# Finite volume effects for masses and decay constants

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Benasque, 10.8.04

# Outline

- Introduction: CHPT in finite volume
- Lüscher's formula for masses
- Asymptotic formula for decay constants
- Numerics
- Summary

Work done in collaboration with S. Dürr, A. Fuhrer and C. Haefeli

# Introduction

CHPT: expansion in $m_{q_l}/\Lambda$ and $p/\Lambda$

In finite volume the momentum is quantized: $p = \frac{2\pi}{L} n$

Condition of applicability of CHPT:

$$m_{q_l} \ll \Lambda \quad \text{and} \quad \frac{2\pi}{L} \ll \Lambda$$

$$\Lambda \sim 4\pi F_\pi \quad \Rightarrow \quad 2LF_\pi \gg 1$$

Once this condition is respected we still have two different physical situations

$$LM_\pi \lesssim 1 \;\; \Rightarrow \;\; \epsilon\text{–regime} \quad M_\pi \sim \frac{1}{L^2} \sim O(\epsilon^2)$$

$$LM_\pi \gg 1 \;\; \Rightarrow \;\; p\text{–regime} \quad M_\pi \sim \frac{1}{L} \sim O(p)$$

# $p$– or $\epsilon$–regime?

Two alternatives:

- Chiral limit on the lattice $\Rightarrow$ $\epsilon$–regime

  (unless one can simulate enormous volumes)

  $\Rightarrow$ Rely on CHPT to relate unphysical observables to physical quantities (cf. M. Laine's talk)

- $M_\pi > M_\pi^{\rm phys}$: choose $L \gg 1/M_\pi$, $\Rightarrow$ $p$–regime

  (e.g. $M_\pi = 300$ MeV, $L = 2$ fm, $M_\pi L \sim 3$)

  $\Rightarrow$ Rely on CHPT to make the chiral and the large volume extrapolation

# $p$-regime

Calculational rule in CHPT for isotropic finite box with periodic boundary conditions:

- the Lagrangian is the same as in infinite volume
- the propagators must be made periodic:

$$G_L(\vec{x}, t) = \sum_{\vec{n}} G_\infty(\vec{x} + \vec{n}L, t)$$

# $p$-regime

Calculational rule in CHPT for isotropic finite box with periodic boundary conditions:
Examples:

Gasser and Leutwyler (88)

$$M_\pi(L) = M_\pi \left[1 + \frac{1}{2N_f}\xi \, g_1(\lambda) + O(\xi^2)\right]$$
$$F_\pi(L) = F_\pi \left[1 - \frac{N_f}{2}\xi \, g_1(\lambda) + O(\xi^2)\right]$$

with

$$\lambda = M_\pi L, \quad \xi = (M_\pi / 4\pi F_\pi)^2$$

$$g_1(\lambda) = \sum_{\vec{n}}' \int_0^\infty dz \, e^{-\frac{1}{z} - \frac{z}{4}\vec{n}^2\lambda^2} = \sum_{\vec{n} \neq \vec{0}} G_\infty(\vec{x} + \vec{n}L, t)|_{t=\vec{x}=0}$$

# Finite volume effects in the $p$-regime

Foundations: Gasser and Leutwyler (87)

quenched CHPT: Sharpe, Bernard and Golterman (90's)

Recent applications:

- two–pion states — Lin, Martinelli, Pallante, Sachrajda and Villadoro (03)
- $F_K$ and $B_K$ — Becirevic and Villadoro (03)
- $m_p$ — QCDSF (03)
- $m_N$, $\mu_N$ and $g_A$ — Beane and Savage (03-04)
- $f_B$ and $B_B$ — Arndt and Lin (04)
- $m_p$ — Koma and Koma (04)

# Finite volume effects in the $p$-regime

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quenched CHPT: Sharpe, Bernard and Golterman (90's)

Talks at Lattice 2004:

- $M_\pi$, $F_\pi$ and $\langle r^2 \rangle_V$ — R. Lewis
- Lüscher Formula for $m_p$ — Y. Koma
- $m_p$ and $g_A$ — M. Goeckeler
- $f_B$ and $B_B$ — D. Lin
- Weak matrix elements in the $\epsilon$-regime — H. Wittig
- Plenary — GC

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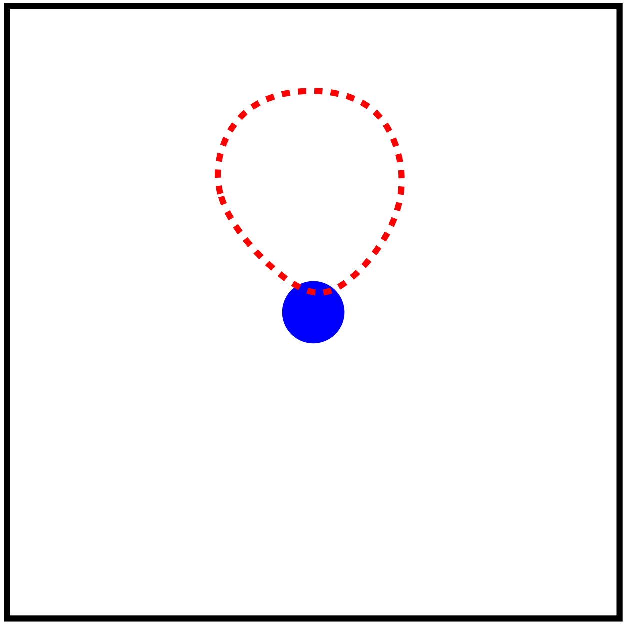
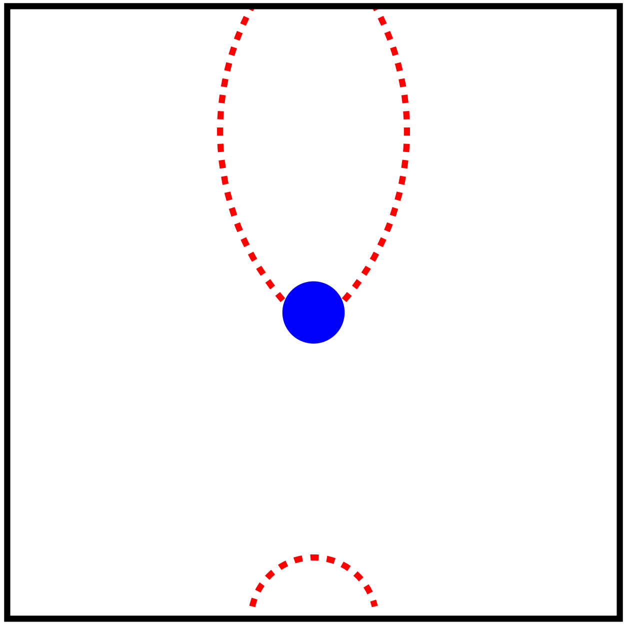
# Masses in finite volume



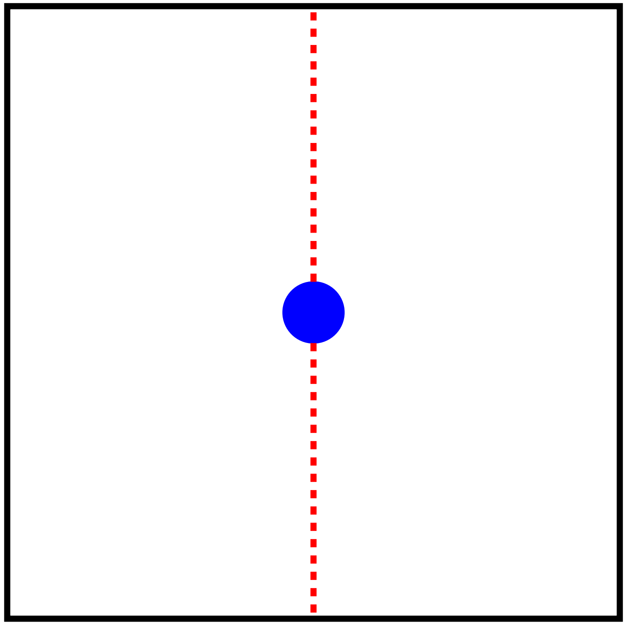
Loop-diagram

# Masses in finite volume



Loop diagram with
periodic
boundary conditions

# Masses in finite volume



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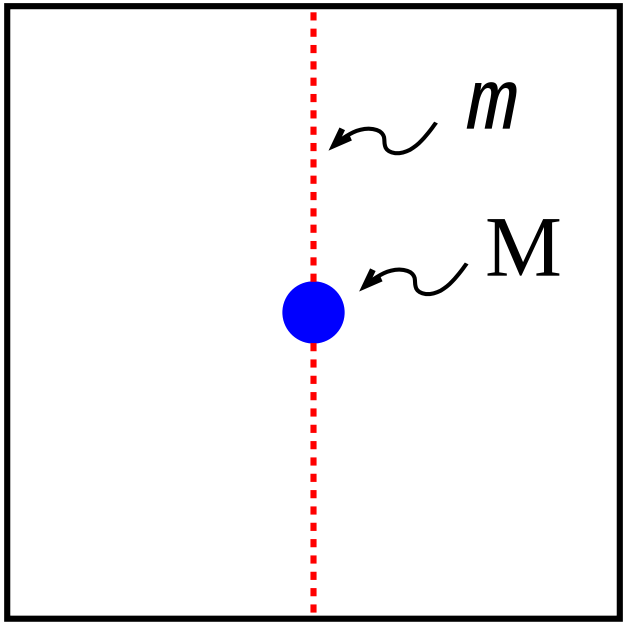
# Masses in finite volume

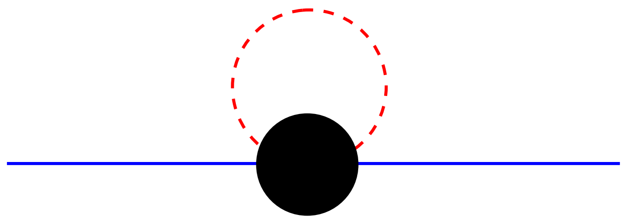



Loop diagram with periodic boundary conditions

This diagram exists only for $L \neq \infty$

Its effect is of the order $\exp\left[-mL\right]$

# Lüscher's Formula

 $= \int d\ell \, \Gamma(p,\ell,-\ell,-p) G_L(\ell)$

$$G_L(\ell) = \sum_{\vec{n}} G_\infty(\ell) e^{i\vec{\ell}\cdot\vec{n}L} \qquad G_\infty(\ell) \sim \frac{1}{\ell^2 + m^2}$$

$$\begin{aligned} M_L - M_\infty &= \int d\ell \, \Gamma(p,\ell,-\ell,p) \left[ G_L(\ell) - G_\infty(\ell) \right] \\ &= \sum_{\vec{n} \neq \vec{0}} \int d\ell \, \Gamma(p,\ell,-\ell,p) G_\infty(\ell) e^{i\vec{\ell}\cdot\vec{n}L} \end{aligned}$$

# Lüscher's Formula

Leading correction for $mL \gg 1$: (Lüscher 86)

$$M_L - M_\infty = C \int_{-\infty}^{\infty} dy\, e^{-\sqrt{m^2+y^2}L} F(iy) + \ldots$$

where $F(\nu)$ is the scattering amplitude between the red ($m$) and blue ($M$) particle, and $C$ a constant that depends from $L,\ m$ and $M$

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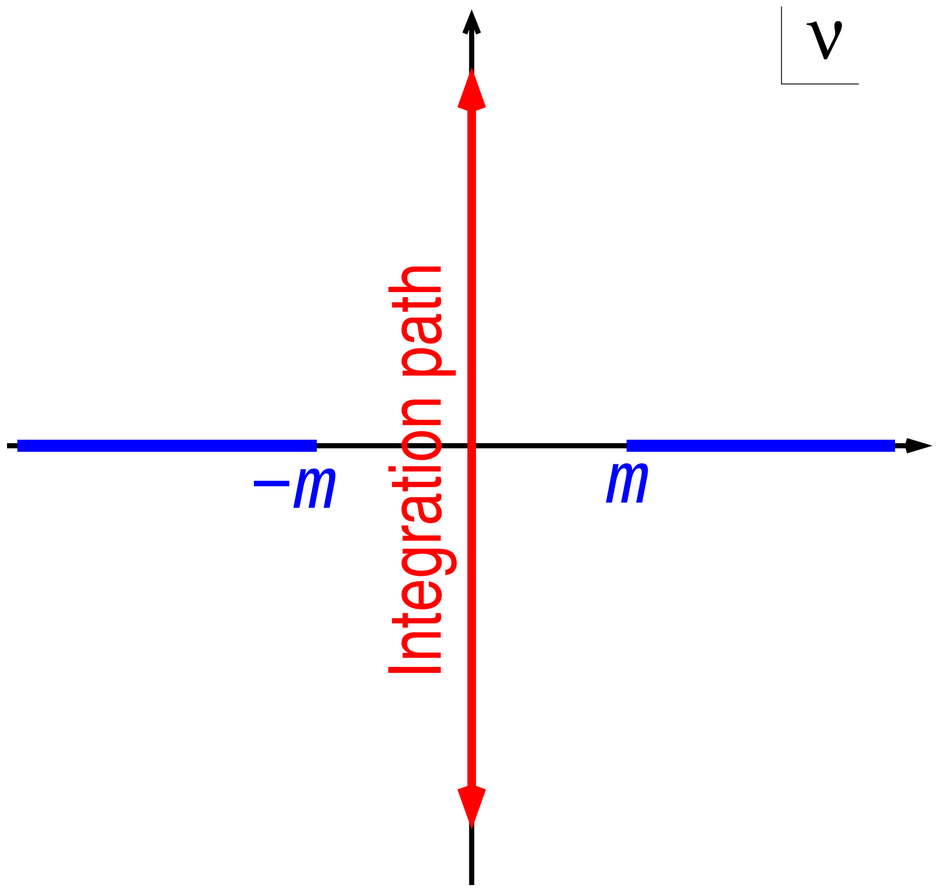

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The formula expresses the corrections as an integral over a (analytically continued) physical amplitude

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What matters for the behaviour of the corrections is not the mass of the particle itself, but rather the mass of the lightest particle to which it is coupled

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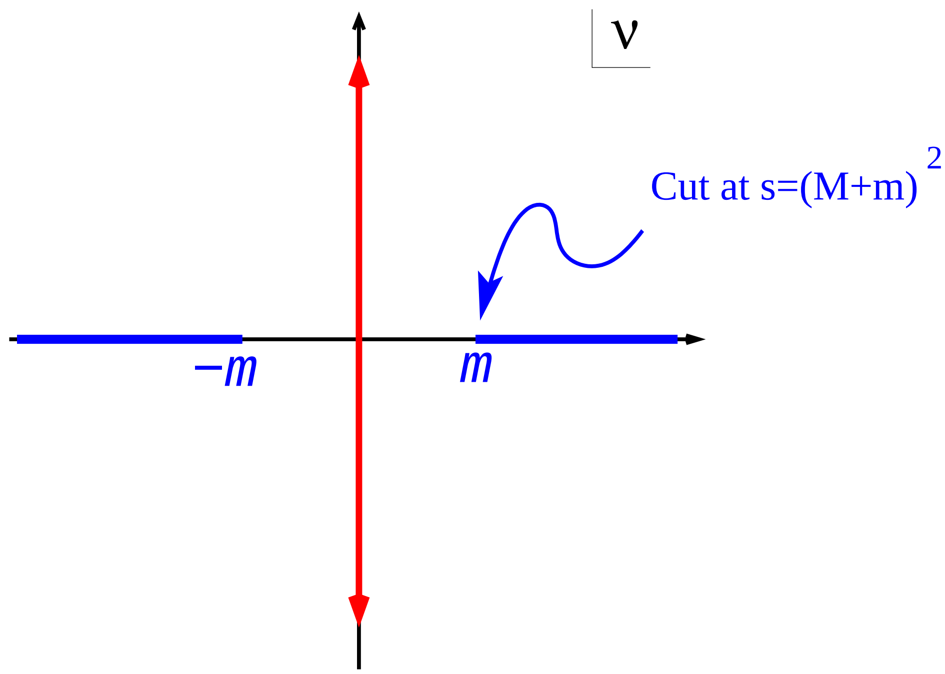
What matters for the behaviour of the corrections is not the mass of the particle itself, but rather the mass of the lightest particle to which it is coupled

e.g. both the corrections for the pion as well as those for the proton mass depend exponentially on $M_\pi L$

# Cuts and poles in the scattering amplitude

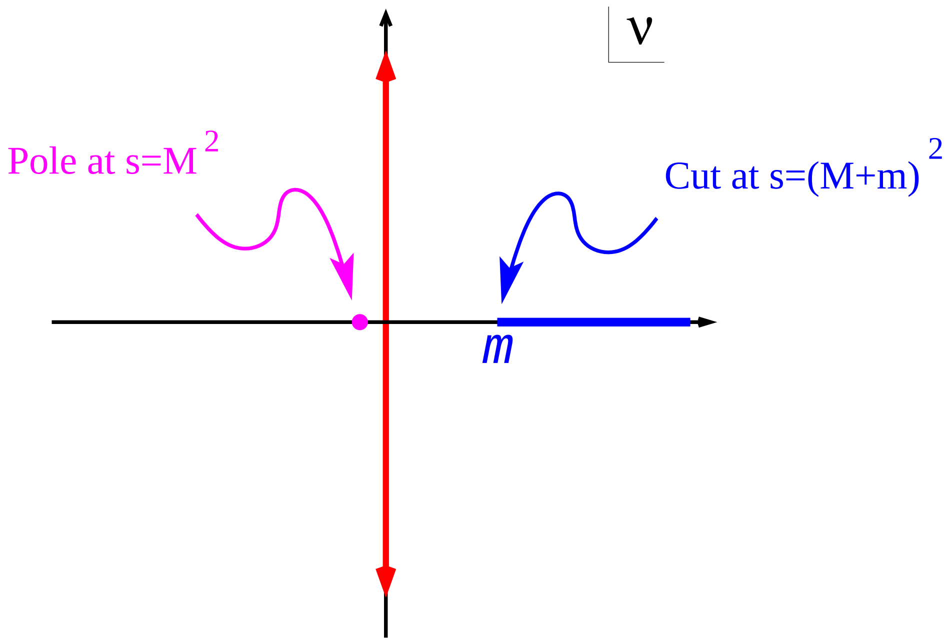
Any scattering amplitude must have a cut at
$s, u = (M+m)^2 \Rightarrow \nu = \frac{s-u}{4M} = \pm m$



# Cuts and poles in the scattering amplitude

In addition it may have poles, e.g. at

$s, u = M^2 \Rightarrow \nu = \frac{s-u}{4M} = \mp\frac{m^2}{2M}$

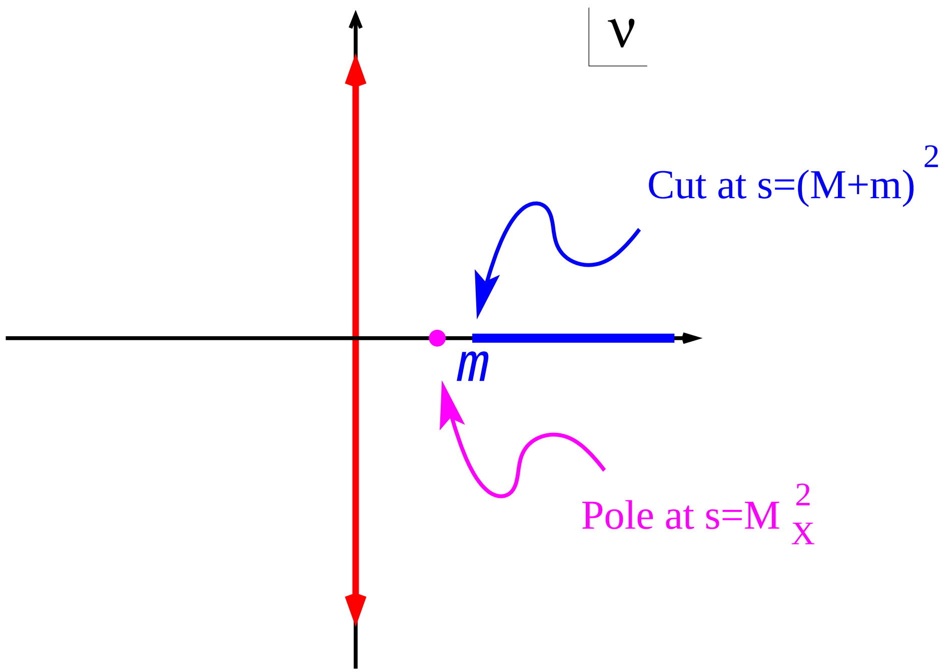



Poles on the lhs of the imaginary axis generate an extra term in the Lüscher's formula (cf. the formula for the nucleon mass)

# Cuts and poles in the scattering amplitude

In addition it may have poles, or at

$$s, u = M_X^2 \Rightarrow \nu = \frac{s-u}{4M} = \mp \frac{m^2}{2M} + \Delta M \left(1 + \frac{\Delta M}{2M}\right) \qquad \Delta M = M_X - M$$

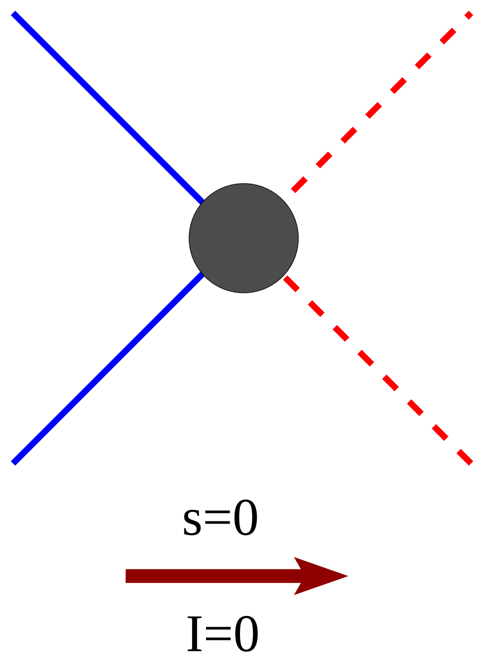



Poles on the rhs of the imaginary axis do not generate an extra term in the Lüscher's formula (cf. D. Lin's talk on heavy mesons)

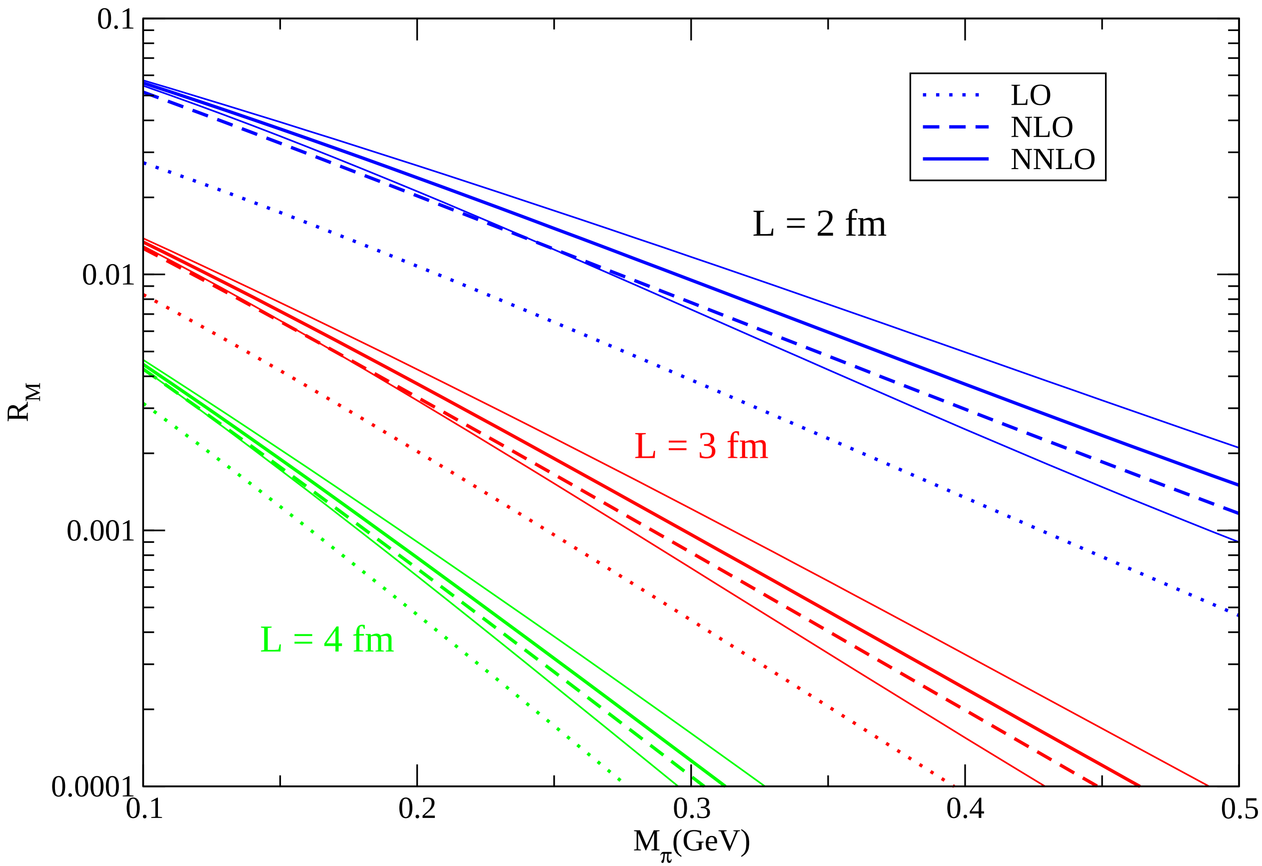
# Corrections for $M_\pi$

$\pi\pi$ scattering amplitude with forward kinematics

$$F_{\pi\pi}(\nu) = T^{I=0}\left[0, 2M_\pi(M_\pi+\nu), 2M_\pi(M_\pi-\nu)\right] = -\frac{M_\pi^2}{F_\pi^2} + O(p^4)$$



# Corrections for $M_\pi$




$$R_M = M_{\pi\,L}/M_\pi - 1$$

GC and S. Dürr 03

# Lüscher's Formula or CHPT?

$$\Delta M^L_{\pi\,\text{Lüscher}} = \frac{-3}{16\pi^2\lambda}\int_{-\infty}^{\infty} dy\; F(\mathrm{i}y)\, e^{-\sqrt{M_\pi^2+y^2}\,L} + O(e^{-\overline{M}L})$$

$$\Delta M^L_{\pi\,\text{CHPT}} = \frac{1}{4}\xi\, g_1(\lambda) + O(\xi^2)$$

$$g_1(\lambda) = \sum_{\vec{n}\neq\vec{0}} G_\infty(\vec{x}+\vec{n}L,t)_{|_{t=\vec{x}=0}} = \sum_{|\vec{n}|=1}^{\infty} \frac{4m(|\vec{n}|)}{|\vec{n}|\,\lambda}\; K_1\left(|\vec{n}|\,\lambda\right)$$

where $m(|\vec{n}|)$ is the multiplicity of a vector of length $|\vec{n}|$ in 3-dimensional discretized space

# Lüscher's Formula or CHPT?

$$\Delta M_{\pi\,\text{Lüscher}}^{L} = \frac{-3}{16\pi^2\lambda}\int_{-\infty}^{\infty} dy\, F(\mathrm{i}y)\, e^{-\sqrt{M_\pi^2+y^2}\,L} + O(e^{-\overline{M}L})$$

$$\Delta M_{\pi\,\mathrm{CHPT}}^{L} = \frac{1}{4}\xi\, g_1(\lambda) + O(\xi^2)$$

The two formulas give the leading term in two different expansions

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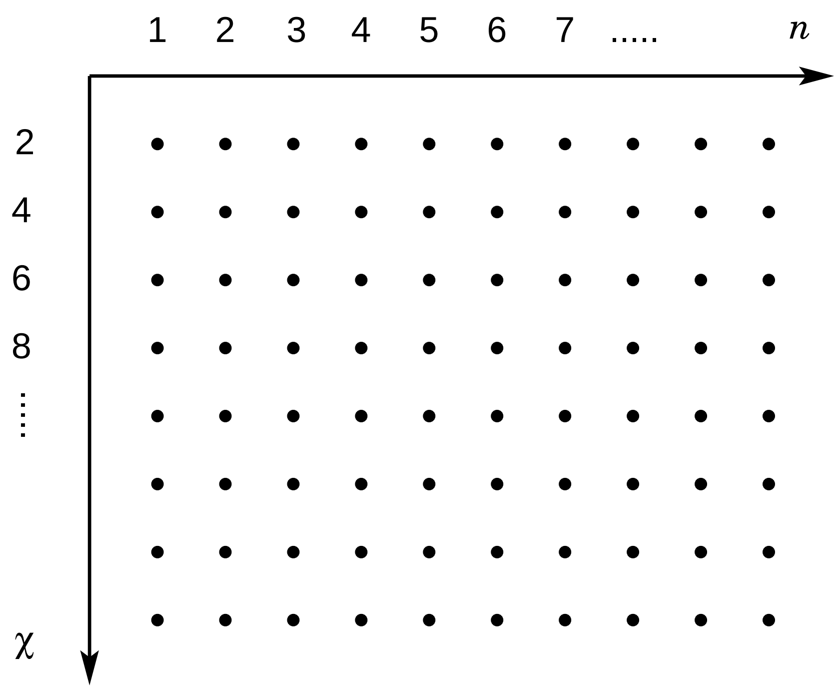

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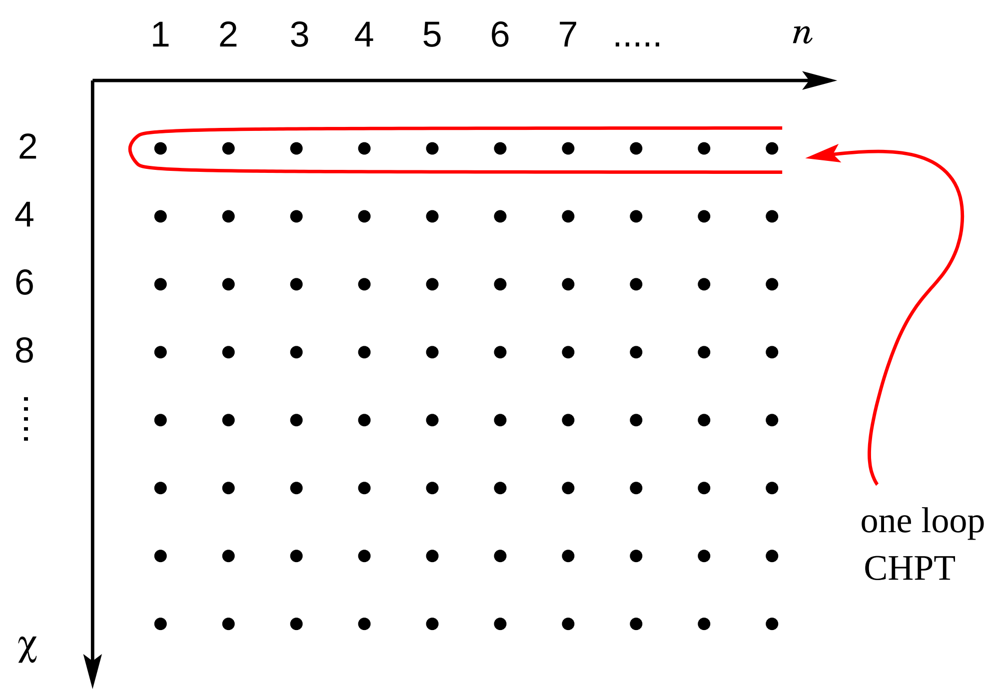

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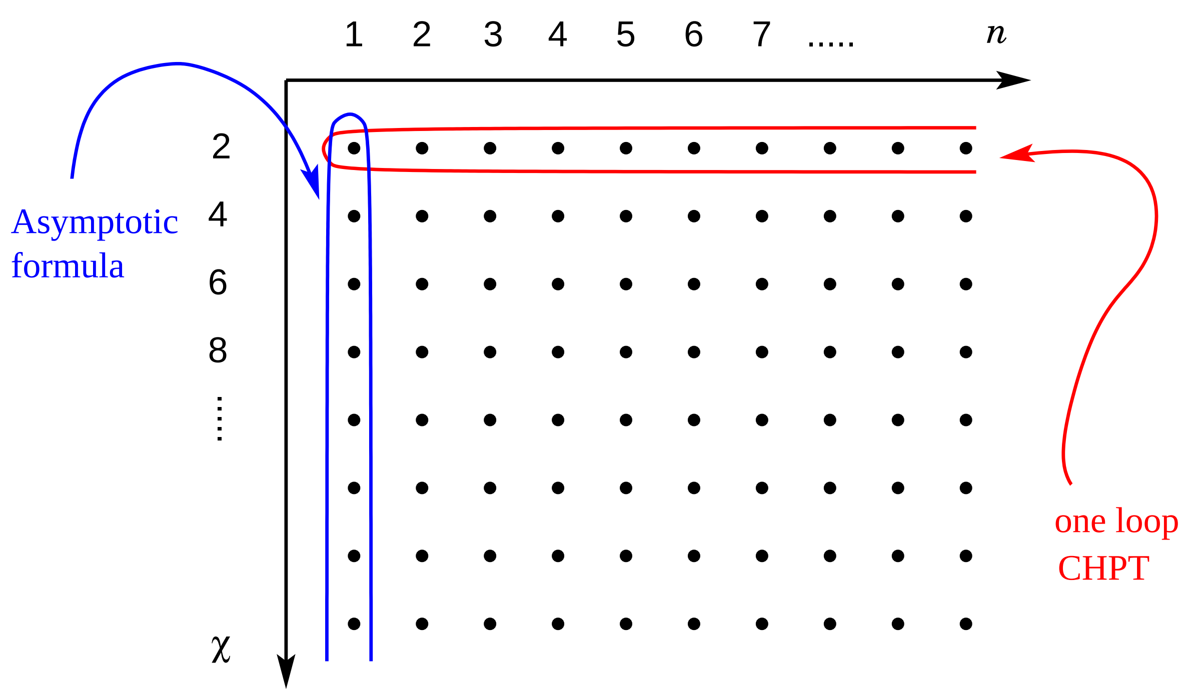

# Extension of the Lüscher's Formula

One-loop CHPT corrections

$$g_1(\lambda) = \sum_{\vec{n}\neq\vec{0}} G_\infty(\vec{x}+\vec{n}L,t)_{|_{t=\vec{x}=0}} = \sum_{|\vec{n}|=1}^{\infty} \frac{4m(|\vec{n}|)}{|\vec{n}|\,\lambda}\; K_1\left(|\vec{n}|\,\lambda\right)$$

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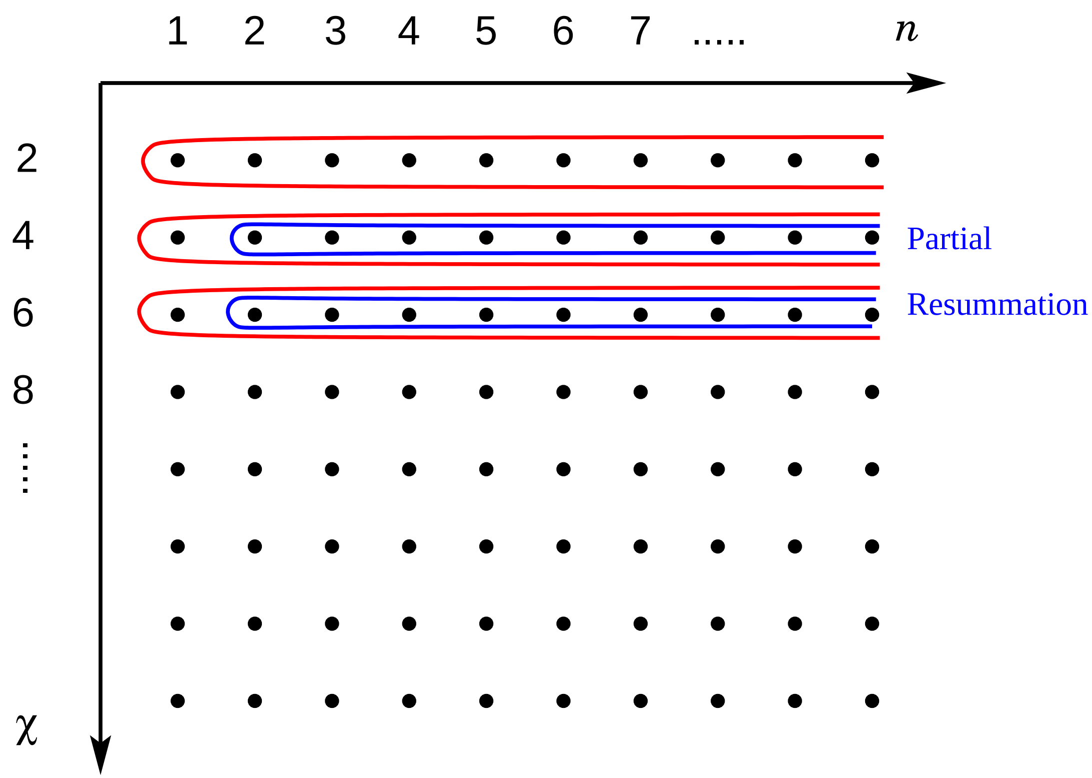
Analogously one can extend the Lüscher's Formula so that it contains contributions from all $|\vec{n}|$ of a single propagator:

$$M_{\pi,L} - M_\pi = -\frac{1}{32\pi^2\lambda} \sum_{|\vec{n}|=1}^{\infty} \frac{m(|\vec{n}|)}{|\vec{n}|} \int_{-\infty}^{\infty} dy F(iy) e^{-\sqrt{\vec{n}^2(M_\pi^2+y^2)}L}$$

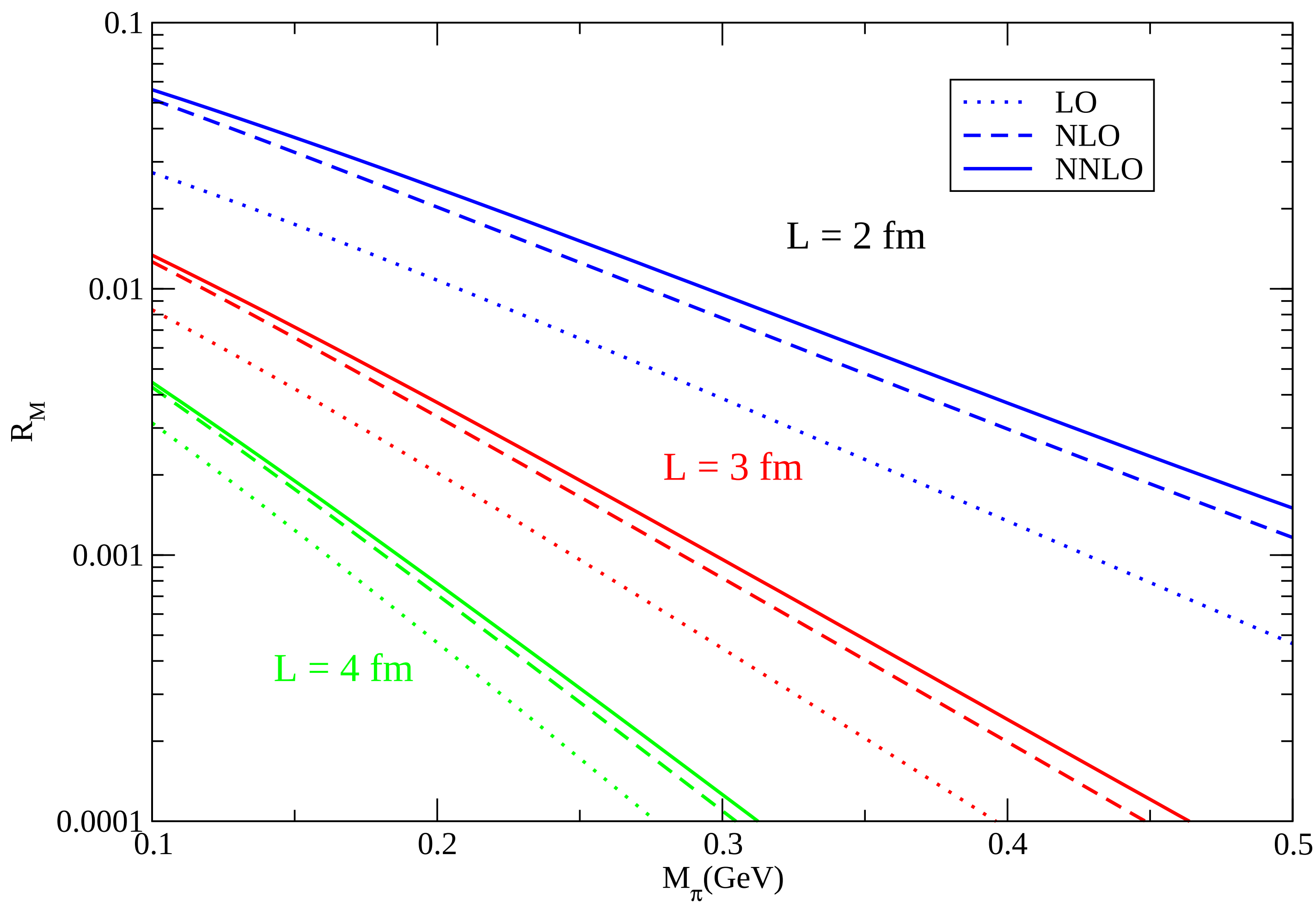
The extension does not provide all exponentially subleading terms!

# Extension of the Lüscher's Formula

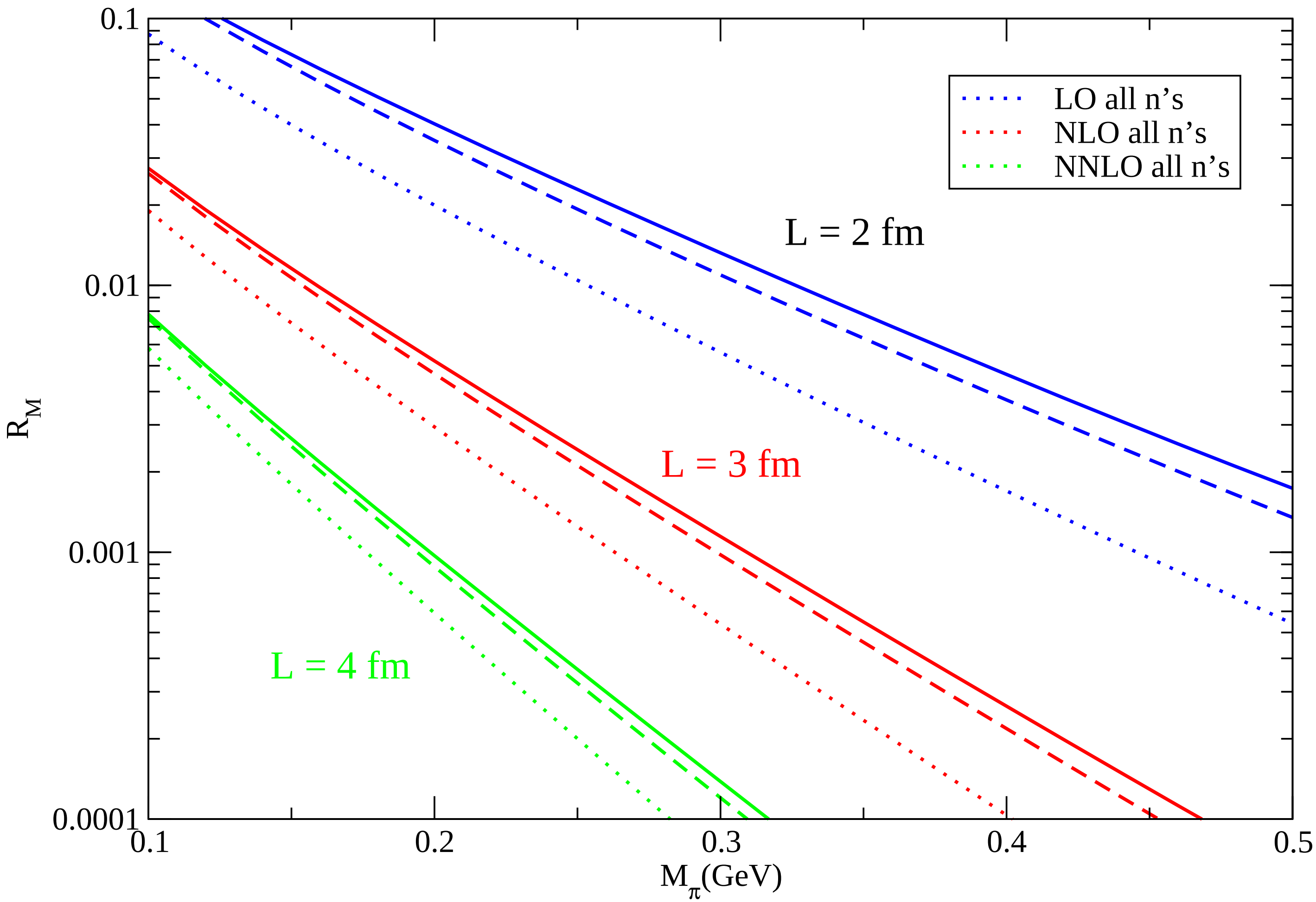
$$M_{\pi,L} - M_\pi = -\frac{1}{32\pi^2\lambda} \sum_{|\vec{n}|=1}^{\infty} \frac{m(|\vec{n}|)}{|\vec{n}|} \int_{-\infty}^{\infty} dy F(iy) e^{-\sqrt{\vec{n}^2(M_\pi^2+y^2)}L}$$



# Nonleading exp. terms in $M_{\pi,L}$



# Nonleading exp. terms in $M_{\pi,L}$



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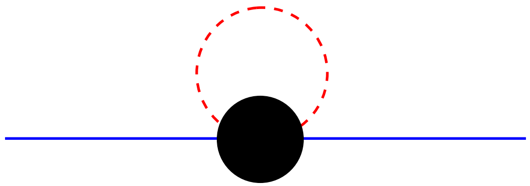
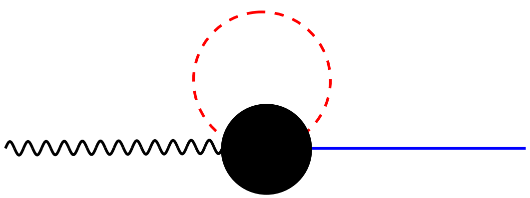
# Extension to decay constants



$$\Rightarrow \quad \Delta M \propto \int_{-\infty}^{\infty} dy\; e^{-\sqrt{M_\pi^2+y^2}L} F(iy) + \ldots$$

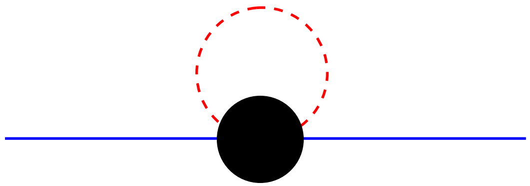
$$F(\nu) \Leftrightarrow \langle \pi\pi | T | \pi\pi \rangle$$



$$\Rightarrow \quad \Delta F \propto \int_{-\infty}^{\infty} dy\; e^{-\sqrt{M_\pi^2+y^2}L} N(iy) + \ldots$$

$$N(\nu) \Leftrightarrow \langle 0 | A_\mu | \pi\pi\pi \rangle \sim A(\tau \rightarrow 3\pi\nu_\tau)$$

# Extension to decay constants



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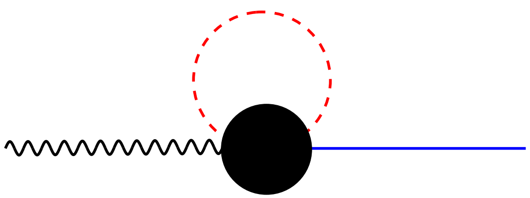

$$\Rightarrow \quad \Delta F \propto \int_{-\infty}^{\infty} dy \; e^{-\sqrt{M_\pi^2+y^2}L} N(iy) + \ldots$$

$$N(\nu) \Leftrightarrow \langle 0 | A_\mu | \pi\pi\pi \rangle \sim A(\tau \to 3\pi\nu_\tau)$$

The $\langle 0 | A_\mu | \pi\pi\pi \rangle$ amplitude must be subtracted:

$$N(\nu) = \langle (2\pi)_{I=0}\pi | A_\mu(0) | 0 \rangle - iQ_\mu \frac{F_\pi F(\nu)}{M_\pi^2 - Q^2}$$

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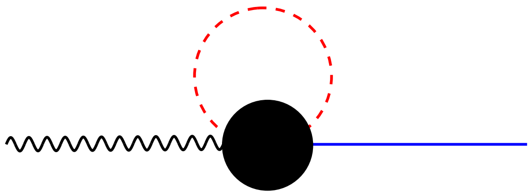
# Extension to decay constants



$$\Rightarrow \quad \Delta F \propto \int_{-\infty}^{\infty} dy\, e^{-\sqrt{M_\pi^2+y^2}L} N(iy) + \ldots$$

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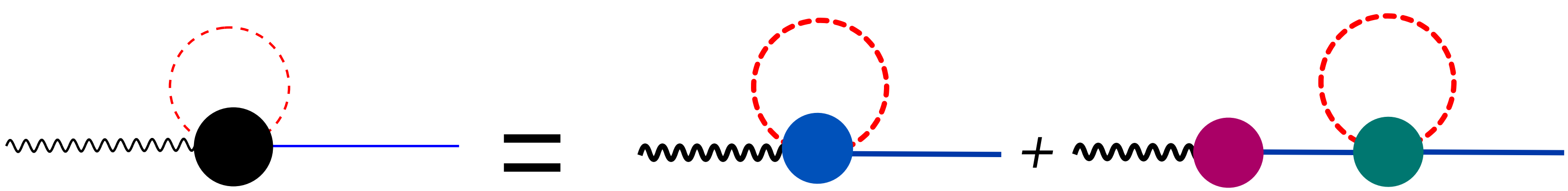

# Ward identity in finite volume

$$\langle 0|A_\mu^i(0)|\pi^k(p)\rangle = i\delta^{ik}F_\pi p_\mu \qquad \langle 0|P^i(0)|\pi^k(p)\rangle = i\delta^{ik}G_\pi$$

Ward identity

$$F_\pi M_\pi^2 = \hat{m}G_\pi$$

The identity is valid also in finite Volume

$$R_G = R_F + 2R_M \qquad (R_X := \Delta X_\pi^L / X_\pi)$$

# Ward identity in finite volume

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$$R_G = R_F + 2R_M \qquad (R_X := \Delta X_\pi^L / X_\pi)$$

and must be satisfied by the asymptotic formulae :

$$C\int_{-\infty}^{\infty} dy e^{-\sqrt{M_\pi^2+y^2}L}\left[N_G(iy) - N_F(iy) + \frac{F_\pi}{M_\pi}F(iy)\right] = 0$$

# Ward identity in finite volume

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The identity is valid also in finite Volume

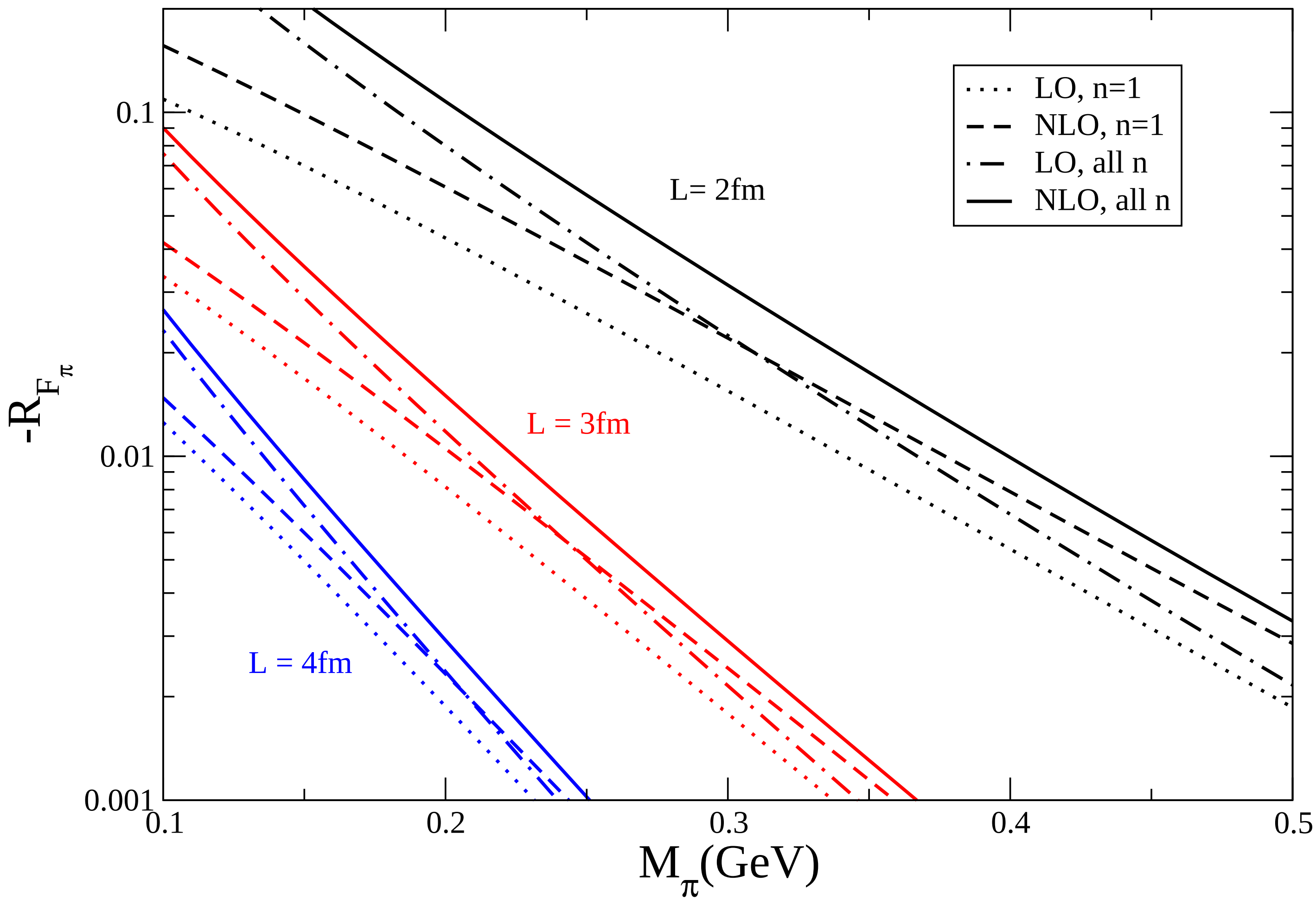
$$R_G = R_F + 2R_M \qquad (R_X := \Delta X_\pi^L / X_\pi)$$

and must be satisfied by the asymptotic formulae – in particular for the integrands:

$$N_G(\nu) - N_F(\nu) + \frac{F_\pi}{M_\pi} F(\nu) = 0$$

This is a Ward identity for 4-point functions!

# Corrections for $F_\pi$

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# Other applications

| Quantity | Amplitude | Theory status |
|---|---|---|
| $M_K$ | $A(\pi K \to \pi K)$ | $O(p^6)$ (Bijnens et al.) |
| $F_K$ | $A(K_{l4})$ | $O(p^6)$ (Bijnens et al.) |
| $M_\eta$ | $A(\pi\eta \to \pi\eta)$ | $O(p^4)$ (Bernard et al.) |
| $F_\eta$ | $A(\eta_{l4})$ | ? |
| $M_N$ | $A(\pi N \to \pi N)$ | $O(p^4)$ various Authors |
| $M_B$ | $A(\pi B \to \pi B)$ | ? |
| $F_B$ | $A(B_{l4})$ | ? |

Work in progress: GC, S. Dürr, C. Haefeli, A. Fuhrer

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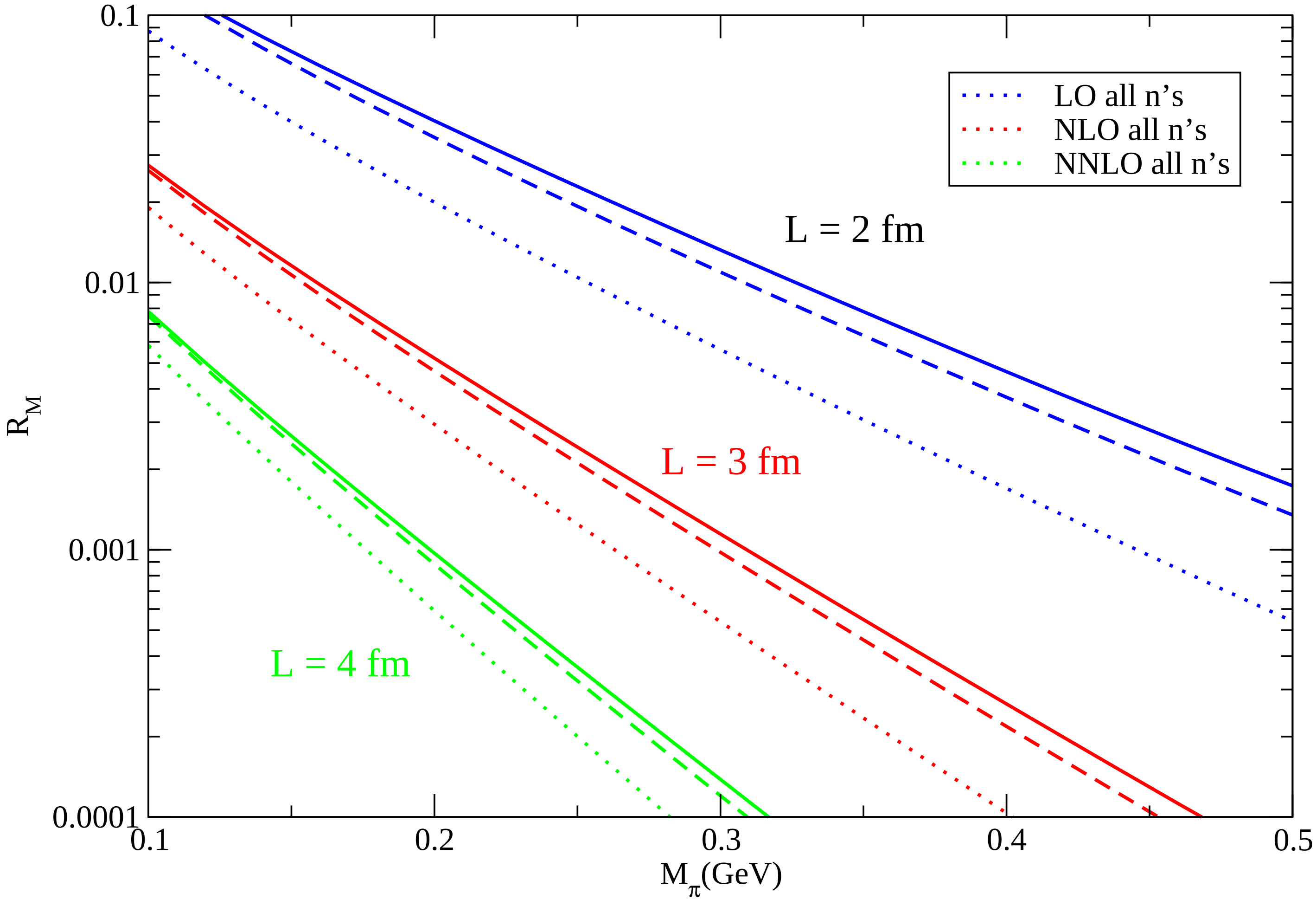
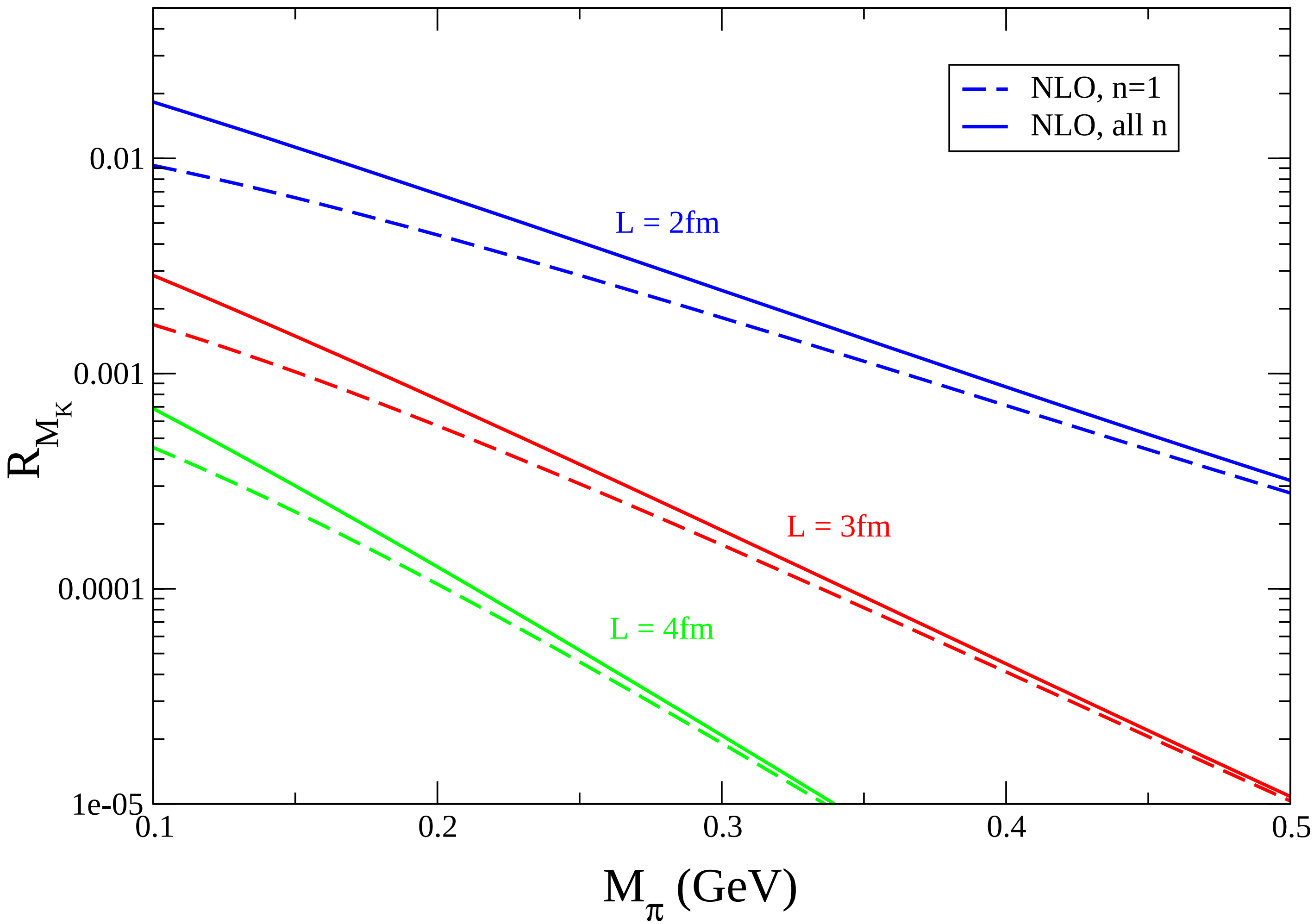
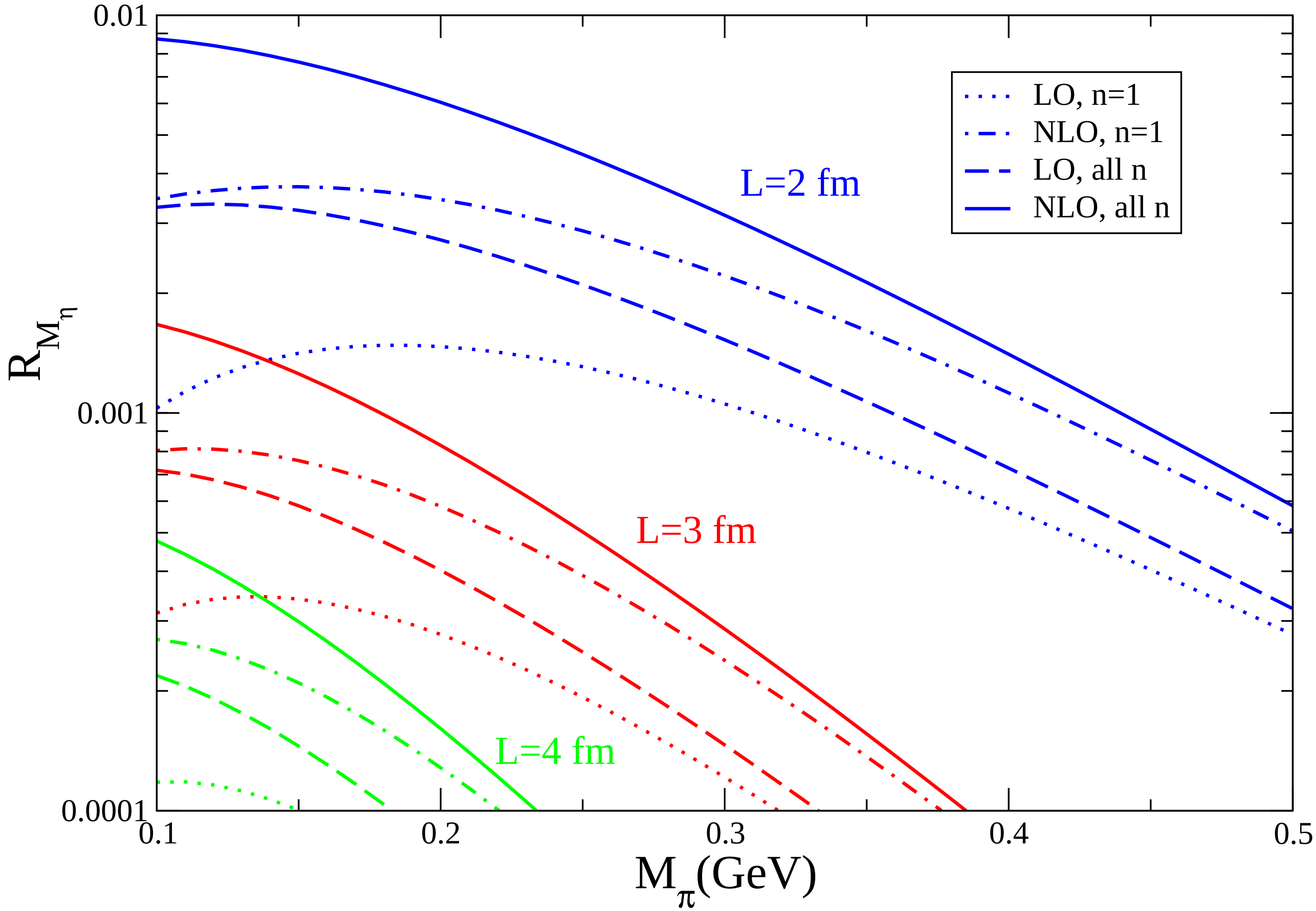


# Corrections for $M_K$



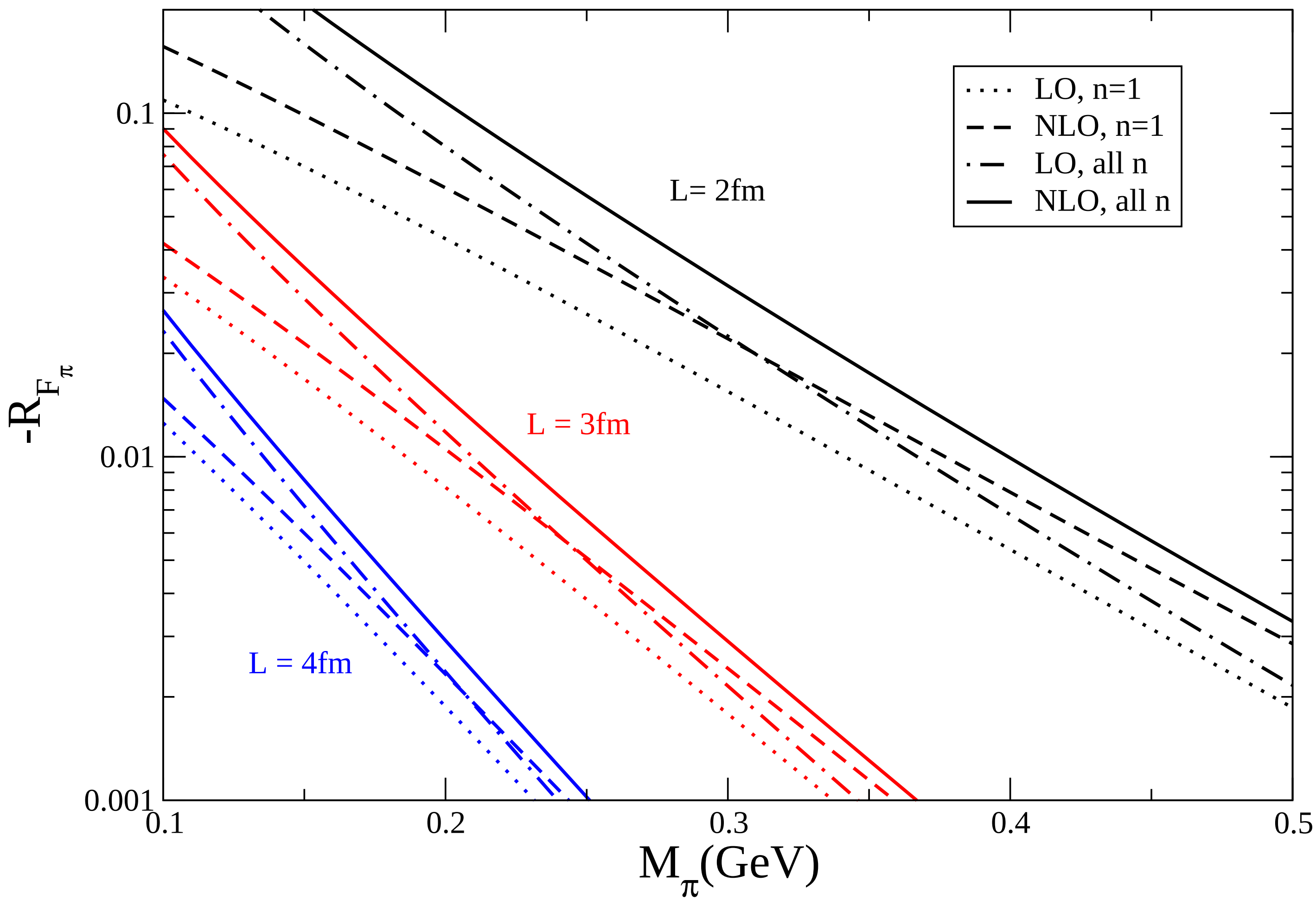

work in progress, G.C. and C. Haefeli

# Corrections for $M_\eta$



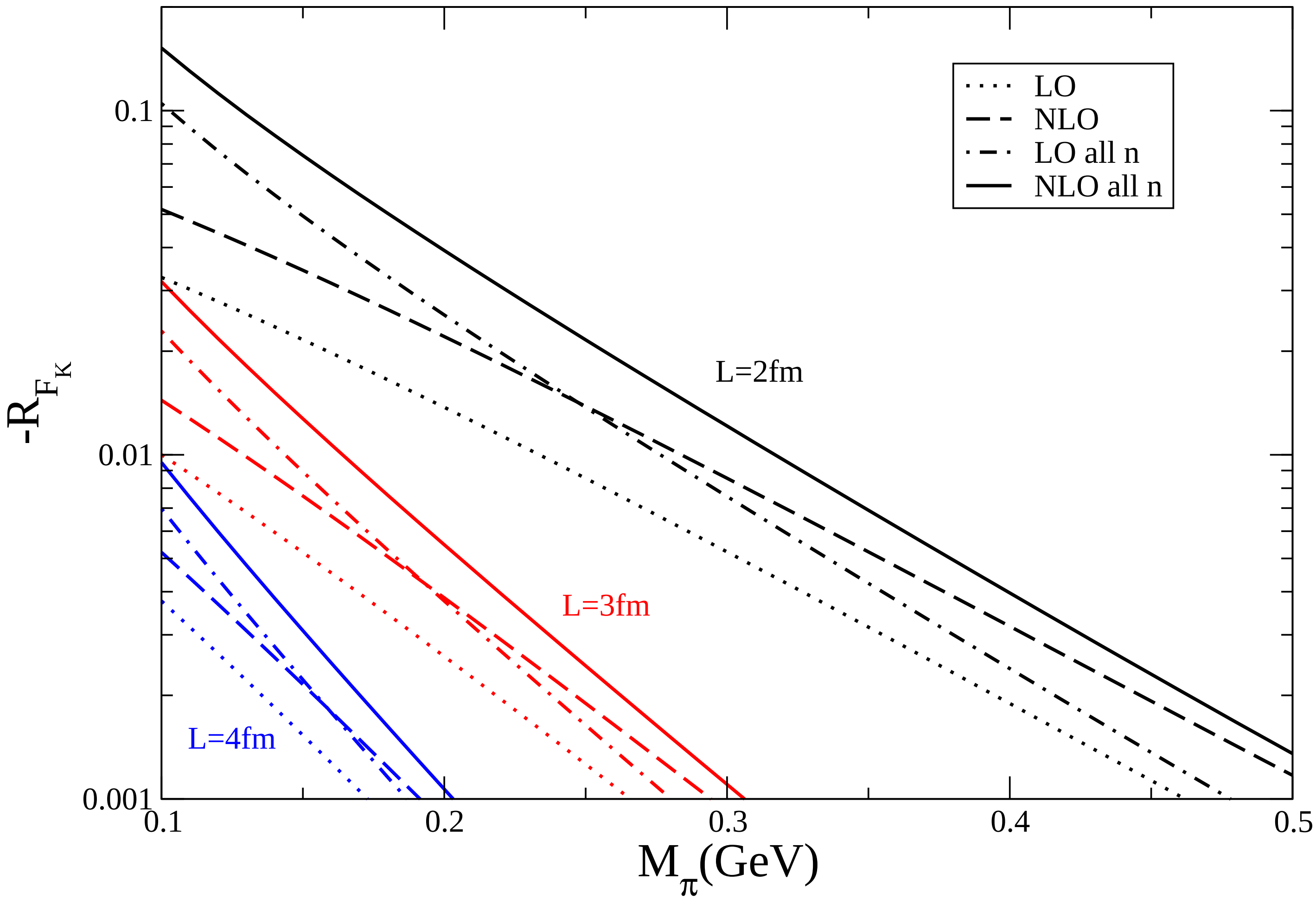

work in progress, G.C. and C. Haefeli

# Corrections for $F_\pi$




G.C. and C. Haefeli 04

# Corrections for $F_K$




LO agrees with Becirevic and Villadoro

work in progress, GC and C. Haefeli

# Summary

- For large volumes ($2LF_\pi \gg 1$), finite–volume effects can be calculated analytically within CHPT
- several one-loop CHPT calculations in the $p$–regime ($M_\pi L \gg 1$) have appeared in the recent literature
- the combined use of CHPT and asymptotic formulae *à la* Lüscher offers the most efficient way to get to higher orders in CHPT
- I have presented numerical evaluations of these finite volume corrections for all pseudoscalar masses and decay constants, as well as for the nucleon (after the talk of T.Hemmert 10 days ago)

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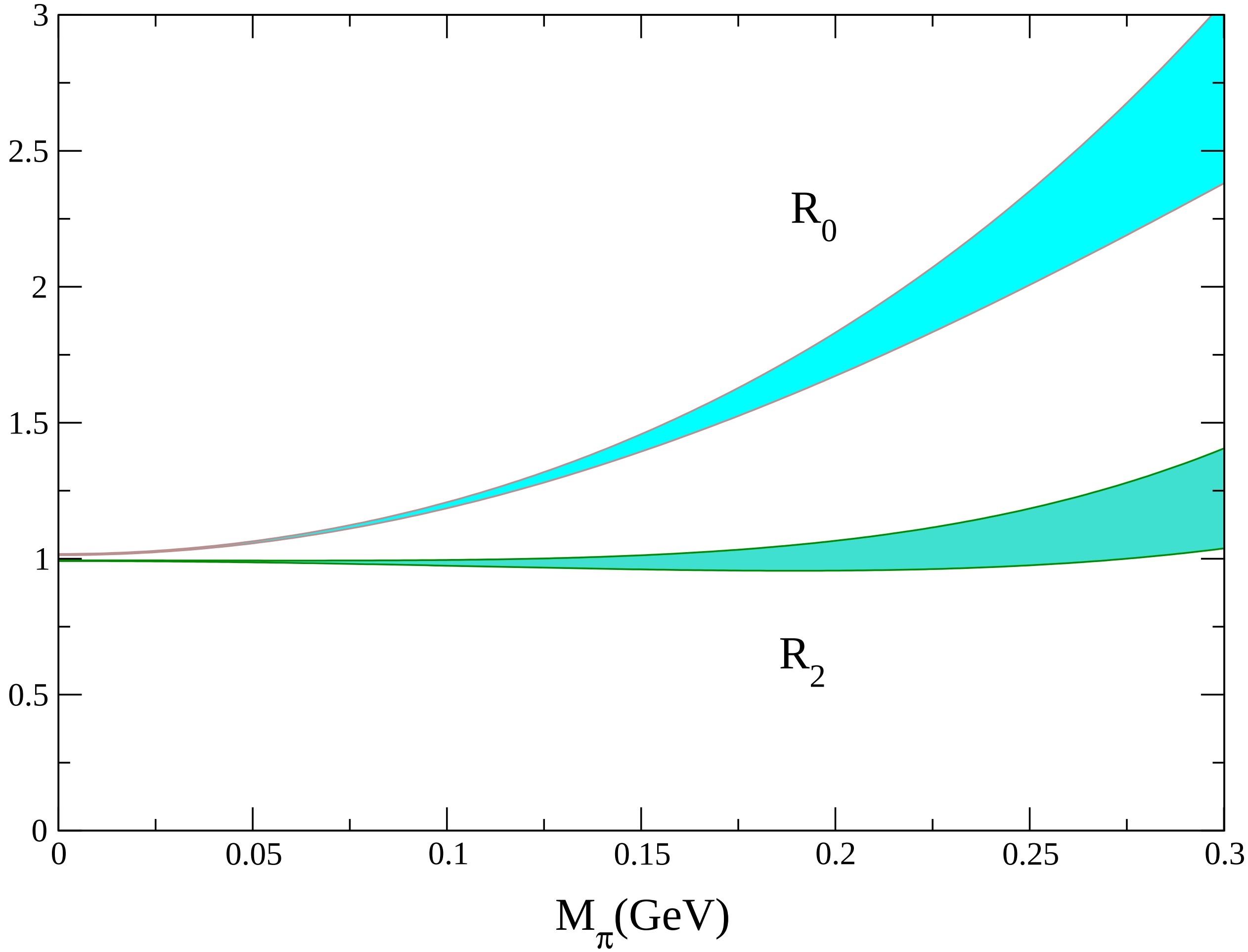
- the extrapolation $L \to \infty$ can be made analytically!

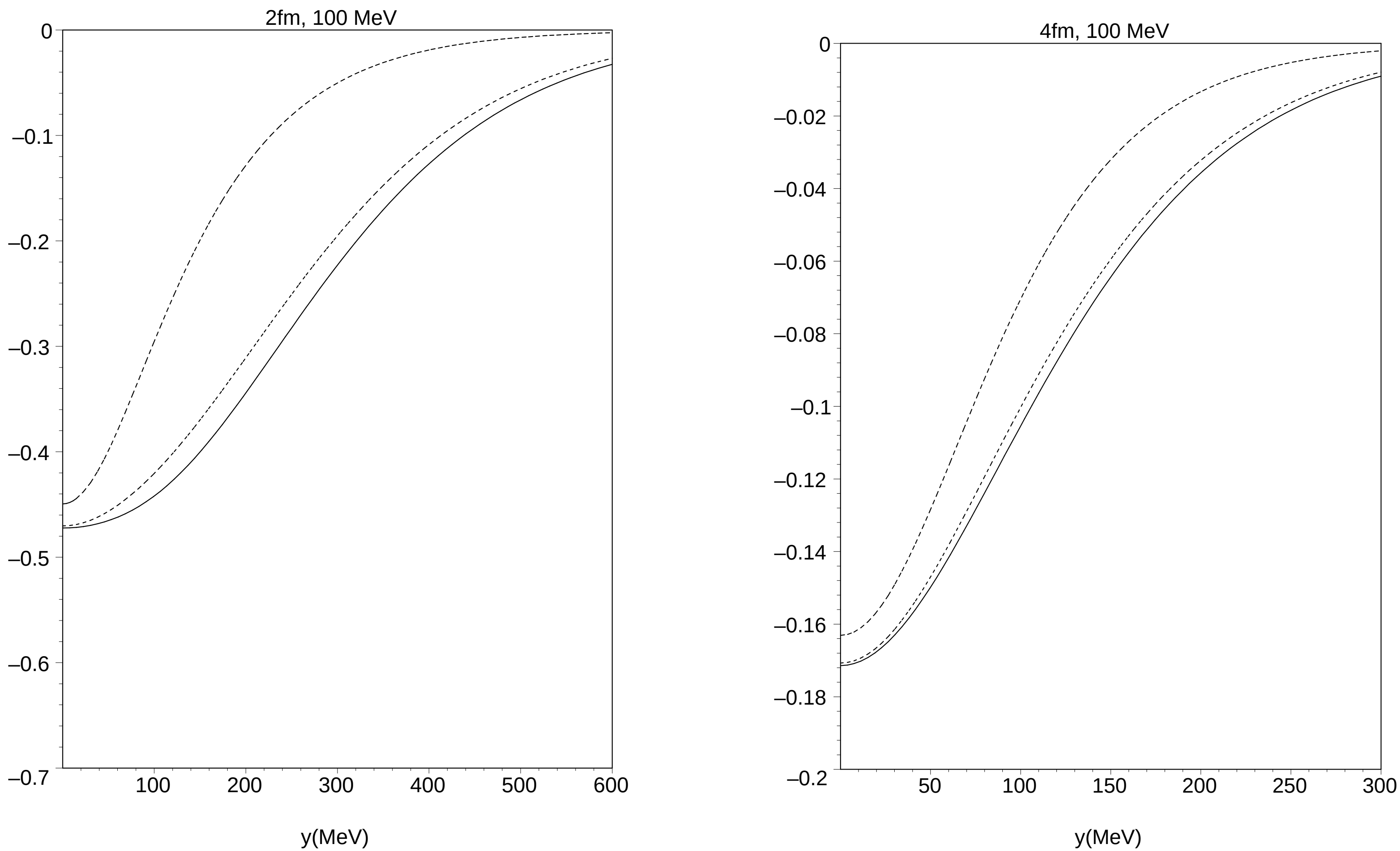
# A remark/proposal

- When lattice calculation will become precise enough one will be able to use these effects to get information on the infinite-volume amplitudes
- From the corrections to the pion mass, e.g. one will be able to extract interesting information on the $I = 0$ amplitude in an unphysical region – or, in other words, on the relevant low-energy constants

# The scattering lengths



# Integrands for the pion mass



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