

Lattice Calculation of $K \rightarrow \pi\pi$
with On-shell Pions

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OUTLINE

- Circumvent the Maini-Testa theorem by imposing anti-periodic boundary conditions on the final-state pions.
- Use Lellouch-Lüscher to relate the finite volume matrix elements with those at infinite volume.
- Preliminary $I = 2$ phase shifts.
- Status of $\Delta I = 3/2$ K -decay calculation.

G-parity Boundary Conditions

- G-parity operation on the pion:

$$\begin{aligned}G|\pi^\pm\rangle &= -|\pi^\pm\rangle \\G|\pi^0\rangle &= -|\pi^0\rangle\end{aligned}$$

- G-parity operation on the quark fields:

$$G\begin{pmatrix} \mathbf{u} \\ \mathbf{d} \end{pmatrix} = \begin{pmatrix} -\mathbf{d}^C \\ \mathbf{u}^C \end{pmatrix}$$

- Must impose charge-conjugate boundary conditions on the gauge field to preserve gauge invariance.
- G-parity commutes with isospin but not with the chiral generators.

H-parity Boundary Conditions

- H-parity operation on the quark fields (definition):

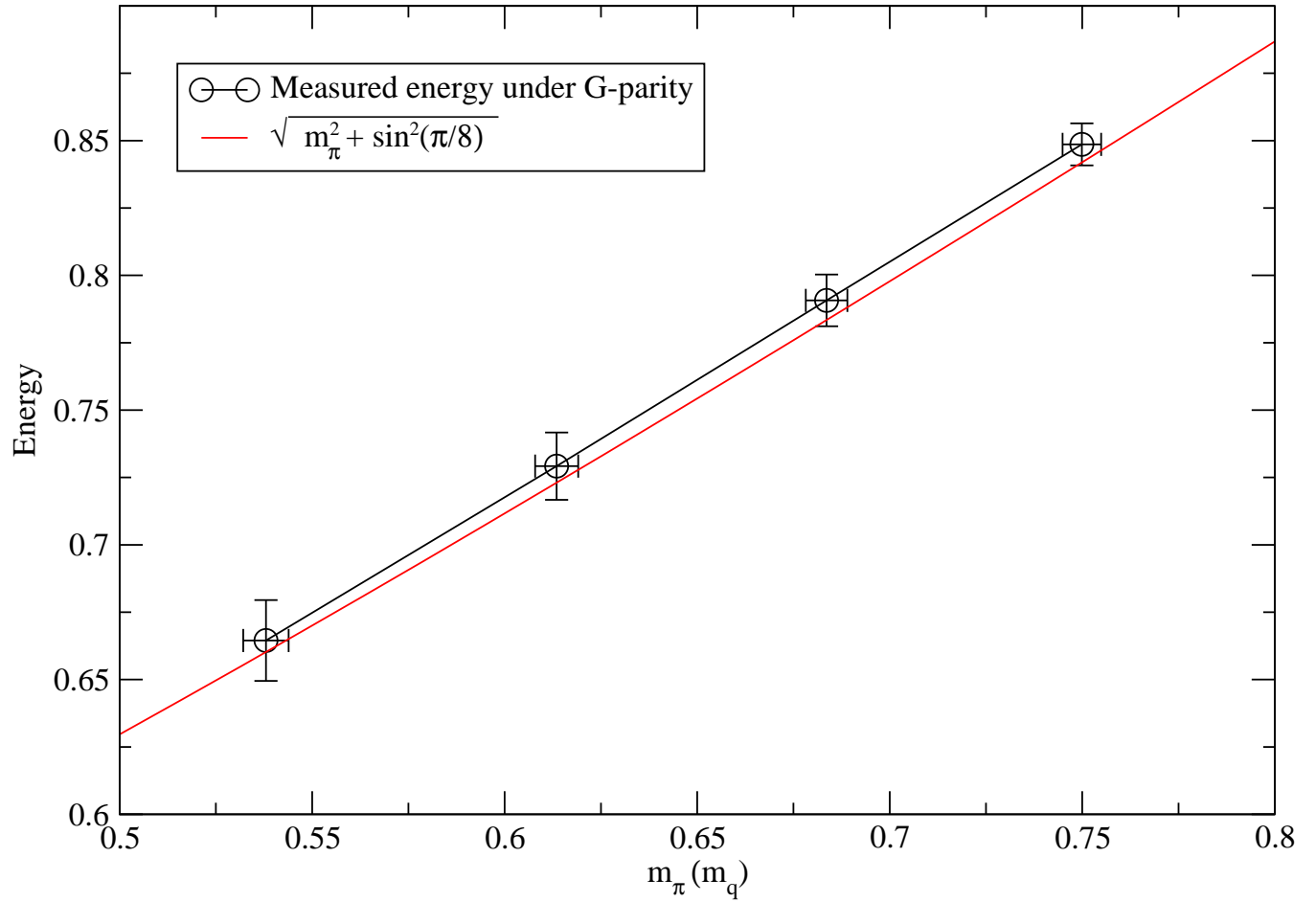
$$H \begin{pmatrix} \mathbf{u} \\ \mathbf{d} \end{pmatrix} = \begin{pmatrix} -\mathbf{u} \\ \mathbf{d} \end{pmatrix}$$

- H-parity operation on the pion:

$$\begin{aligned} H|\pi^\pm\rangle &= -|\pi^\pm\rangle \\ H|\pi^0\rangle &= +|\pi^0\rangle \end{aligned}$$

- $I_z = 2$, $\pi^+\pi^+$ state contains anti-symmetric pions with non-zero momenta.
- Not true for the $I = 0$ $\pi\pi$ state.
- No modification of the usual gauge configurations is required— an advantage.

Single Pion Result



$1/a = 0.978(14)\text{GeV}$, $\beta = 5.7$, $8^3 \times 32$, $L_s = 10$
Wilson gauge action, domain wall fermions
H-parity boundary conditions

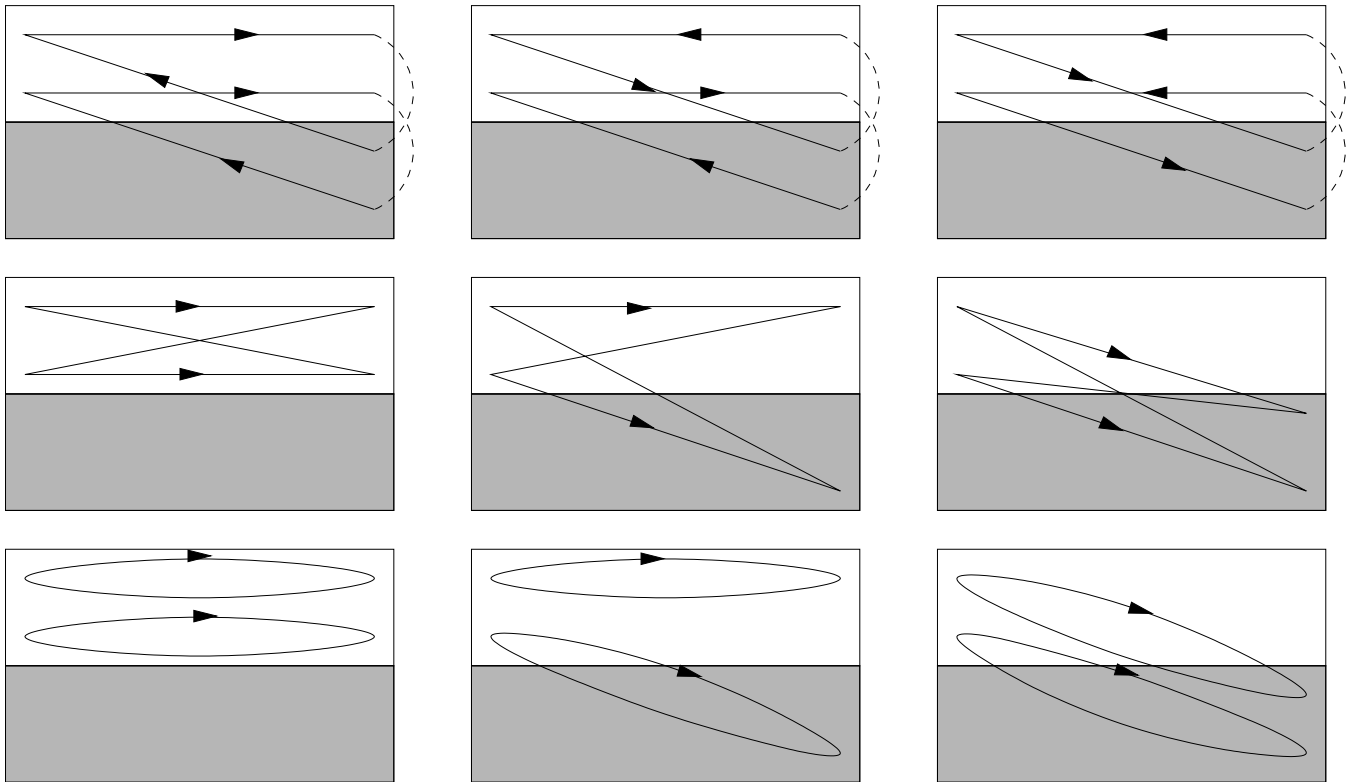
Two-pion states and the $I = 2$ phase shift

- Before studying $K \rightarrow \pi\pi$ decays we examine the $I = 2$ phase shift.
- We use Lüscher's method to extract phase shift from the energy levels in a finite box.

$$n\pi - \delta_0(k) = \phi(q) \quad q \equiv \frac{kL}{2\pi}$$

- The new boundary conditions modify only the functional form of $\phi(q)$.

Some of the diagrams entering the $\pi\pi - \pi\pi$ propagator

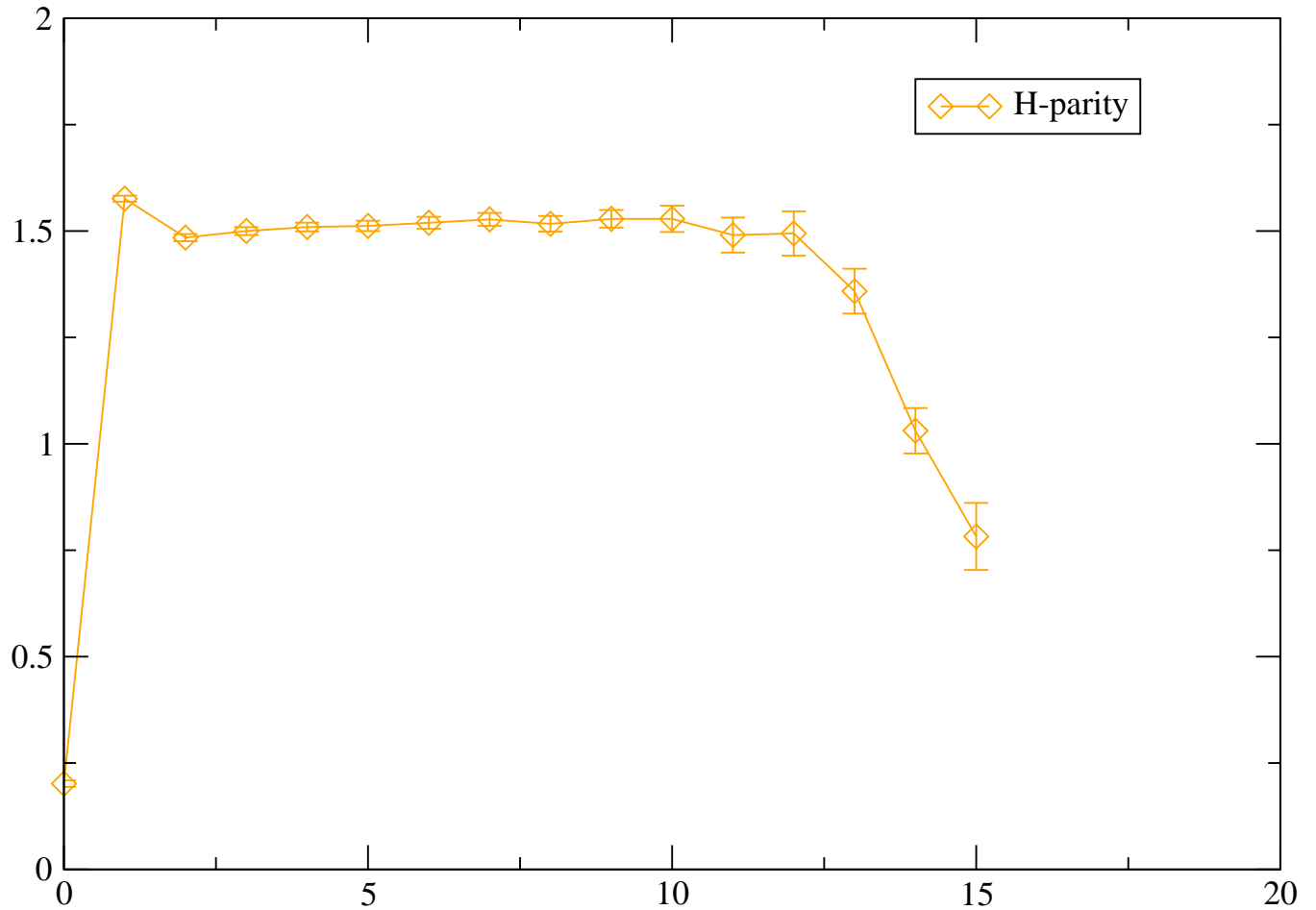


Open: usual gauge links $U_\mu(x)$

Shaded: charge-conjugate gauge links $U_\mu(x)^*$

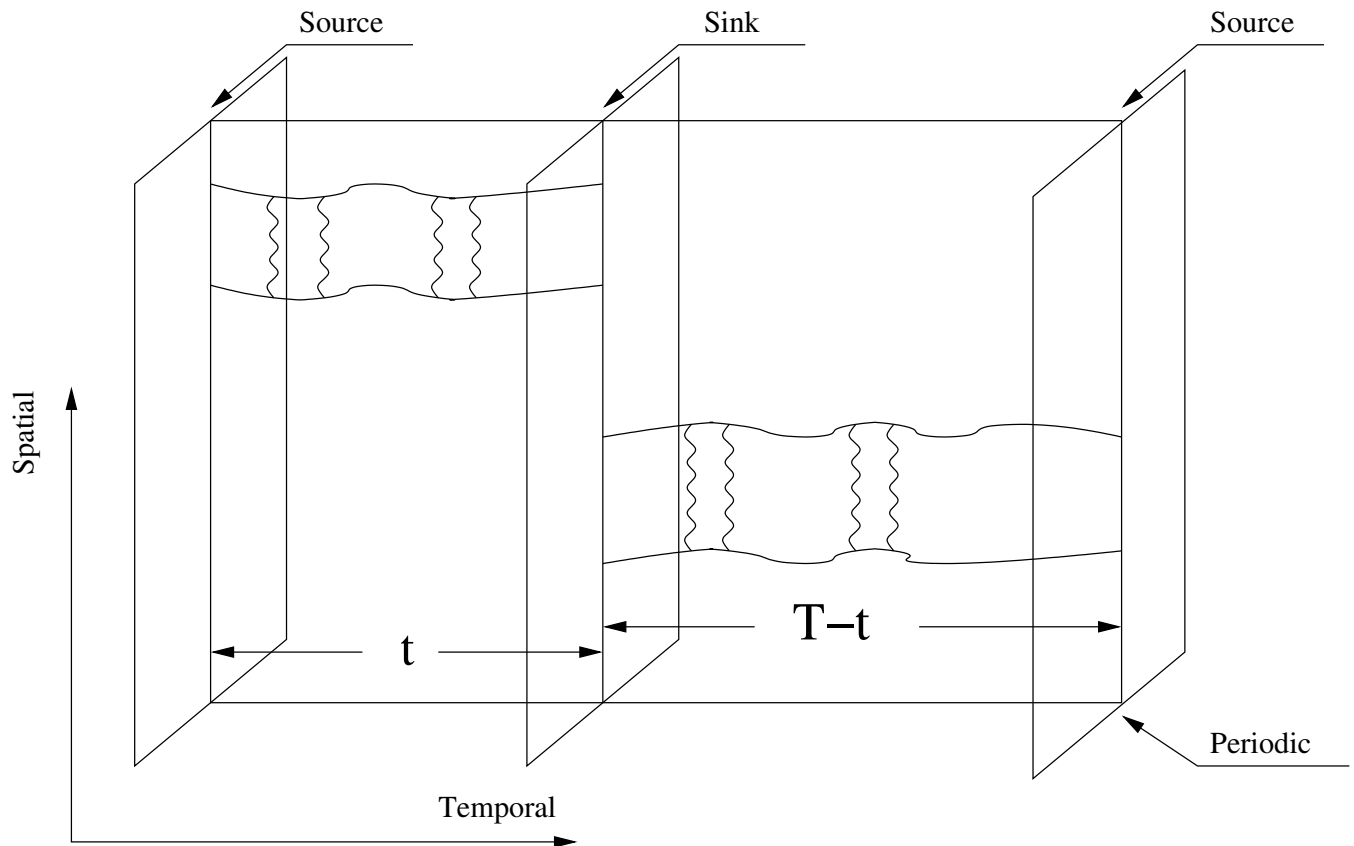
Effective Mass Plot

H-parity



$1/a = 0.978(14)\text{GeV}$, $\beta = 5.7$, $8^3 \times 32$, $L_s = 10$
Wilson gauge action, domain wall fermions
H-parity boundary conditions

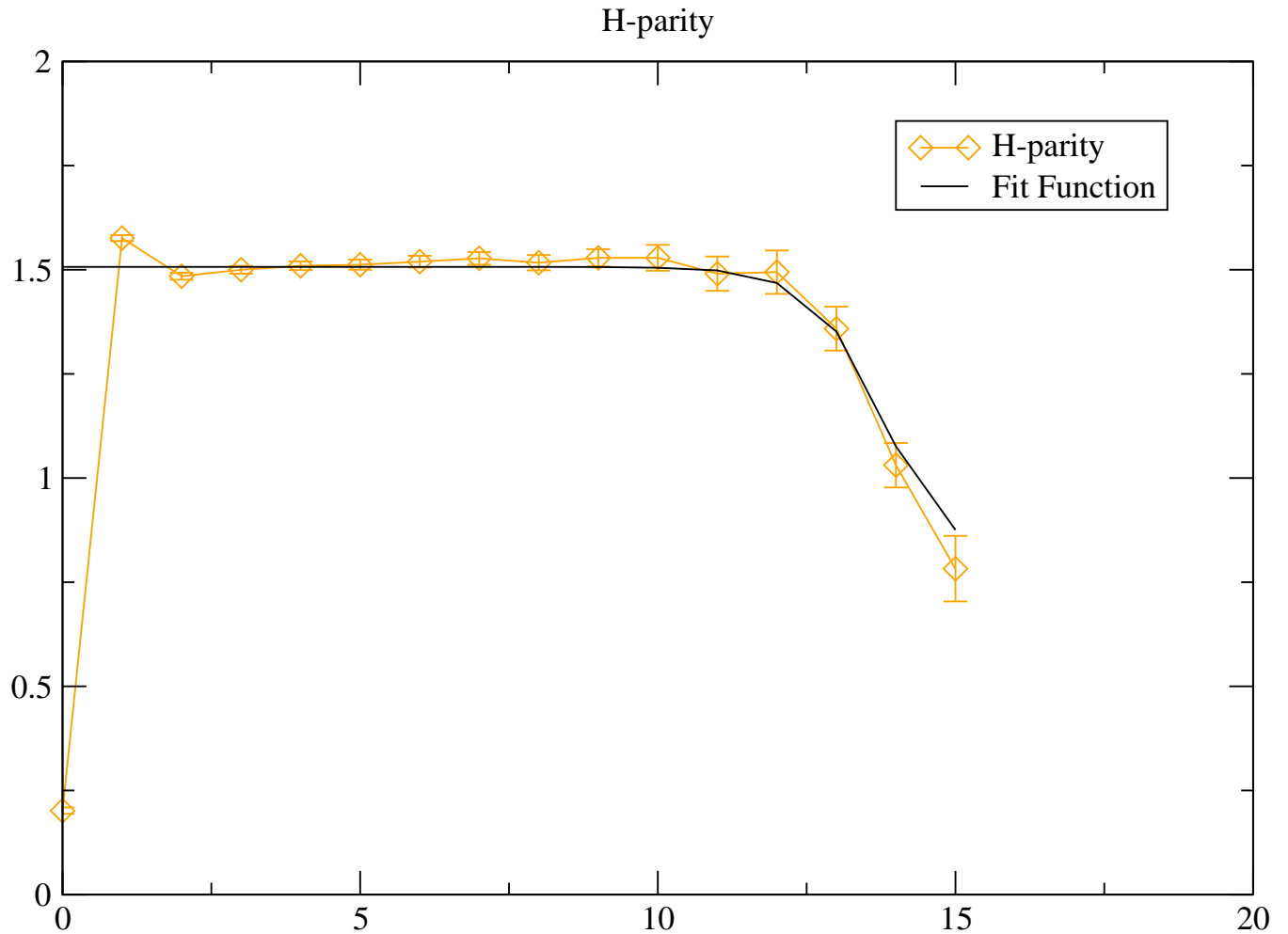
Explanation of constant term



A single pion propagates for all time:
$$G(t) \approx e^{-m_\pi(T-t)} e^{-m_\pi t} = e^{-m_\pi T} = e^{-2m_\pi \frac{T}{2}}$$

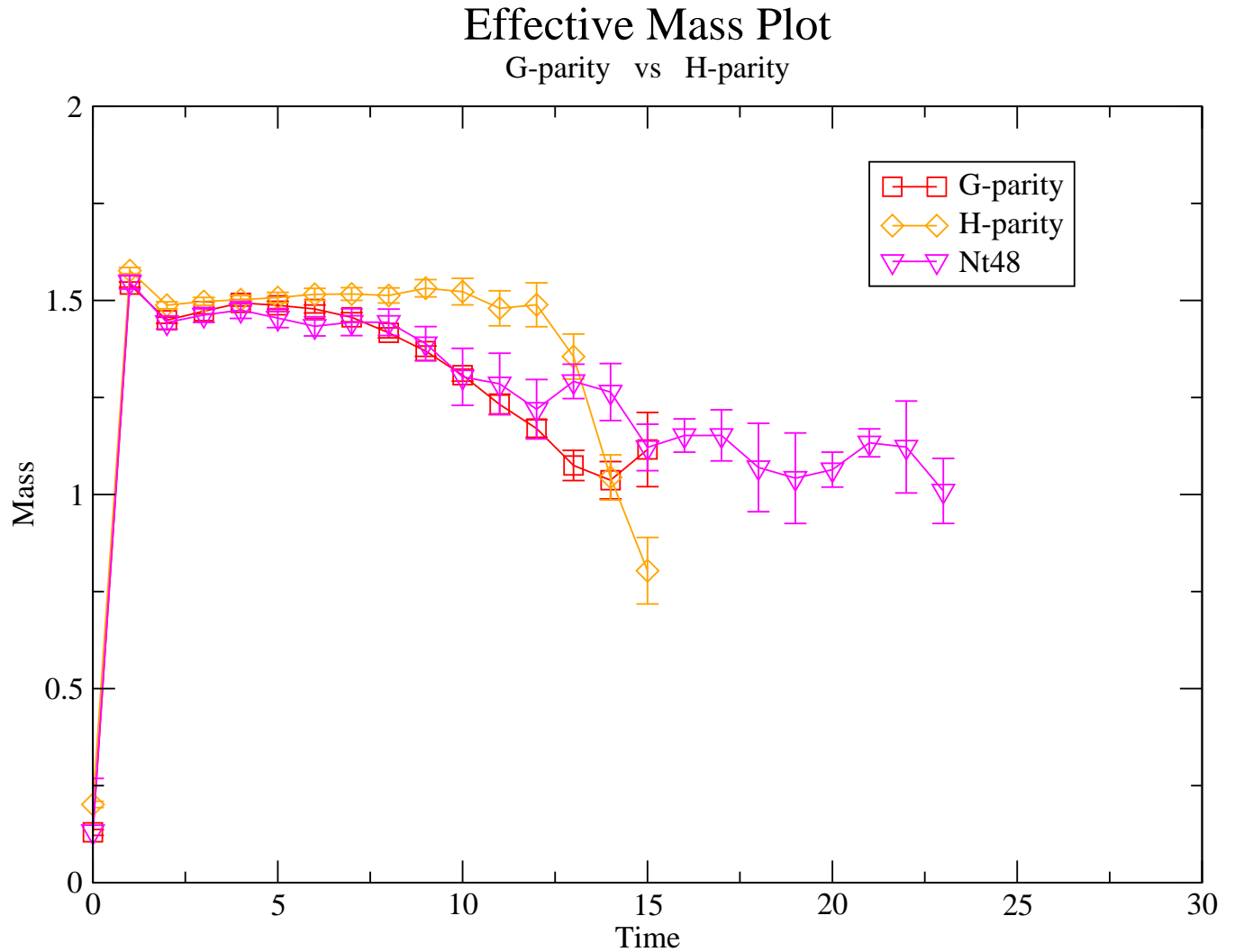
Fit including a constant

Effective Mass Plot



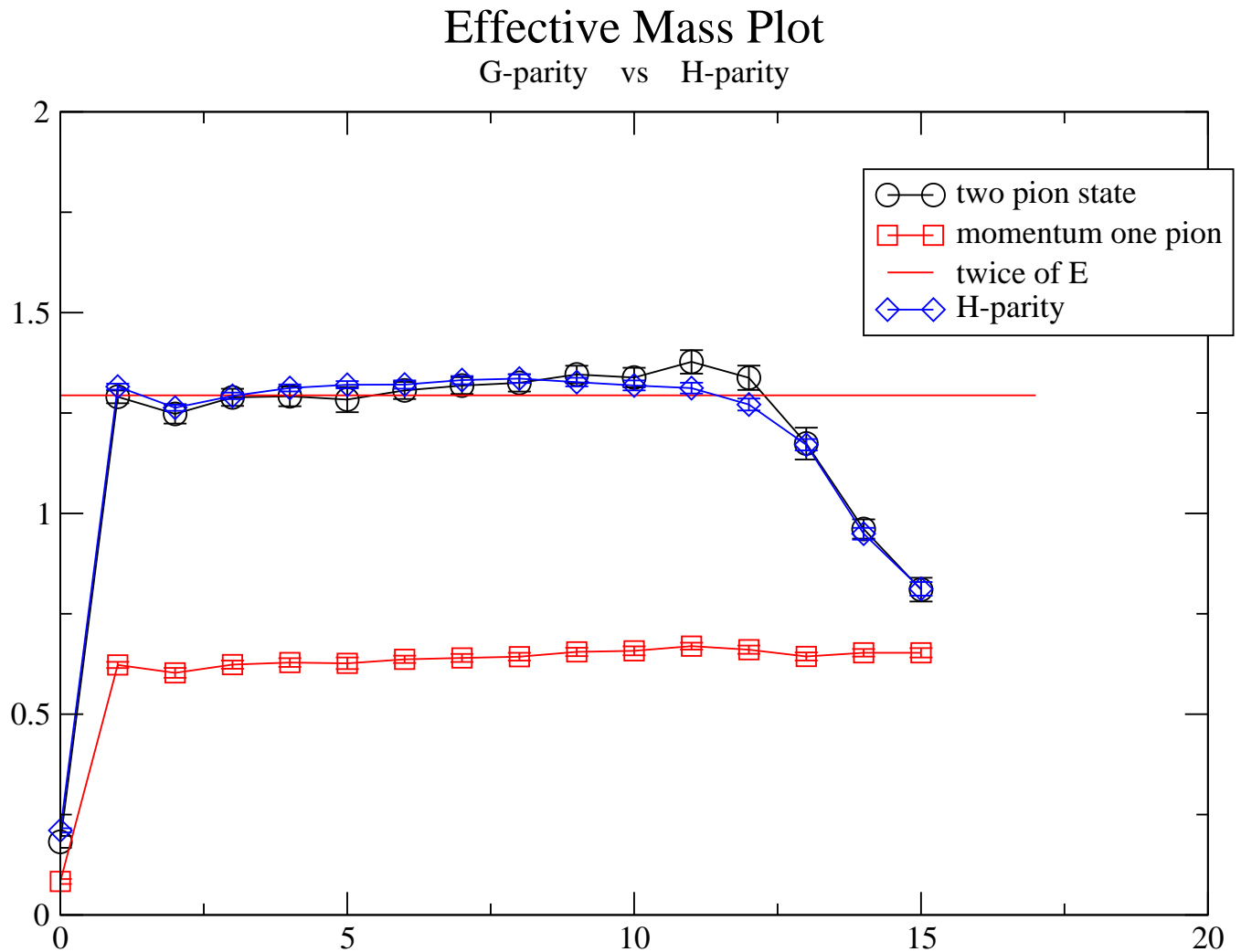
$1/a = 0.978(14)\text{GeV}$, $\beta = 5.7$, $8^3 \times 32$, $L_s = 10$
Wilson gauge action, domain wall fermions
H-parity boundary conditions

G-parity boundary conditions



$1/a = 0.978(14)\text{GeV}$, $\beta = 5.7$, $8^3 \times 32$ and $\times 48$
 $L_s=10$, Wilson gauge action, domain wall
fermions, H-parity boundary conditions

Examine a larger $8^2 \times 16 \times 32$ volume



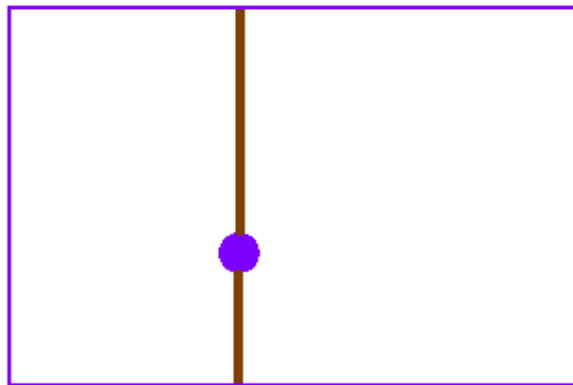
$$1/a = 0.978(14)\text{GeV}, \beta = 5.7$$

$$8^2 \times 16 \times 32 \text{ and } \times 48, L_s = 10$$

Wilson gauge action, domain wall fermions
G- and H-parity boundary conditions

Finite-volume sensitivity of G-parity

- G-parity allows color flux tube going from q to q as well as q to \bar{q} if it passes through the boundary.
- Additional interactions between quarks and their finite volume images.
- A single quark can propagate bound to its image with energy increasing linearly with the size of the z direction:

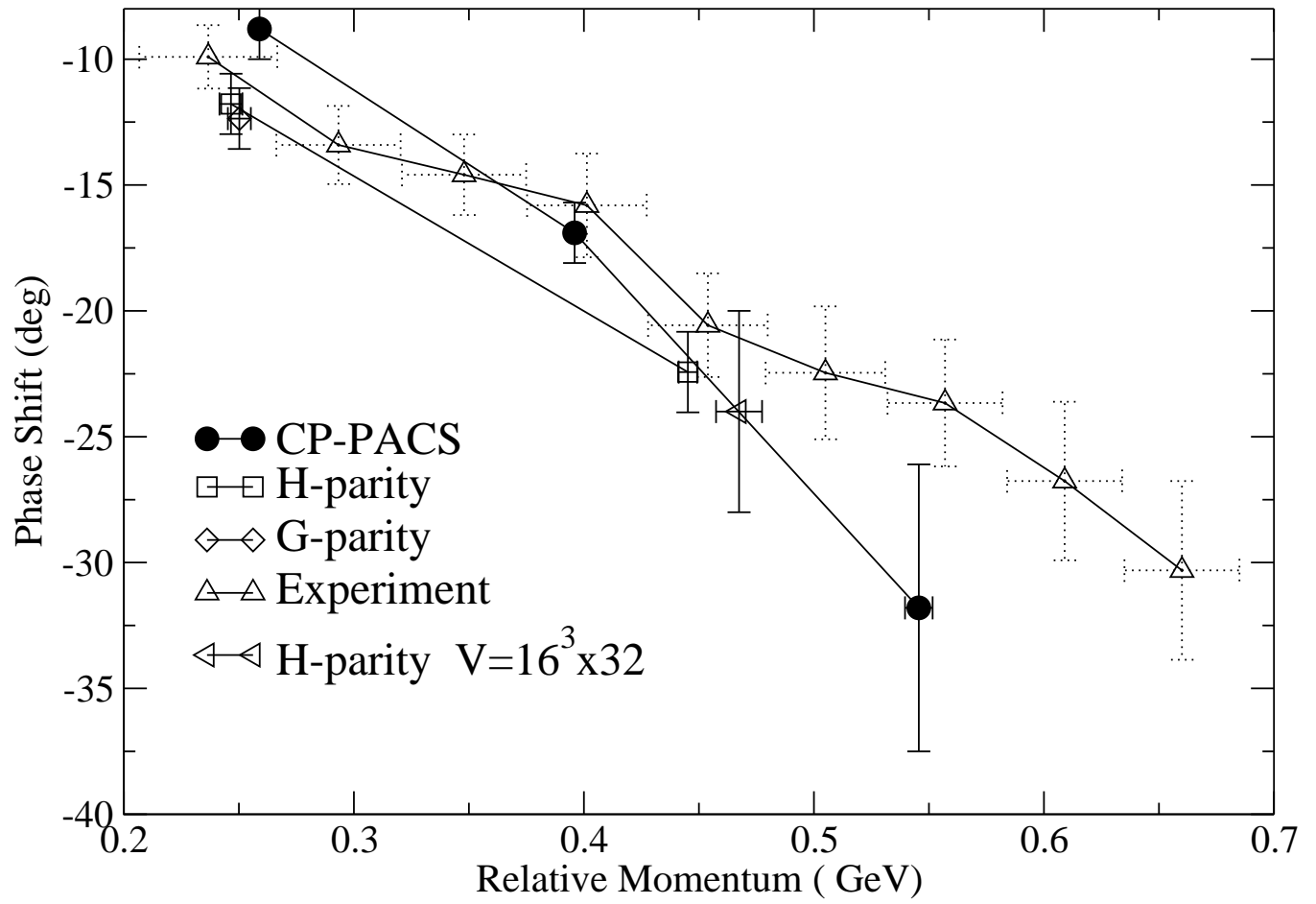


$I = 2$ $\pi\pi$ phase shift results (domain wall fermions)

	Vol	$1/a(\text{GeV})$	#conf's.
p = 250MeV			
G-parity	$8^2 \times 16 \times 32$	0.978(14)	91
H-parity	$8^2 \times 16 \times 32$	0.978(14)	172
p = 450MeV			
H-parity	$8^3 \times 32$	0.978(14)	270
H-parity	$16^3 \times 32$	1.98(3)	80

- Running parameters : $L_s = 10$, $M_5 = 1.65$

Results for $\delta_{\pi\pi}^{I=2}$



$K \rightarrow (\pi\pi)_{I=2}$ Decay

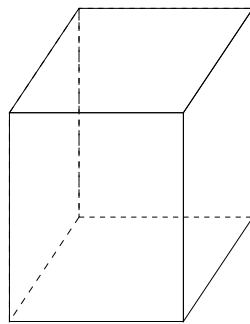
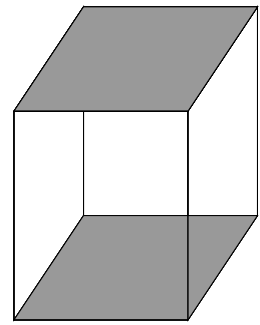
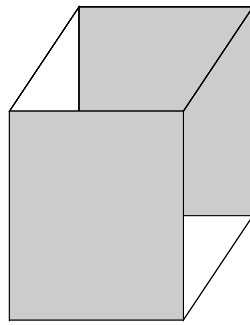
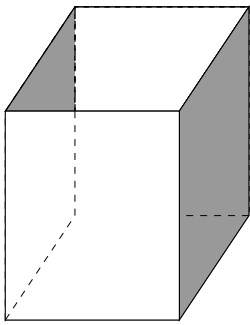
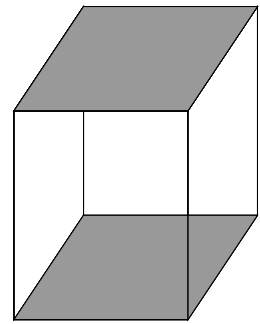
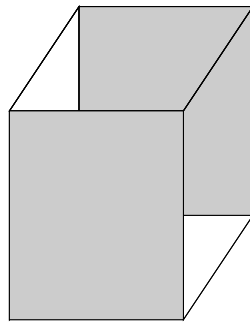
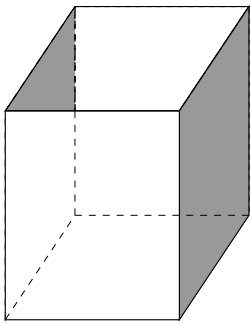
- Examine more recent results obtained over the past year.
- Choose more realistic parameters.
- Use the simpler H-parity boundary conditions.
- Examine matrix elements of the three $\Delta I = 3/2$ operators between $|K\rangle$ and physical $|\pi(p)\pi(p)\rangle$ states.
- Adjust m_K to achieve $m_K = E_{\pi\pi}$.
- Show the character of data and errors — physically normalized results should be available in a few weeks.

Simulation Parameters

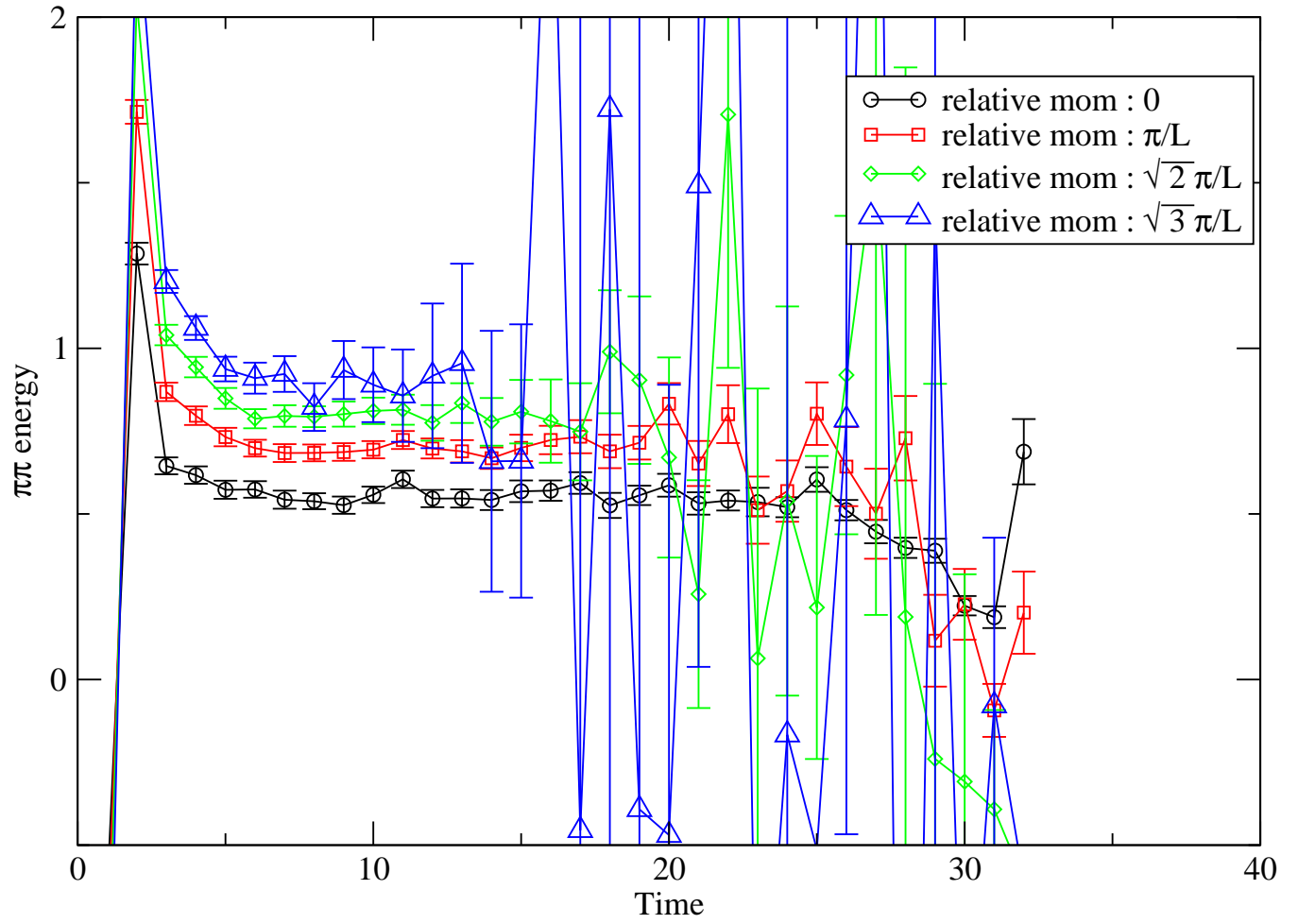
- Lattice size: $16^3 \times 32$
- Pion mass: 352 MeV
- Kaon mass: 712 MeV - 1.29 GeV
- Lattice spacing: $a^{-1}=1.3\text{GeV}$
- Action: DBW2
- Number of Configurations: 120
- Domain Wall Fermions: $M_5=1.8$, $L_s=12$
- Resulting kinematics:

	m_K	m_π	p_π
Simulation	910 MeV	352 MeV	290 MeV
Nature	496 MeV	138 MeV	206 MeV

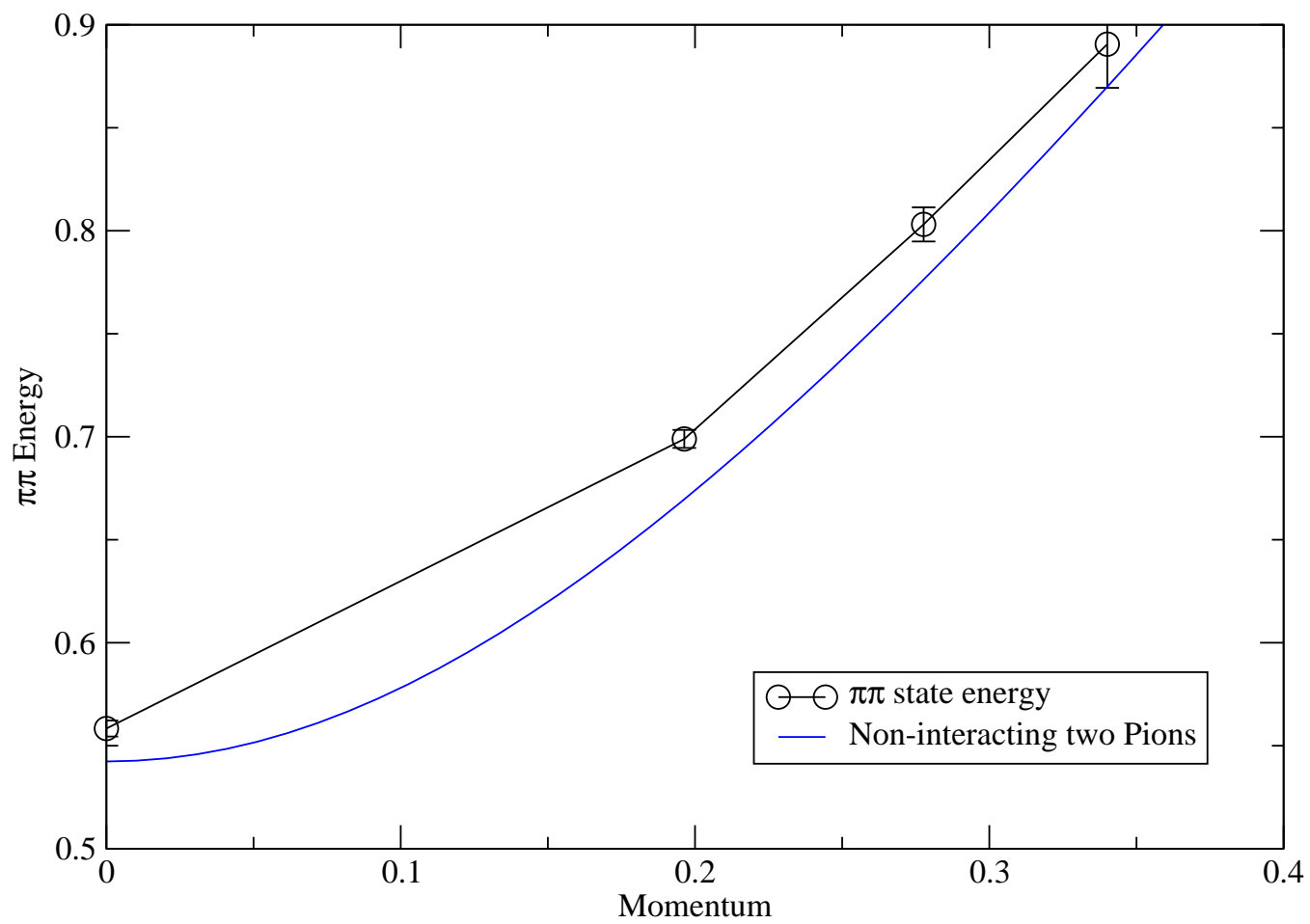
Obtaining three possible momenta



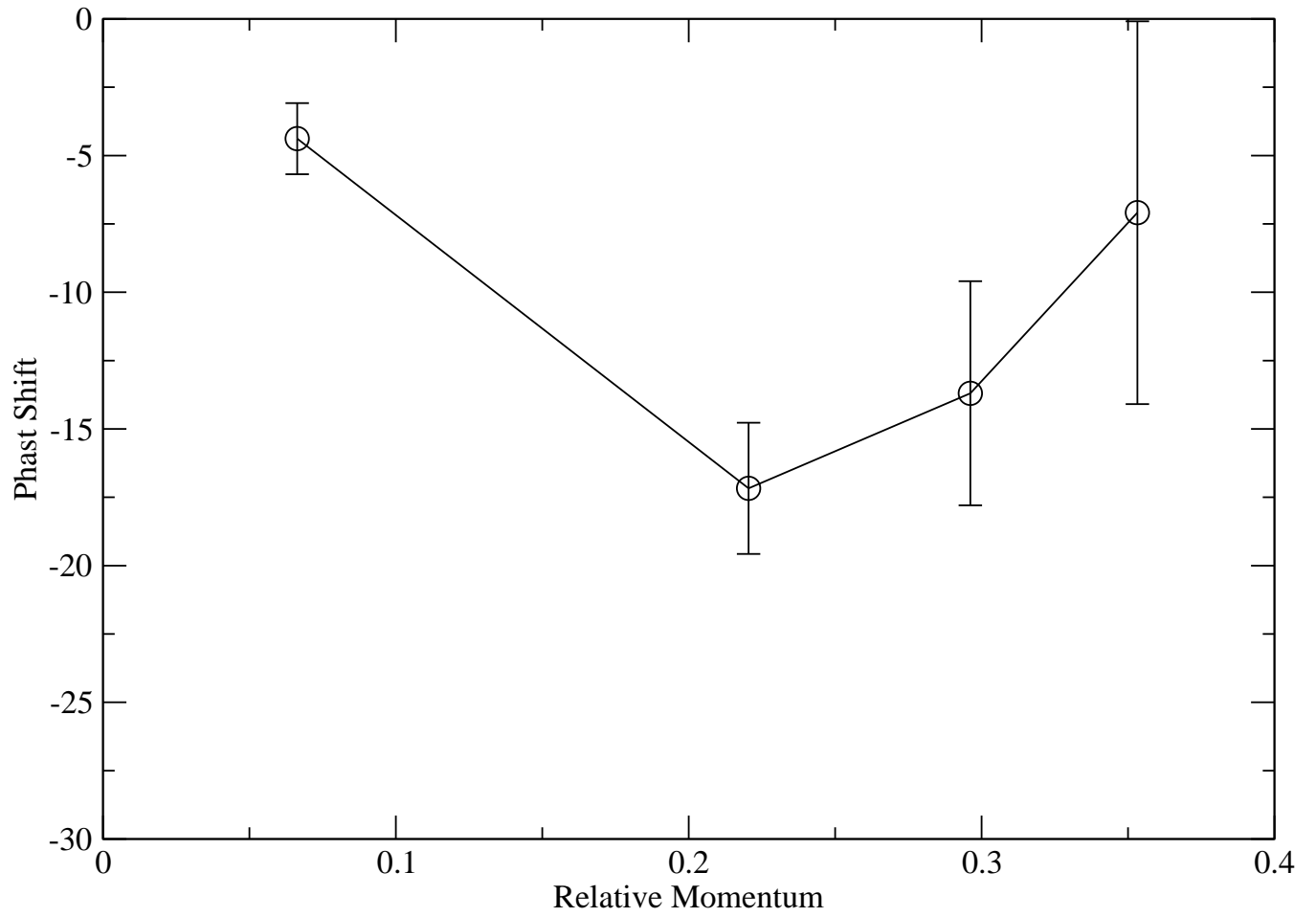
$\pi\pi$ effective mass



$\pi\pi$ energy



$\pi - \pi$ phase shifts



Operators with $\Delta I = 3/2$ and H-parity

- Kaon isospin : $I_z = 1/2$.
- $\pi\pi$ state with relative momentum under H-parity : $I_z = 2$.
- No terms in the effective Hamiltonian have $\Delta I_z = 3/2$.
- Use Wigner-Eckart theorem:

$$\langle K | O_{I_z=1/2}^{I=3/2} | \pi\pi \rangle = \frac{\langle 2, \frac{3}{2}; 1, \frac{1}{2} | 2, \frac{3}{2}; \frac{1}{2}, \frac{1}{2} \rangle}{\langle 2, \frac{3}{2}; 2, \frac{3}{2} | 2, \frac{3}{2}; \frac{1}{2}, \frac{1}{2} \rangle} \langle K | O_{I_z=3/2}^{I=3/2} | \pi\pi \rangle.$$

Normalized matrix elements of lattice operators

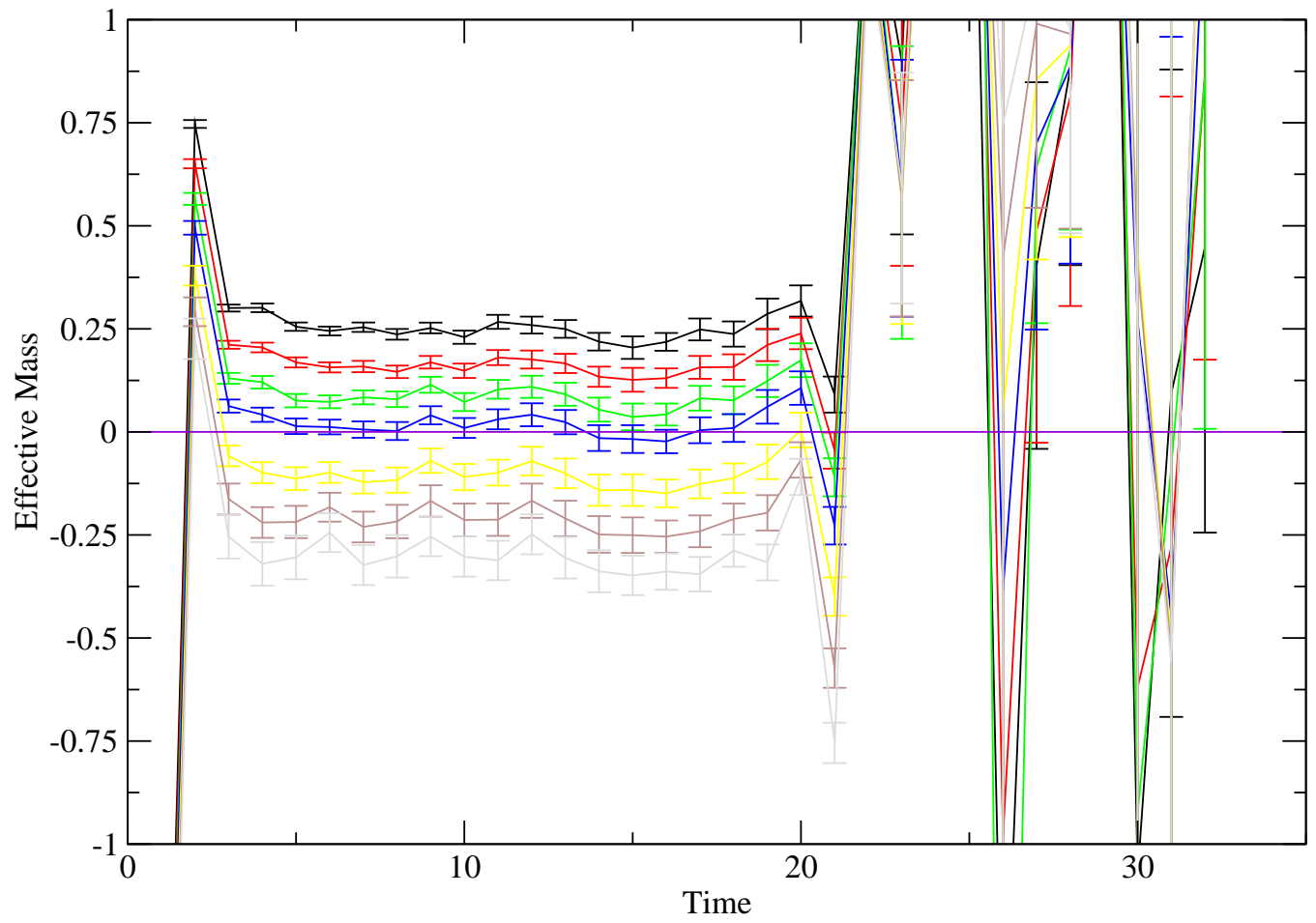
Evaluate three Greens functions in the
usual way:

$$\lim_{t_{\pi\pi} \gg t \gg t_K} G_{(\pi\pi) O K}(t) \rightarrow \langle 0 | P_{\pi\pi} | \pi^+ \pi^+ \rangle \langle \pi^+ \pi^+ | O | K^+ \rangle \langle K^+ | P_K | 0 \rangle e^{-m_K(t-t_K)} e^{-E_{\pi\pi}(t_{\pi\pi}-t)}$$

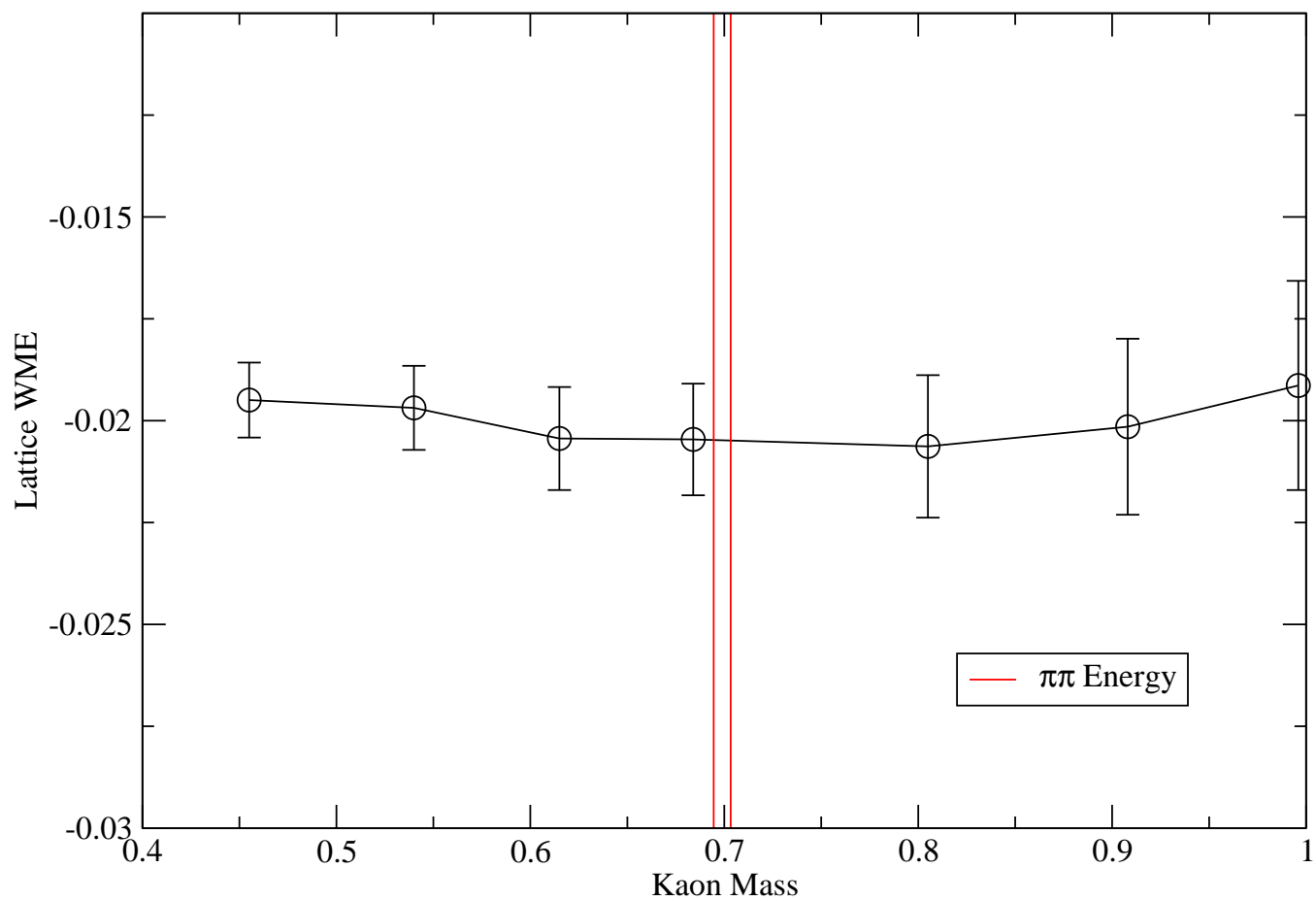
$$\lim_{t_{\pi\pi} \gg 0} G_{(\pi\pi) (\pi\pi)}(t_{\pi\pi}) \rightarrow \langle 0 | P_{\pi\pi} | \pi^+ \pi^+ \rangle \langle \pi^+ \pi^+ | P_{\pi\pi} | 0 \rangle e^{-E_{\pi\pi}(t_{\pi\pi})}$$

$$\lim_{t_K > 0} G_{K K}(t_K) \rightarrow \langle K^+ | P_K | 0 \rangle \langle 0 | P_K | K^+ \rangle e^{-m_K(t_K)}$$

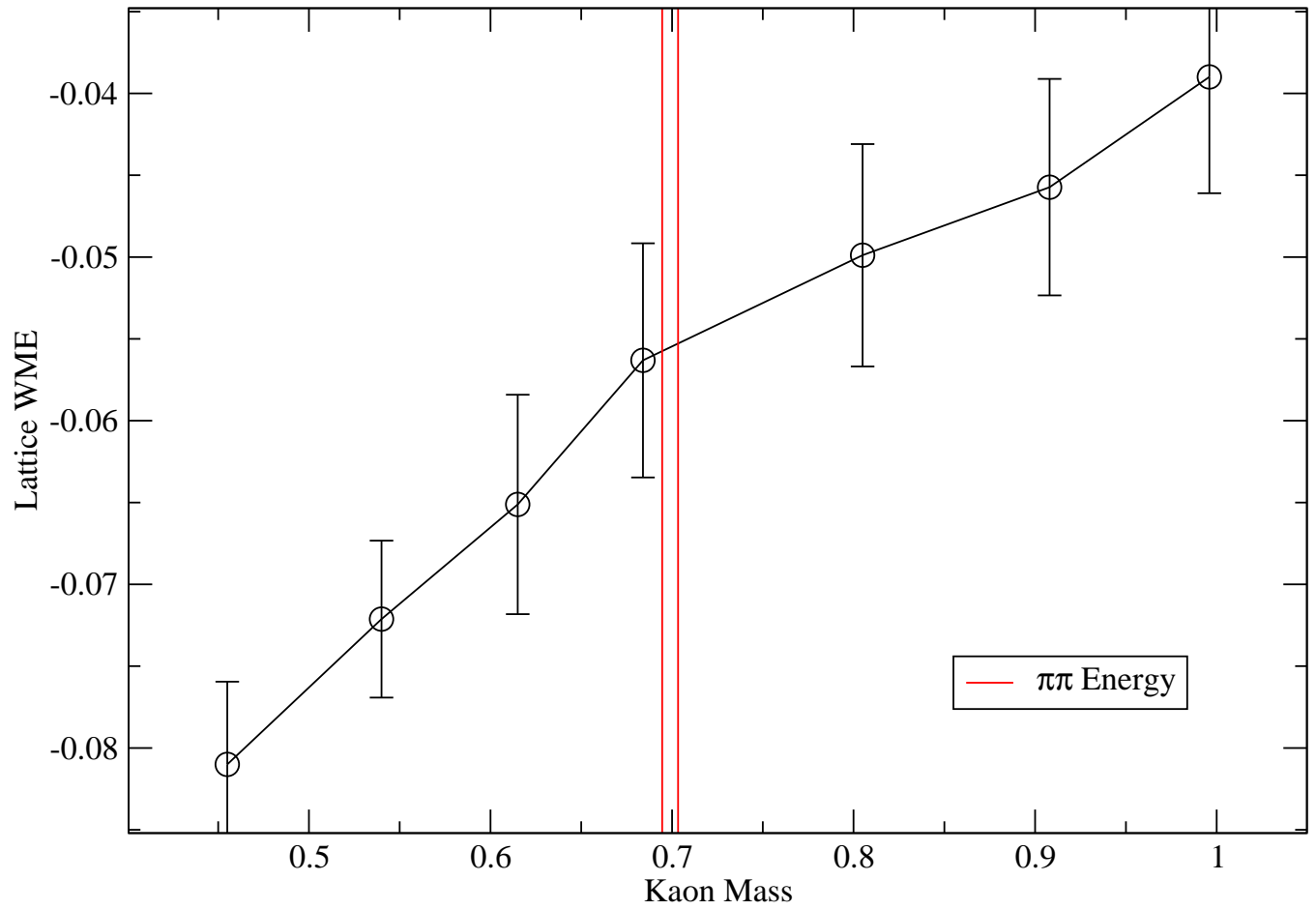
Effective mass difference from $G_{\pi\pi O27K}$



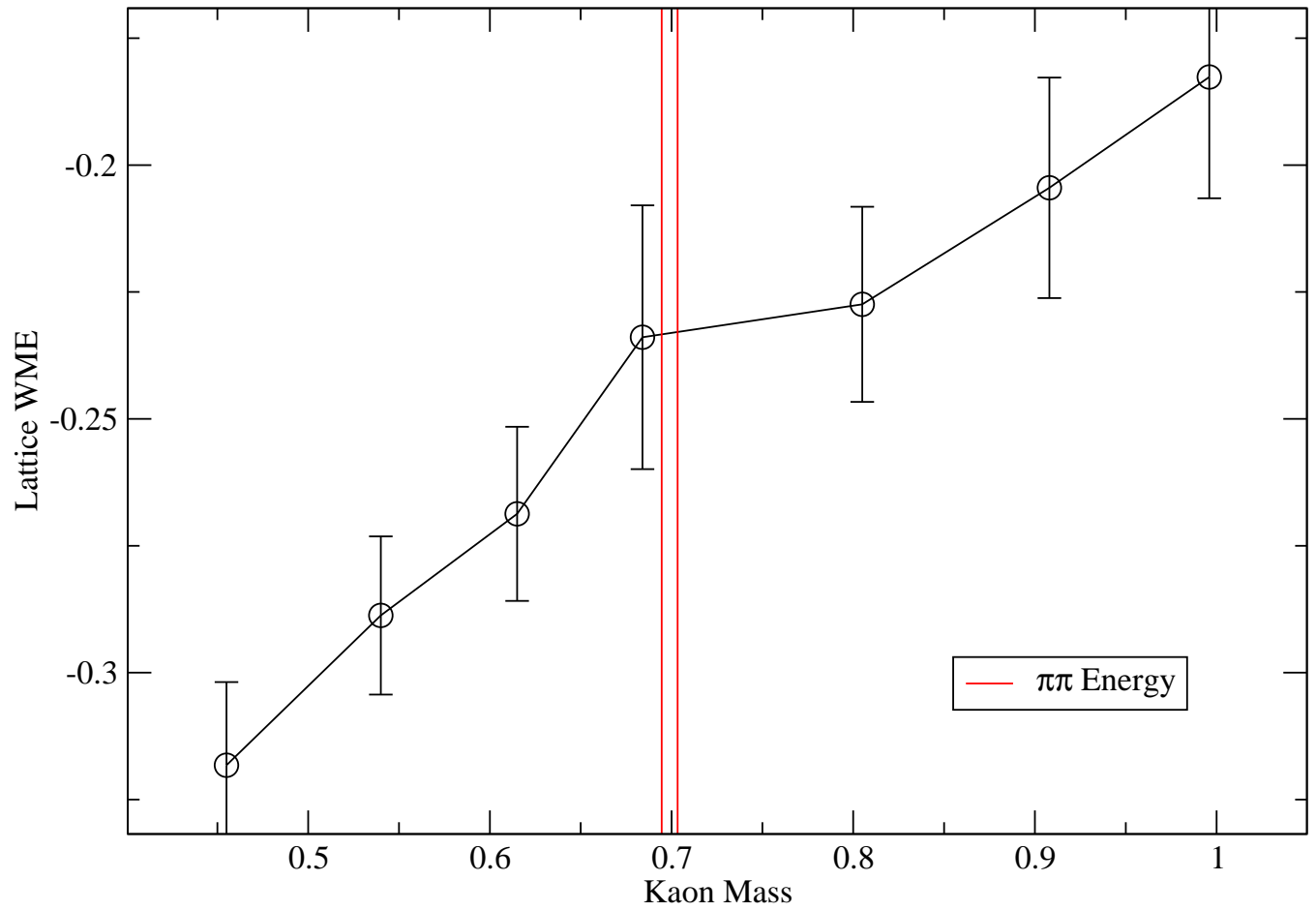
O^{27} matrix element versus Kaon mass



$O^{(8,8)}$ matrix element versus Kaon mass



$O_m^{(8,8)}$ matrix element versus Kaon mass



Preliminary lattice matrix elements

	O^{27}	$O^{(8,8)}$	$O_m^{(8,8)}$
0	-8.379e-3(4.34e-4)	-5.267e-2(3.43e-3)	-1.928e-1(1.19e-2)
1	-2.048e-2(1.42e-3)	-5.553e-2(7.11e-3)	-2.331e-1(2.52e-2)
2	-2.299e-2(3.35e-3)	-4.696e-2(2.09e-2)	-1.687e-1(1.00e-1)
3	-1.772e-2(5.16e-3)	-3.546e-2(3.01e-2)	-2.321e-1(8.28e-2)

Remaining steps

- Apply Lellouch-Lüscher finite-volume correction (**done**).
- Compute renormalization matrix for $1/a = 1.3\text{GeV}$ case (**done**).
- Evaluate needed Wilson coefficients (**underway**).
- Extract physically normalized matrix elements (**soon**).

Conclusion and Outlook

- Calculation of $\Delta I = 3/2$ amplitudes is practical with an **on-shell** π - π final state. For physical parameters:
 - $1/a = 2$ GeV, $a = 0.1$ fm.
 - $L = 64$, ≈ 6 fm, $\approx 4/m_\pi$.
 - Impose anti-periodic conditions on each face.
- Calculation of $\Delta I = 1/2$ amplitudes is possible using G-parity boundary conditions.
 1. Quenched calculations are not possible because zero-momentum, η' - η' states will dominate.
 2. Charge conjugation of the gauge fields on the boundary requires special configurations.
 3. Decay to the vacuum is allowed and must be subtracted.
- Using a K -meson with $\vec{p} \neq 0$ (Rummukainen-Gottlieb) would address 2. and 3. above.