on square ladder

+ find that for PBC, $L=4l$, Witten index $W=3$
+ intuition: charge order (along legs) combined with Ising order (along rungs)

+ expect $k=2$ $N=2$ SCFT, $c=3/2$, as for $M_2$ model
+ however, finite size spectra puzzling
A supersymmetric multicritical point in a model of lattice fermions

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We study a model of spinless fermions with infinite nearest-neighbor repulsion on the square ladder which has microscopic supersymmetry. It has been conjectured that in the continuum the model is described by the superconformal minimal model with central charge $c = 3/2$. Thus far it has not been possible to confirm this conjecture due to strong finite-size corrections in numerical data. We trace the origin of these corrections to the presence of unusual marginal operators that break Lorentz invariance, but preserve part of the supersymmetry. By relying mostly on entanglement entropy calculations with the density-matrix renormalization group, we are able to reduce finite-size effects significantly. This allows us to unambiguously determine the continuum theory of the model. We also study perturbations of the model and establish that the supersymmetric model is a multicritical point. Our work underlines the power of entanglement entropy as a probe of the phases of quantum many-body systems.
Adding terms to $H_{\text{susy}}$ to stabilize I or C (or CI) order

$$H_{\text{pert}} = t_\perp \sum_i \left( d_{i,\circ}^\dagger d_{i,\circ} + d_{i,\circ}^\dagger d_{i,\circ}^\circ \right)$$

$$+ J \sum_i \left( n_{i,\circ} n_{i+1,\circ} + n_{i,\circ} n_{i+1,\circ} \right)$$

Choosing $J >> 0$, $t_\perp << 0$ stabilizes both orders (CI)
M$_1$ model on square ladder

Susy point is multicritical point
M₁ model on square ladder

Numerical (MPS) measurement of density 1-point functions

\[
\langle n^+_k \rangle = \langle \psi | n^+_k | \psi \rangle \equiv \langle \psi | c^\dagger_k c^\dagger + c^\dagger c^\dagger | \psi \rangle \\
\langle n^-_k \rangle = \langle \psi | n^-_k | \psi \rangle \equiv \langle \psi | c^\dagger_k c^\dagger - c^\dagger c^\dagger | \psi \rangle
\]

Dingerink, MSc thesis 2019
(a) The $\mathbb{Z}_2$ structure of $\langle n_k^+ \rangle$.  

(b) The $\mathbb{Z}_4$ structure of $\langle n_k^- \rangle$. 

$L=40$, $f=20$
$M_1$ model on square ladder

\[ \langle n_k^- \rangle \text{ data collapse with exponent } \nu = 1/4, \]

CFT expressions for fitting these curves expressed through

\[ \langle \sigma \sigma \sigma \sigma \rangle \text{ 4-point function} \]
5. $M_k$ models: Witten index, CFT

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8. PH symmetric model and coupled fermion chains
Particle Hole symmetric $M_1$ model

Supercharge

\[ Q = \sum_i (d_i + e_i^\dagger) \]

\[ d_i = p_{i-1} c_i p_{i+1} \]
\[ p_i = 1 - n_i \]

\[ e_i^\dagger = n_{i-1} c_i^\dagger n_{i+1} \]
\[ n_i = c_i^\dagger c_i \]

de Gier-Feher-Nienhuis-Rusaczonek 2015
Susy groundstates in PH M\textsubscript{1} model

- degenerate groundstates at \( E=0 \)

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*Table 1:* The degeneracy of the groundstate, the number of energy levels, and the smallest degree of degeneracy for periodic and antiperiodic boundary conditions.

- in addition: extensive degeneracies of excited states
Coupled free fermion chains

**supercharge**

moves particle from bottom to top chain

\[ Q = c_2^\dagger c_1 + \sum_{k=1}^{L-1} \left( e^{i\alpha_{2k-1} \pi/2} c_{2k}^\dagger + e^{i\alpha_{2k} \pi/2} c_{2k+2}^\dagger \right) c_{2k+1} \]

\[ \alpha_k \equiv \sum_{j=1}^{k} (-1)^j n_j \]

Fendley-KjS 2007
Coupled free fermion chains

- coupled free fermion chains

\[
H_c = - \sum_{j=2}^{2L-1} \left[ (1 - i) c^\dagger_{j+1} n_j c_{j-1} + (1 + i) c^\dagger_{j-1} n_j c_{j+1} \right] - 2 \sum_{j=1}^{2L-1} n_j n_{j+1}
\]

Fendley-KjS, 2007
Coupled free fermion chains

on $L+L$ sites, OPB:

find $2^L E=0$ susy groundstates understood via possibility to place up to $L$ tightly bound pairs between top and bottom chains
Coupled free fermion chains

Particles on lower (upper) chain have $F = \pm 1/2$ (semions)

\[ F = -\frac{1}{2} \quad \cdots \quad 2k \quad 2k+2 \quad \cdots \]

\[ F = \frac{1}{2} \quad \cdots \quad 2k+1 \quad \cdots \]

\[ \Delta F = -1 \]

Witten index

\[ W = \text{Tr} \left[ (-1)^F \right] = \prod_{1}^{L} (1 + i) \prod_{1}^{L} (1 - i) = 2^L \]
• mechanism: condensation of $E=0$ `Cooper pairs’
• in Bethe Equations:

\[
z^L_j = \pm i^{-L/2} \prod_{l=1}^{k} \frac{u_l - (z_j - 1/z_j)^2}{u_l + (z_j - 1/z_j)^2}, \quad j = 1, \ldots, m
\]

\[
1 = \prod_{j=1}^{m} \frac{u_l - (z_j - 1/z_j)^2}{u_l + (z_j - 1/z_j)^2}, \quad l = 1, \ldots, k.
\]

can add pairs of rapidities $(u, -u)$ at nested level without affecting BE at top level
A curious mapping

coupled fermion chain model (FS) and particle-hole symmetric $M_1$ model (FGNR) turn out to be equivalent

empty ladder: $|000000000000\rangle_{FS} \leftrightarrow 0|110011001100\rangle_{FGNR}$
single FS semion: $|000010000000\rangle_{FS} \leftrightarrow 0|110000110011\rangle_{FGNR}$
single FS pair: $|000011000000\rangle_{FS} \leftrightarrow 0|110001001100\rangle_{FGNR}$
lower leg filled: $|101010101010\rangle_{FS} \leftrightarrow 0|000000100000\rangle_{FGNR}$
single FGNR particle: $|101001101010\rangle_{FS} \leftrightarrow 0|000010000000\rangle_{FGNR}$
upper leg filled: $|010101010101\rangle_{FS} \leftrightarrow 0|101010101010\rangle_{FGNR}$
upper leg plus semion: $|110101010101\rangle_{FS} \leftrightarrow 0|010101010101\rangle_{FGNR}$
filled ladder: $|111111111111\rangle_{FS} \leftrightarrow 0|011001100110\rangle_{FGNR}$

Feher-Garbal-de Gier-KjS 2017
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