

Measuring the strong and weak low-energy constants of QCD in a small volume?

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General ideas:

L. Giusti, C. Hoelbling, M. Lüscher, H. Wittig,
Comput. Phys. Commun. 153 (2003) 31 [hep-lat/0212012]

Further work:

M.L., P. Hernández, L. Giusti, P. Weisz, H. Wittig,
P.H. Damgaard, K. Jansen, L. Lellouch, C. Pena, J. Wennekers
[hep-lat/0211020,0212014,0312012,0402002,0407007; hep-ph/0407086; ...]

The strong interaction part of the chiral Lagrangian:

$$\mathcal{L}_E = \frac{F^2}{4} \text{Tr}[\partial_\mu U \partial_\mu U^\dagger] - \frac{\Sigma}{2} \text{Tr}[UM + M^\dagger U^\dagger] + \dots,$$

$U \in \text{SU}(3)$, $M = \text{diag}(m_u, m_d, m_s)$, and F, Σ are low-energy constants.

The weak interaction Hamiltonian:

$$\mathcal{H}_w^{\chi\text{PT}} = 2\sqrt{2}G_F V_{ud}V_{us}^* \{g_{27}\mathcal{O}_{27} + g_8\mathcal{O}_8 + g'_8\mathcal{O}'_8\} + \text{H.c.} ,$$

where G_F is the Fermi constant, V_{ij} are elements of the CKM-matrix, g_{27} , g_8 and g'_8 are dimensionless low-energy constants, and

$$\mathcal{O}_{27} \equiv \frac{1}{4}F^4 \left[(\partial_\mu U U^\dagger)_{ds} (\partial_\mu U U^\dagger)_{uu} + \frac{2}{3} (\partial_\mu U U^\dagger)_{du} (\partial_\mu U U^\dagger)_{us} \right] ,$$

$$\mathcal{O}_8 \equiv \frac{1}{4}F^4 \sum_{k=u,d,s} (\partial_\mu U U^\dagger)_{dk} (\partial_\mu U U^\dagger)_{ks} ,$$

$$\mathcal{O}'_8 \equiv F^2 \Sigma (UM + M^\dagger U^\dagger)_{ds} .$$

Reproducing $g_8/g_{27} \gg 1$ from lattice QCD is a long-standing challenge.

What's new (i): To match for g_{27}, g_8 , we can carry out simulations in a “small” volume, $2\pi/M_{\text{glueball}} \ll L \ll 2\pi/M_\pi$, with M_π physically light.

Why? χ PT applies as soon as the momentum scales are below the QCD scale, e.g. $L \sim 2.0$ fm. The usual counting rules for χ PT just need to be modified.

Gasser, Leutwyler 1987;
Neuberger 1988;
Hasenfratz, Leutwyler 1990;
Hansen, Leutwyler 1990, 1991

On a finite periodic lattice ($V = L^3T$, $L_0 \equiv T$, $L_i \equiv L$),

$$p_\mu = \frac{2\pi}{L_\mu} n_\mu, \quad n_\mu \in \mathbb{Z} .$$

Writing $U = \exp(2i\xi/F)$, the propagator is

$$\langle \xi_p \xi_{-p} \rangle \sim \frac{1}{p^2 + M_\pi^2} .$$

For $L^2 \ll (2\pi/M_\pi)^2$, the zero-modes $p = 0$ become dominant and have to be summed to all orders. This is the so-called ϵ -regime of χ PT.

In the ϵ -regime, one can write

$$U = \exp\left(i\frac{2\xi}{F}\right) U_0, \quad \int_x \xi(x) = 0 .$$

The non-zero momentum modes are treated perturbatively as in usual χ PT.

Left over are non-perturbative zero-mode integrals. Going from the θ -vacuum to a fixed topology ν , they are of the type

$$\frac{1}{2} \left\langle \text{Tr} \left[U_0 + U_0^\dagger \right] \right\rangle_{U_0 \in U(N_f)} = \frac{d}{d\mu} \ln \det [I_{\nu+j-i}(\mu)]|_{i,j=1,\dots,N_f} ,$$
$$\mu \equiv m\Sigma V \sim 1 .$$

The great strength of the ϵ -regime is that NLO corrections can be computed without introducing any new low-energy constants, unlike in the usual “ p -regime” where $L \gtrsim 2\pi/M_\pi!$

What's new (ii): Start with $m_c = m_u = m_d = m_s$, so that the theory has an exact $SU(4)_L \times SU(4)_R$ symmetry in the chiral limit.

Why? To disentangle the role of the charm quark, i.e., tell apart effects due to the mass scale m_c (~ 1 GeV) from soft gluon exchange (~ 250 MeV).

Furthermore, group theory becomes simpler: “the GIM cancellation takes place”, i.e. no penguin contractions are needed, and there are only two operators rather than three ($27 \leftrightarrow 80 \equiv +$; $8 \leftrightarrow 20 \equiv -$).

$$H_w^{\text{QCD}} = \sqrt{2}G_F V_{ud}V_{us}^* \sum_{\sigma=\pm} k_1^\sigma O_1^\sigma ,$$

$$O_1^\pm = \{(\bar{s}\gamma_\mu P_L u)(\bar{u}\gamma_\mu P_L d) \pm (\bar{s}\gamma_\mu P_L d)(\bar{u}\gamma_\mu P_L u)\} - (u \rightarrow c) ,$$

$$\mathcal{H}_w^{\text{XPT}} = \sqrt{2}G_F V_{ud}V_{us}^* \sum_{\sigma=\pm} g_1^\sigma O_1^\sigma ,$$

$$O_1^\pm = \frac{F^4}{4} \{ (U\partial_\mu U^\dagger)_{us} (U\partial_\mu U^\dagger)_{du} \pm (U\partial_\mu U^\dagger)_{ds} (U\partial_\mu U^\dagger)_{uu} \} \\ - (u \rightarrow c) .$$

To match for g_1^\pm , define the correlators

$$[\mathcal{C}_1^\pm(x_0, y_0)]^{ab} = \int d^3x \int d^3y \langle \mathcal{J}_0^a(x) [\mathcal{O}_1^\pm(0)] \mathcal{J}_0^b(y) \rangle ,$$

where \mathcal{J}_μ^a is the left-handed current, $(\mathcal{J}_\mu^a)^{\text{QCD}} = \bar{\psi} \gamma_\mu P_L T^a \psi$.

On the chiral theory side, we obtain in the ϵ -regime (for $x_0, y_0 \neq 0$)

$$H(x_0, y_0) \equiv \frac{\mathcal{C}_1^-(x_0, y_0)}{\mathcal{C}_1^+(x_0, y_0)} = 1 - \frac{4}{F^2 T^2} \rho^3 \left\{ \beta_1(\rho) \rho^{-3/2} - k_{00}(\rho) \right\} ,$$

where $\rho = T/L$, and β_1, k_{00} are certain known shape coefficients.

$$\implies \frac{g_1^-}{g_1^+} = \frac{k_1^- (C_1^-)^{\text{QCD}}(x_0, y_0)}{k_1^+ (C_1^+)^{\text{QCD}}(x_0, y_0)} \frac{1}{H(x_0, y_0)} .$$

What's new (iii): Use Ginsparg-Wilson fermions (the Neuberger Dirac operator).

The renormalisation and mixings are like in the continuum: no power-divergent subtractions, and measurements can be easily carried out at a fixed topology.

What's new (iv): "Low-mode averaging".

Low eigenvalues ($|\lambda_1| \sim (\Sigma V)^{-1}$) of the massless Dirac operator tend to make the signal noisy (or "spikey"), if $m \lesssim (\Sigma V)^{-1}$:

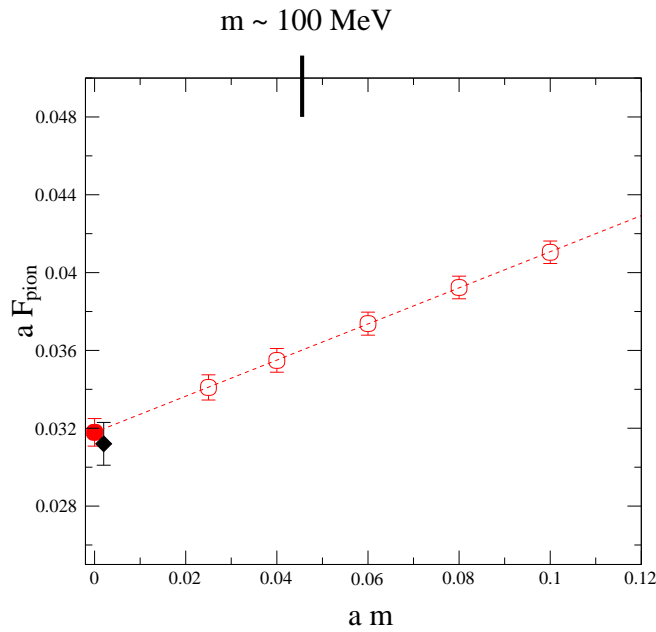
$$\langle \psi(x) \bar{\psi}(y) \rangle = \sum_n \frac{v_n(x) v_n^\dagger(y)}{\lambda_n + m} .$$

To avoid these fluctuations, a certain number of low modes, n_{low} , are treated separately: we take the volume average of their contributions to the correlators.

See also: Edwards 2002;
DeGrand, Schaefer 2003

Result for F from $\int d^3x \langle \mathcal{J}_0^a(x) \mathcal{J}_0^b(0) \rangle$

	β	T/L	L [fm]	am	configs.
ϵ -regime	6.0	16/16	1.49	0.005...0.010	203
p -regime	6.0	24/16	1.49	0.025...0.100	113

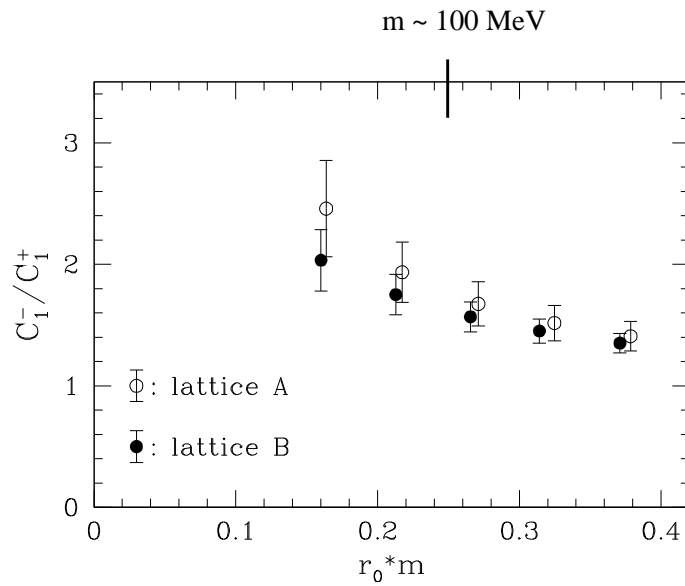


\Rightarrow Numerical signal is good, and the determinations in the p -regime (large m followed by chiral extrapolation) and ϵ -regime (directly at small m) agree.

In physical units, at this V and a , the quenched $F \sim 103(4)$ MeV.

Result for g_1^-/g_1^+ ?

	β	T/L	$L[\text{fm}]$	am	configs.
A	6.0	40/12	1.12	0.030...0.070	751
B	5.8485	30/12	1.49	0.040...0.092	638

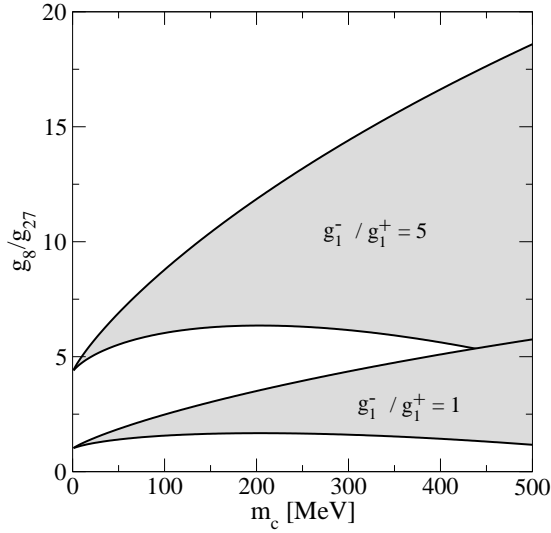


Simulating directly in the ϵ -regime here requires a higher n_{low} in the low-mode averaging procedure \Rightarrow in progress. On the other hand, for a fixed volume, small m , and NLO in ChPT, the ratio could also be fit to a Taylor series in m !

\Rightarrow One gets an enhancement, but this is at least partly cancelled by the Goldstone-mode factor $H(x_0, y_0)$ (≈ 2.3 for lattice B). Further systematics needed to see whether there is an effect in the SU(4) limit already.

How does the system behave for $m_c > m_u = m_d = m_s$? In ChPT:

$$\frac{g_8}{g_{27}} = \frac{1}{6} \left\{ \left[1 + \frac{15m_c \Sigma}{32\pi^2 F^4} \ln \frac{\Lambda_1}{m_c} \right] + \frac{g_-}{g_+} \left[5 + \frac{15m_c \Sigma}{32\pi^2 F^4} \ln \frac{\Lambda_2}{m_c} \right] \right\}.$$



Here the higher order LECs enter, and there is no firm prediction from χ PT. Lattice needed: domain wall fermion measurements from RBC collaboration suggest (?) m_c has little effect, so maybe SU(4) is all we need!

Conclusions

Philosophy: in order to understand from which scale the enhancement comes from, let us try to factorise the problem into parts and inspect one physics scale at a time, with controlled systematic errors, rather than everything at once.

Conceptual points: (i) Usually $V \rightarrow \infty$, $m \rightarrow 0$, here $m \rightarrow 0$, $V \rightarrow \infty$.
(ii) Start with the SU(4) degenerate limit.

Technical points: (iii) Use Ginsparg-Wilson fermions.
(iv) Implement low-mode averaging.

Initial tests suggest that a numerical signal can be obtained this way.

Challenges: (i) Three-point functions at smaller m .
(ii) $L \gtrsim 2.0$ fm for ChPT convergence in the ϵ -regime.
(iii) Check the effect of $m_c > m_u, m_d, m_s$.
(iv) Unquenching...