



EFFECTIVE LAGRANGIANS IN THE RESONANCE REGION

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- The Large N_C Limit
- Chiral Perturbation Theory (χ PT)
- Resonance Chiral Theory ($R\chi T$)
- ε'/ε
- Quantum Loops in $R\chi T$



Benasque, 3 August 2004

Energy Scale

Fields

Effective Theory

M_W

W, Z, γ, g
 τ, μ, e, ν_i
 t, b, c, s, d, u

Standard Model

↓ OPE

$\lesssim m_c$

$\gamma, g ; \mu, e, \nu_i$
 s, d, u

$\mathcal{L}_{\text{QCD}}^{(n_f=3)}, \mathcal{L}_{\text{eff}}^{\Delta S=1,2}$

↓ $N_C \rightarrow \infty$

M_K

$\gamma ; \mu, e, \nu_i$
 π, K, η

χPT

Energy Scale

Fields

Effective Theory

M_W

t, b, c
 $s, d, u ; G^a$

$\text{QCD}^{N_f=6}$

$\lesssim m_c$

$s, d, u ; G^a$

$\text{QCD}^{N_f=3}$

Λ_χ

V, A, S, P
 π, K, η

$\text{R}\chi\text{PT}$

$\lesssim M_K$

π, K, η

$\chi\text{PT}^{N_f=3}$

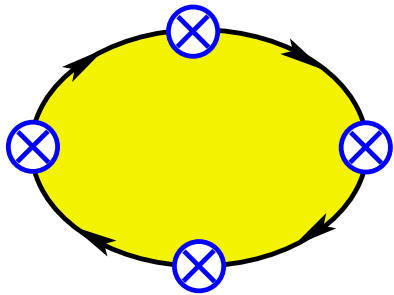
$\lesssim M_\pi$

π

$\chi\text{PT}^{N_f=2}$

N_C COUNTING RULES

$$g_s \sim 1/\sqrt{N_C} \quad ; \quad \alpha_s \sim 1/N_C \quad ; \quad \langle T(J_1 \cdots J_n) \rangle \sim N_C$$



- Dominance of planar gluonic exchanges
- Non-planar diagrams suppressed by $1/N_C^2$
- Internal quark loops suppressed by $1/N_C$

Colour Confinement $\rightarrow J|0\rangle \sim |1 \text{ Meson}\rangle$

$$\langle J(k) J(-k) \rangle = \sum_n \frac{f_n^2}{k^2 - M_n^2}$$

- **Infinite** number of mesons ($\sim \ln k^2$)
- $f_n = \langle 0|J|n\rangle \sim \sqrt{N_C}$; $M_n \sim O(1)$
- Mesons are **free, stable** and **non-interacting**

$$\langle J J J \rangle = \sum \text{[Diagram 1]} + \sum \text{[Diagram 2]}$$

Diagram 1: A vertex with three external lines, one horizontal to the right and two diagonal (top-left and bottom-left).
Diagram 2: A vertex with three external lines, one vertical down, one horizontal to the right, and one diagonal (top-right).

$$\langle J J J J \rangle = \sum \text{[Diagram 3]} + \sum \text{[Diagram 4]}$$

Diagram 3: A central horizontal line with two vertices, each having two external lines (one diagonal, one horizontal).
Diagram 4: A central vertical line with two vertices, each having two external lines (one horizontal, one diagonal).

$$+ \sum \text{[Diagram 5]} + \sum \text{[Diagram 6]}$$

Diagram 5: A central vertex with four external lines in a cross configuration.
Diagram 6: A vertex with four external lines, one vertical down, one horizontal to the right, and two diagonal (top-left and bottom-right).


 $\sim N_C^{1-\frac{n}{2}}$


 $\sim N_C^{1-\frac{n}{2}}$

Crossing + Unitarity ➔ Tree Approximation to some Local Effective Meson Lagrangian

χ PT

Low-Energy Expansion (p^{2n}, m_q^n) : $\mathcal{L} = \sum_n \mathcal{L}_{2n}$

$$\begin{aligned}\mathcal{L}_2 &= \frac{f^2}{4} \langle D_\mu U^\dagger D^\mu U + 2B_0 (U^\dagger \mathcal{M} + \mathcal{M}^\dagger U) \rangle \\ &= D_\mu \pi^+ D_\mu \pi^- - M_\pi^2 \pi^+ \pi^- + \dots + \frac{1}{6f^2} (\pi^+ \overleftrightarrow{D}_\mu \pi^-) (\pi^+ \overleftrightarrow{D}^\mu \pi^-) + \dots\end{aligned}$$

$$\frac{M_\pi^2}{m_u + m_d} = \frac{M_{K^0}^2}{m_s + m_d} = \frac{M_{K^+}^2}{m_s + m_u} = B_0 = -\frac{\langle \bar{q}q \rangle}{f^2}$$

N_C Counting:

- $f^2 \approx f_\pi^2 \sim N_C$; $B_0 \sim O(1) \rightarrow M_\varphi \sim O(1)$
- $U \rightarrow \sum_n (\varphi/f)^n \sim O(1)$; $V_n \sim f^{2-n} \sim N_C^{1-n/2}$
- $\mathcal{L} \sim N_C$; n-Loop $\sim 1/(16\pi^2 f^2)^n \sim 1/N_C^n$

O(p⁴) χPT

i) \mathcal{L}_4 at tree level (Gasser–Leutwyler)

$$\begin{aligned}
 \mathcal{L}_4 = & L_1 \langle D_\mu U^\dagger D^\mu U \rangle^2 + L_2 \langle D_\mu U^\dagger D_\nu U \rangle \langle D^\mu U^\dagger D^\nu U \rangle + L_3 \langle D_\mu U^\dagger D^\mu U D_\nu U^\dagger D^\nu U \rangle \\
 & + L_4 \langle D_\mu U^\dagger D^\mu U \rangle \langle U^\dagger \chi + \chi^\dagger U \rangle + L_5 \langle D_\mu U^\dagger D^\mu U (U^\dagger \chi + \chi^\dagger U) \rangle \\
 & + L_6 \langle U^\dagger \chi + \chi^\dagger U \rangle^2 + L_7 \langle U^\dagger \chi - \chi^\dagger U \rangle^2 + L_8 \langle \chi^\dagger U \chi^\dagger U + U^\dagger \chi U^\dagger \chi \rangle \\
 & - i L_9 \langle F_R^{\mu\nu} D_\mu U D_\nu U^\dagger + F_L^{\mu\nu} D_\mu U^\dagger D_\nu U \rangle + L_{10} \langle U^\dagger F_R^{\mu\nu} U F_{L\mu\nu} \rangle
 \end{aligned}$$

$$F_J^{\mu\nu} \equiv \partial^\mu J^\nu - \partial^\nu J^\mu - i[J^\mu, J^\nu] \quad ; \quad J^\mu = v^\mu \pm a^\mu \quad ; \quad \chi \equiv 2 B_0 \mathcal{M}$$

ii) \mathcal{L}_2 at one loop (unitarity) $T_4 \sim p^4 \{ a \log(p^2/\mu^2) + b(\mu) \}$

- Chiral Logarithms unambiguously predicted
- L_i 's fixed by QCD dynamics [1-loop divergences $\rightarrow L_i^r(\mu)$]

iii) Wess–Zumino–Witten term (anomaly): $\pi^0 \rightarrow \gamma\gamma, \eta \rightarrow \gamma\gamma, \dots$

$O(p^4)$ χ PT COUPLINGS

i	$L_i^r(M_\rho) \times 10^3$	$O(N_C)$	Source	Γ_i
1	0.4 ± 0.3	N_C	$K_{e4}, \pi\pi \rightarrow \pi\pi$	$\frac{3}{32}$
2	1.4 ± 0.3	N_C	$K_{e4}, \pi\pi \rightarrow \pi\pi$	$\frac{3}{16}$
3	-3.5 ± 1.1	N_C	$K_{e4}, \pi\pi \rightarrow \pi\pi$	0
4	-0.3 ± 0.5	1	Zweig rule	$\frac{1}{8}$
5	1.4 ± 0.5	N_C	F_K/F_π	$\frac{3}{8}$
6	-0.2 ± 0.3	1	Zweig rule	$\frac{11}{144}$
7	-0.4 ± 0.2	1	GMO, $L_{5,8}$	0
8	0.9 ± 0.3	N_C	$M_{K^0} - M_{K^+}, L_5,$ $(m_s - \hat{m})/(m_d - m_u)$	$\frac{5}{48}$
9	6.9 ± 0.7	N_C	$\langle r^2 \rangle_V^\pi$	$\frac{1}{4}$
10	-5.5 ± 0.7	N_C	$\pi \rightarrow e\nu\gamma$	$-\frac{1}{4}$

$$2L_1 - L_2 = (-0.6 \pm 0.6) \times 10^{-3} \sim O(1)$$

$$c \langle D_\mu U D_\nu U^\dagger D^\mu U D^\nu U^\dagger \rangle \rightarrow 2\delta L_1 = \delta L_2 = -\frac{1}{2}\delta L_3 = c$$

R_χT

Include Resonance Nonet Multiplets

(Ecker, Gasser, Pich, de Rafael)

V(1⁻⁻), A(1⁺⁺), S(0⁺⁺), P(0⁻⁺)

$$\mathcal{L}_2^V = \frac{F_V}{2\sqrt{2}} \langle V_{\mu\nu} f_+^{\mu\nu} \rangle + \frac{i G_V}{\sqrt{2}} \langle V_{\mu\nu} u^\mu u^\nu \rangle$$

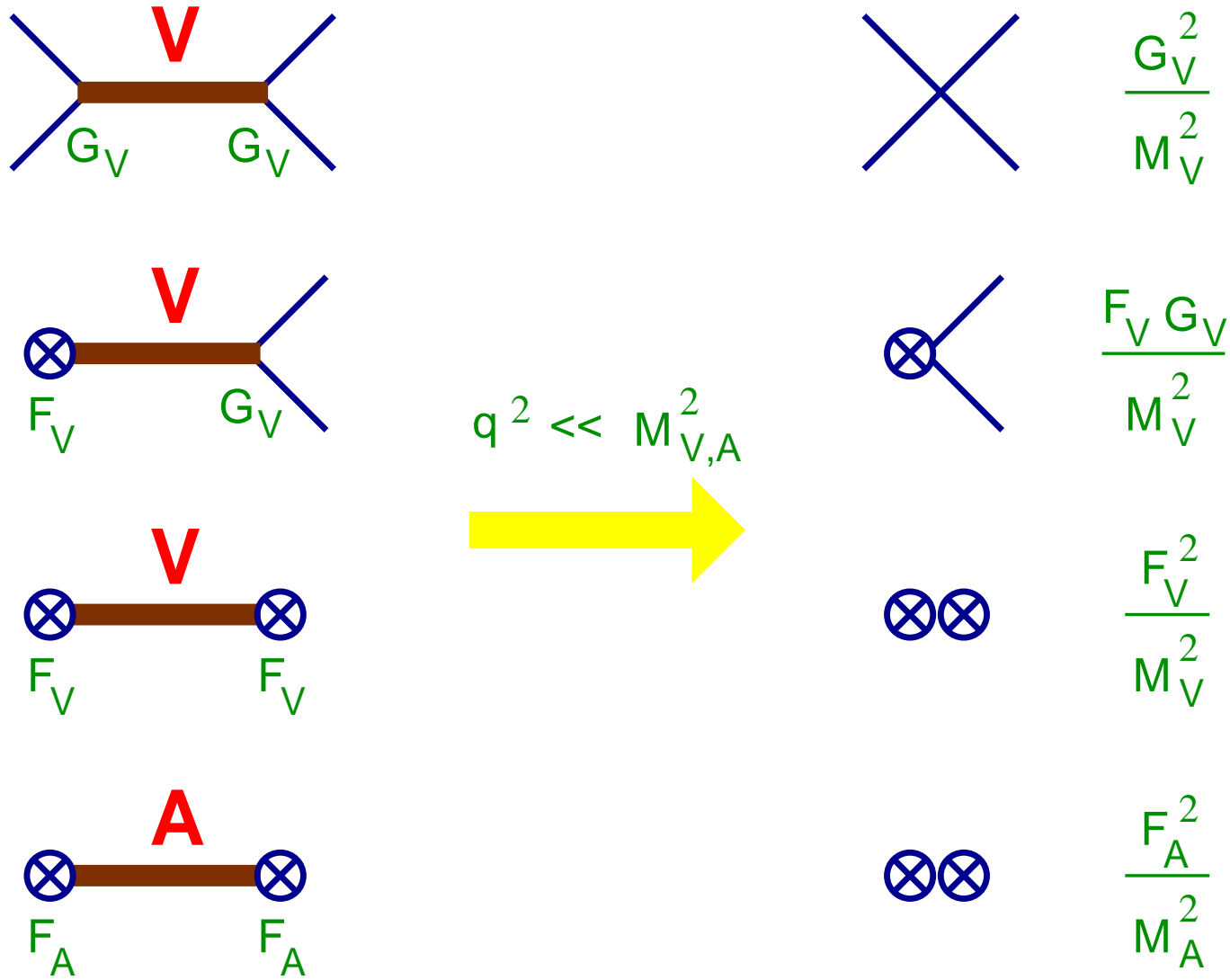
$$\mathcal{L}_2^A = \frac{F_A}{2\sqrt{2}} \langle A_{\mu\nu} f_-^{\mu\nu} \rangle$$

$$\mathcal{L}_2^S = c_d \langle S u^\mu u^\nu \rangle + c_m \langle S \chi_+ \rangle$$

$$\mathcal{L}_2^P = i d_m \langle P \chi_- \rangle$$

$$u_\mu = i u^\dagger D_\mu U u^\dagger = u_\mu^\dagger \quad ; \quad U = u^2$$

$$f_\pm^{\mu\nu} = u F_L^{\mu\nu} u^\dagger \pm u^\dagger F_R^{\mu\nu} u \quad ; \quad \chi_\pm = u^\dagger \chi u^\dagger \pm u \chi^\dagger u$$



$$\left\{ \mathbf{V} [M_V, G_V, F_V] , \mathbf{A} [M_A, F_A] \right\} \longleftrightarrow \left\{ \mathbf{S} [M_S, c_d, c_m] , \mathbf{P} [M_P, d_m] \right\}$$

$O(N_C)$:

$$2L_1 = L_2 = \sum_i \frac{G_{V_i}^2}{4M_{V_i}^2} \quad ; \quad L_3 = \sum_i \left\{ -\frac{3G_{V_i}^2}{4M_{V_i}^2} + \frac{c_{d_i}^2}{2M_{S_i}^2} \right\}$$

$$L_5 = \sum_i \frac{c_{d_i} c_{m_i}}{M_{S_i}^2} \quad ; \quad L_8 = \sum_i \left\{ \frac{c_{m_i}^2}{2M_{S_i}^2} - \frac{d_{m_i}^2}{2M_{P_i}^2} \right\}$$

$$L_9 = \sum_i \frac{F_{V_i} G_{V_i}}{2M_{V_i}^2} \quad ; \quad L_{10} = \frac{1}{4} \sum_i \left\{ \frac{F_{A_i}^2}{M_{A_i}^2} - \frac{F_{V_i}^2}{M_{V_i}^2} \right\}$$

$O(1)$:

$$2L_1 - L_2 = L_4 = L_6 = 0 \quad ; \quad L_7 = -\frac{\tilde{d}_m^2}{2M_{\eta_1}^2}$$

BUT

$$M_{\eta_1}^2 \sim O\left(\frac{1}{N_C}, \mathcal{M}\right)$$

SHORT-DISTANCE CONSTRAINTS

Vector Form Factor $\langle \pi | v_\mu | \pi \rangle$:

$$F_V(t) = 1 + \sum_i \frac{F_{V_i} G_{V_i}}{f^2} \frac{t}{M_{V_i}^2 - t}$$

$$\lim_{t \rightarrow \infty} F_V(t) = 0$$



$$\sum_i F_{V_i} G_{V_i} = f^2$$

Axial Form Factor $\langle \gamma | a_\mu | \pi \rangle$:

$$G_A(t) = \sum_i \left\{ \frac{2 F_{V_i} G_{V_i} - F_{V_i}^2}{M_{V_i}^2} + \frac{F_{A_i}^2}{M_{A_i}^2 - t} \right\}$$

$$\lim_{t \rightarrow \infty} G_A(t) = 0$$



$$\sum_i (2 F_{V_i} G_{V_i} - F_{V_i}^2) / M_{V_i}^2 = 0$$

Weinberg Sum Rules:

$$\Pi_{LR}(t) = -\frac{f^2}{t} + \sum_i \frac{F_{V_i}^2}{M_{V_i}^2 + t} - \sum_i \frac{F_{A_i}^2}{M_{A_i}^2 + t}$$

$$\lim_{t \rightarrow \infty} t \Pi_{LR}(t) = 0$$

$$\lim_{t \rightarrow \infty} t^2 \Pi_{LR}(t) = 0$$



$$\sum_i (F_{V_i}^2 - F_{A_i}^2) = f^2$$

$$\sum_i (M_{V_i}^2 F_{V_i}^2 - M_{A_i}^2 F_{A_i}^2) = 0$$

Scalar Form Factor:

$$F_{K\pi}^S(s) = 1 + \sum_i \frac{4c_{m_i}}{f^2} \left[c_{d_i} + (c_{m_i} - c_{d_i}) \frac{M_K^2 + M_\pi^2}{M_{S_i}^2} \right] \frac{s}{M_{S_i}^2 - s}$$

$$\lim_{s \rightarrow \infty} F_{K\pi}^S(s) = 0 \quad \rightarrow$$

$$4 \sum_i c_{d_i} c_{m_i} = f^2 \quad ; \quad \sum_i \frac{c_{m_i}}{M_{S_i}^2} (c_{m_i} - c_{d_i}) = 0$$

SS – PP Sum Rules:

$$\Pi_{SS-PP}(t) = 16 B_0^2 \left\{ \sum_i \frac{c_{m_i}^2}{M_{S_i}^2 + t} - \sum_i \frac{d_{m_i}^2}{M_{P_i}^2 + t} - \frac{f^2}{8t} \right\}$$

$$\lim_{t \rightarrow \infty} t \Pi_{SS-PP}(t) = 0 \quad \rightarrow$$

$$8 \sum_i (c_{m_i}^2 - d_{m_i}^2) = f^2$$

Pseudoscalar Nonet:

$$\mathcal{L}_2 \doteq \frac{f^2}{4} \langle \chi U^\dagger + U \chi^\dagger \rangle \sim -i \frac{f}{\sqrt{24}} \eta_1 \langle \chi_- \rangle \quad \rightarrow$$

$$\tilde{d}_m = -\frac{f}{\sqrt{24}}$$

1-Resonance Approximation:

(Ecker, Gasser, Leutwyler, Pich, de Rafael)

$$F_V = 2 G_V = \sqrt{2} F_A = \sqrt{2} f \quad ; \quad M_A = \sqrt{2} M_V \quad ; \quad d_m = \frac{1}{2\sqrt{2}} f$$

$$c_m = c_d = \frac{1}{2} f \quad (\text{Jamin, Oller, Pich})$$



$$2 L_1 = L_2 = \frac{1}{4} L_9 = -\frac{1}{3} L_{10} = \frac{f^2}{8 M_V^2}$$

$$L_3 = -\frac{3 f^2}{8 M_V^2} + \frac{f^2}{8 M_S^2} \quad ; \quad L_5 = \frac{f^2}{4 M_S^2}$$

$$L_8 = \frac{f^2}{8 M_S^2} - \frac{f^2}{16 M_P^2} \quad ; \quad L_7 = -\frac{f^2}{48 M_{\eta_1}^2}$$

L_i'S FROM RESONANCE EXCHANGE

i	$L_i^r(M_\rho)$	V	A	S	η_1	Total	Total ^{b)}
1	0.4 ± 0.3	0.6	0	0	0	0.6	0.9
2	1.4 ± 0.3	1.2	0	0	0	1.2	1.8
3	-3.5 ± 1.1	-3.6	0	0.6	0	-3.0	-4.3
4	-0.3 ± 0.5	0	0	0	0	0.0	0.0
5	1.4 ± 0.5	0	0	1.4 ^{a)}	0	1.4	2.1
6	-0.2 ± 0.3	0	0	0	0	0.0	0.0
7	-0.4 ± 0.2	0	0	0	-0.3	-0.3	-0.3
8	0.9 ± 0.3	0	0	0.9 ^{a)}	0	0.9	0.8
9	6.9 ± 0.7	6.9 ^{a)}	0	0	0	6.9	7.2
10	-5.5 ± 0.7	-10.0	4.0	0	0	-6.0	-5.4

^{a)} Input

^{b)} Short-Distance Constraints

Three–Point Functions:

- B. Moussallam
- M. Knecht, A. Nyffeler
- P.D. Ruiz-Femenía, A.Pich, J. Portolés
- V. Cirigliano, G. Ecker, M. Eidemüller, A.Pich, J. Portolés
- J. Bijnens, E. Gámiz, E. Lipartia, J. Prades



Constraints on $\mathcal{O}(p^6)$ χ PT couplings

Vertices with 2 or 3 Resonances. $\mathcal{O}(p^4)$ Couplings:

- V. Cirigliano, G. Ecker, M. Eidemüller, R. Kaiser, A.Pich, J. Portolés

More Resonance Multiplets:

Minimal Hadronic Ansatz

- M. Knecht, S. Peris, M. Perrottet, B. Phily, E. de Rafael
- P.D. Ruiz-Femenía, J. Portolés

$1/N_C$ Corrections:

- **Resonance Widths**

- F. Guerrero, A. Pich
- D. Gómez–Dumm, A. Pich, J. Portolés

- **Unitarity Corrections**

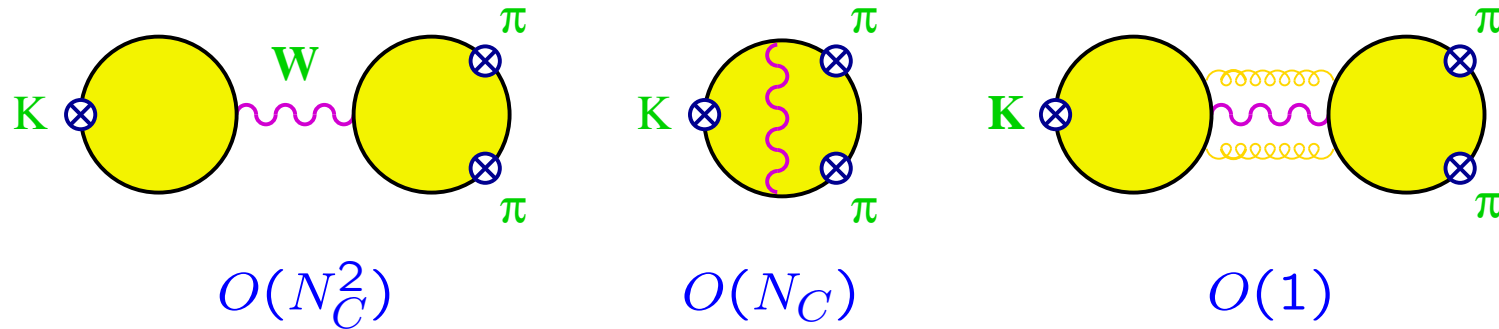
Final State Interactions

- F. Guerrero, A. Pich
- E. Pallante, A. Pich, I. Scimemi
- J.J. Cillero, A. Pich, J. Portolés
- M. Jamin, J.A. Oller, A. Pich

- **Quantum Loops in $R_\chi T$**

- J.J. Cillero, I. Rosell, A. Pich

Weak Currents Factorize at Large N_C



$$A[K^0 \rightarrow \pi^0 \pi^0] = 0 \quad \longrightarrow \quad A_0 = \sqrt{2} A_2$$

No $\Delta I = \frac{1}{2}$ enhancement at leading order in $1/N_C$

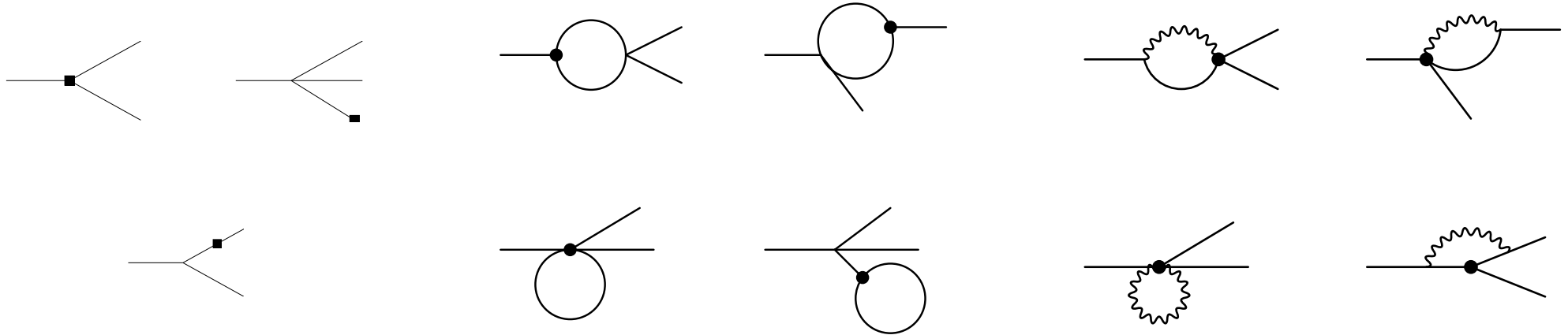
- **Multiscale problem:** **OPE** $\frac{1}{N_C} \log \left(\frac{M_W}{\mu} \right) \sim \frac{1}{3} \times 4$

Short-distance logarithms must be summed

- **Large χ PT logarithms:** **FSI** $\frac{1}{N_C} \log \left(\frac{\mu}{M_\pi} \right) \sim \frac{1}{3} \times 2$

Infrared logarithms must also be included $[\delta_I \sim O(1/N_C) , \delta_0 - \delta_2 \approx 45^\circ]$

$$O [p^4, (m_u - m_d) p^2, e^2 p^2] \quad \chi\text{PT}$$



Nonleptonic weak Lagrangian:

$$O(G_F p^4)$$

$$\mathcal{L}_{\text{weak}}^{(4)} = \sum_i G_8 N_i F^2 O_i^8 + \sum_i G_{27} D_i F^2 O_i^{27} + \text{h.c.}$$

Electroweak Lagrangian:

$$O(G_F e^2 p^2)$$

$$\mathcal{L}_{\text{EW}} = e^2 \sum_i G_8 Z_i F^4 O_i^{\text{EW}} + \text{h.c.}$$

$O(e^2 p^2)$ Electromagnetic + $O(p^4)$ Strong

K_i, L_i

$K \rightarrow \pi\pi, \pi\pi\gamma$

Inclusive, DAPHNE

$$\mathcal{A}_n^{(X)} = a_n^{(X)} \left[1 + \Delta_L \mathcal{A}_n^{(X)} + \Delta_C \mathcal{A}_n^{(X)} \right]$$

1) $O(p^4)$ χ PT Loops: Large correction [NLO in $1/N_C$]

$$\Delta_L \mathcal{A}_{1/2}^{(8)} = 0.27 \pm 0.05 + 0.47 i \quad ;$$

$$\Delta_L \mathcal{A}_{1/2}^{(27)} = 1.02 \pm 0.60 + 0.47 i \quad ; \quad \Delta_L \mathcal{A}_{3/2}^{(27)} = -0.04 \pm 0.05 - 0.21 i$$

$$\Delta_L \mathcal{A}_{1/2}^{(g)} = 0.27 \pm 0.05 + 0.47 i \quad ; \quad \Delta_L \mathcal{A}_{3/2}^{(g)} = -0.50 \pm 0.20 - 0.21 i$$

Pallante-Pich-Scimemi

2) All local $O(p^4)$ couplings fixed at $N_C \rightarrow \infty$ \rightarrow $\Delta_C \mathcal{A}_n^{(X)}$

Small correction to $O(p^2)$ results

3) Isospin Breaking: $O[(m_u - m_d)p^2, e^2 p^2]$ Sizeable corrections

Cirigliano-Ecker-Neufeld-Pich

4) $\text{Re}(g_8), \text{Re}(g_{27}), \chi_0 - \chi_2$ fitted to data

$$\frac{\varepsilon'_K}{\varepsilon_K} \sim \left[\frac{105 \text{ MeV}}{m_s(2 \text{ GeV})} \right]^2 \left\{ B_6^{(1/2)} (1 - \Omega_{\text{eff}}) - 0.4 B_8^{(3/2)} \right\}$$

Delicate Cancellation. Strong Sensitivity to:

- m_s (quark condensate) $m_s(2 \text{ GeV}) = 105 \pm 20 \text{ MeV}$
- Isospin Breaking ($m_u \neq m_d, \alpha$) $\Omega_{\text{eff}} = 0.06 \pm 0.08$
Cirigliano-Ecker-Neufeld-Pich
- Penguin Matrix Elements

χ PT Loops (FSI): $B_{6,\infty}^{(1/2)} \times (1.35 \pm 0.05)$; $B_{8,\infty}^{(3/2)} \times (0.54 \pm 0.20)$

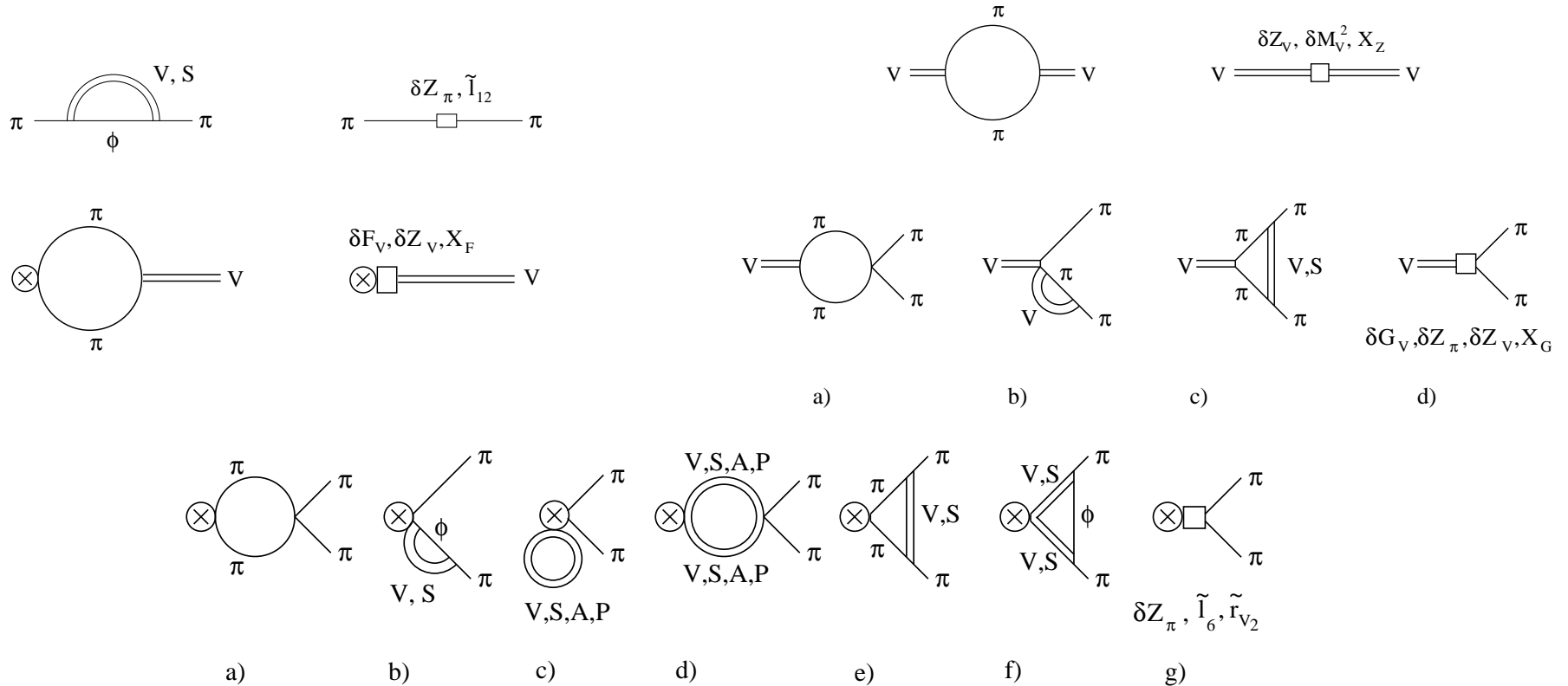
Pallante-Pich-Scimemi '01: $\text{Re}(\varepsilon'/\varepsilon) = (18 \pm 2_{\mu} \pm 8_{m_s} \pm 5_{1/N_C}) \times 10^{-4}$

Experimental world average: $\text{Re}(\varepsilon'/\varepsilon) = (16.7 \pm 1.6) \times 10^{-4}$

Challenge: Control of subleading $1/N_C$ corrections to χ PT couplings

QUANTUM LOOPS IN $R_{\chi T}$: VFF

I. Rosell, J.J. Sanz-Cillero, A.P., arXiv:hep-ph/0407240



- $1/N_C$ Expansion
- Renormalization: Counterterms
- Ultraviolet Behaviour

COUNTERTERMS

$$\tilde{\mathcal{L}}_{4\chi} = \frac{i\tilde{\ell}_6}{4} \langle f_+^{\mu\nu} [u_\mu, u_\nu] \rangle - \tilde{\ell}_{12} \langle \nabla^\mu u_\mu \nabla^\nu u_\nu \rangle \quad ; \quad \tilde{\mathcal{L}}_{6\chi} = i\tilde{c}_{51} \langle \nabla^\rho f_+^{\mu\nu} [h_{\mu\rho}, u_\nu] \rangle + i\tilde{c}_{53} \langle \nabla_\mu f_+^{\mu\nu} [h_{\nu\rho}, u^\rho] \rangle$$

$$F_V(s) = 1 + \frac{F_V G_V}{f^2} \frac{s}{M_V^2 - s} - \tilde{\ell}_6 \frac{s}{F^2} + \tilde{r}_{V2} \frac{s^2}{F^4}$$

$$\lim_{s \rightarrow \infty} F_V(s) = 0 \quad \xrightarrow{N_C \rightarrow \infty} \quad F_V G_V = f^2 \quad , \quad \tilde{\ell}_6 = 0 \quad , \quad \tilde{r}_{V2} \equiv 4f^2 (\tilde{c}_{53} - \tilde{c}_{51}) = 0$$

$$\mathcal{L}_{4Z} = \frac{X_{Z_1}}{2} \langle \nabla^2 V^{\mu\nu} \{ \nabla_\nu, \nabla^\sigma \} V_{\mu\sigma} \rangle + \dots$$

$$X_Z \left\{ \frac{M_V^4}{2} \langle V^{\mu\nu} V_{\mu\nu} \rangle + \frac{M_V^2 F_V}{\sqrt{2}} \langle V_{\mu\nu} f_+^{\mu\nu} \rangle + \dots \right\}$$

$$\mathcal{L}_{4F} = X_{F_1} \langle V_{\mu\nu} \nabla^2 f_+^{\mu\nu} \rangle + \dots$$

EOM

$$-X_F \{ M_V^2 \langle V_{\mu\nu} f_+^{\mu\nu} \rangle + \dots \}$$

$$\mathcal{L}_{4G} = i X_{G_1} \langle \{ \nabla^\alpha, \nabla_\mu \} V^{\mu\nu} [u_\nu, u_\alpha] \rangle + \dots$$

$$-X_G \{ 2i M_V^2 \langle V^{\alpha\nu} [u_\alpha, u_\nu] \rangle + \dots \}$$

$$\tilde{\ell}_6^{\text{eff}} = \tilde{\ell}_6 + 2 X_Z F_V G_V - 2\sqrt{2} X_F G_V - 4\sqrt{2} X_G F_V \quad , \quad F_V^{\text{eff}} = F_V + 2 X_Z M_V^2 F_V - 2\sqrt{2} X_F M_V^2$$

$$G_V^{\text{eff}} = G_V + 2 X_Z M_V^2 G_V - 4\sqrt{2} X_G M_V^2 \quad , \quad (M_V^2)^{\text{eff}} = M_V^2 + 2 X_Z M_V^4 \quad , \quad \tilde{r}_{V2}^{\text{eff}} = \tilde{r}_{V2}$$

Low-Energy Limit: χ PT

$$F_V(s) = 1 - \frac{s}{f^2} \left\{ \ell_6^r(\mu) + \frac{1}{96\pi^2} \left[\ln \left(-\frac{q^2}{\mu^2} \right) - \frac{5}{3} \right] \right\} + \frac{s^2}{F^4} \left\{ r_{V2}^r(\mu) + \dots \right\} + \mathcal{O} \left(\frac{s^3}{F^6} \right)$$

$$-\frac{1}{96\pi^2} \left[\bar{\ell}_6 + \log \frac{m_\pi^2}{M_V^2} \right] \equiv \ell_6^r(\mu) + \frac{1}{96\pi^2} \ln \frac{M_V^2}{\mu^2} = -\frac{f^2}{M_V^2} + \hat{\ell}_6 + \frac{1}{96\pi^2} \left[\ln \frac{M_S^2}{4M_V^2} + \frac{13}{6} \right]$$

$$r_{V2}^r(\mu) - \frac{f^2}{192\pi^2 M_V^2} \left(5 - \frac{M_V^2}{M_S^2} \right) \ln \frac{M_V^2}{\mu^2} =$$

$$\frac{f^4}{M_V^4} + \hat{r}_{V2} + \frac{f^2}{192\pi^2 M_V^2} \left\{ \left(1 - \frac{M_V^2}{M_S^2} \right) \ln \frac{M_S^2}{M_V^2} - \ln 2 - \frac{64}{15} + \frac{9}{5} \frac{M_V^2}{M_S^2} \right\}$$

$$\bar{\ell}_6 = 16.0 \pm 0.5 \pm 0.7 \quad , \quad r_{V2}^r(M_\rho) = (1.6 \pm 0.5) \cdot 10^{-4}$$

Bijnens-Colangelo-Talavera

$$\Rightarrow \hat{\ell}_6 = (-0.2 \pm 0.9) \cdot 10^{-3} \quad , \quad \hat{r}_{V2} = (-0.2 \pm 0.5) \cdot 10^{-4}$$

$$\ell_6^{N_c \rightarrow \infty} = -f^2/M_V^2 = -0.014 \quad , \quad r_{V2}^{N_c \rightarrow \infty} = f^4/M_V^4 = 2.1 \cdot 10^{-4}$$

$$F_V(s) = A(s) \frac{M_V^2}{M_V^2 - s - \Sigma_V^r(s)} + B(s)$$

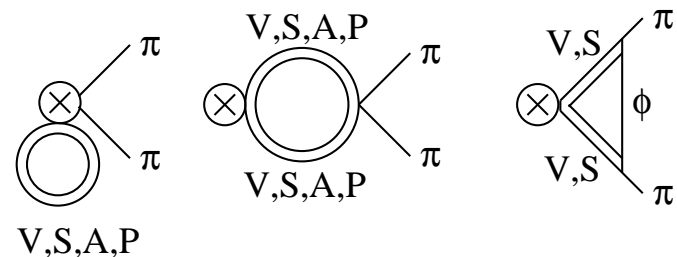
High-Energy Limit: $s \rightarrow \infty$

$$A(s) = \frac{M_V^2}{16\pi^2 F^2} \left\{ -\ln \frac{-s}{M_V^2} \left[\ln \frac{s}{M_V^2} - 2 \right] + \frac{1}{2} \ln^2 \frac{s}{M_V^2} \right\} + \mathcal{O}(1)$$

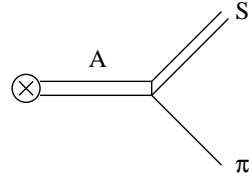
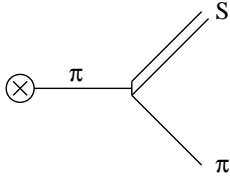
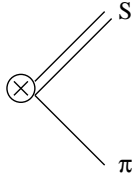
$$\Sigma_V^r(s) = \frac{-s^2}{96\pi^2 F^2} \left\{ \ln \frac{-s}{\mu^2} - \frac{5}{3} + 192\pi^2 F^2 X_Z^r(\mu) \right\}$$

$$B(s) = \frac{1}{16\pi^2 F^2} \left\{ -s^2 \left[\frac{1}{6} \left(\frac{1}{M_V^2} + \frac{1}{M_A^2} \right) \left(\ln \frac{-s}{\mu^2} - \frac{2}{3} \right) - \frac{16\pi^2}{F^2} \tilde{r}_{V2}^r(\mu) \right] + \frac{s}{3} \ln \frac{-s}{\mu^2} + \mathcal{O}(s) \right\}$$

BAD behaviour generated
by the 2-Resonance cuts



- $\langle S_{I=0}^0 \pi^- | \bar{d} \gamma_\mu \gamma_5 u | 0 \rangle = -2i \mathcal{F}_{S\pi}(q^2) \left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right) p_\pi^\nu \quad , \quad q^\mu = (p_\pi + p_S)^\mu$

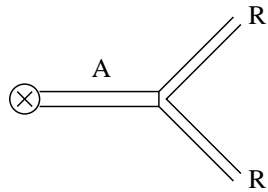
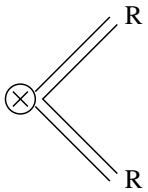


$$\mathcal{L}_{SA}^2 = \lambda_1^{SA} \langle \{ \nabla^\mu S, A_{\mu\nu} \} u^\nu \rangle$$

$$\lambda_1^{SA} = -\frac{\sqrt{2} c_d}{F_A} = -\frac{1}{\sqrt{2}}$$

$$\mathcal{F}_{S\pi}(q^2) = \frac{2c_d}{F} - \sqrt{2} \lambda_1^{SA} \frac{F_A}{F} \frac{q^2}{M_A^2 - q^2} = \frac{M_A^2}{M_A^2 - q^2}$$

- $\langle R_{I=1}^0(p_1) R^-(p_2) | \bar{d} \gamma^\mu u | 0 \rangle = \sqrt{2} (p_2 - p_1)^\mu \mathcal{F}_{RR}(q^2) \quad , \quad R = S, P$



$$\mathcal{L}_{VRR}^2 = i \lambda^{VRR} \langle V^{\mu\nu} \nabla_\mu R \nabla_\nu R \rangle$$

$$\lambda^{VRR} = \frac{\sqrt{2}}{F_V} = \frac{1}{F}$$

$$\mathcal{F}_{RR}(q^2) = 1 + \frac{F_V}{\sqrt{2}} \lambda^{VRR} \frac{q^2}{M_V^2 - q^2} = \frac{M_V^2}{M_V^2 - q^2}$$

SUMMARY

- $N_C \rightarrow \infty$ provides a sensible approximation to the mesonic world
- Useful tool for quantitative non-perturbative analyses at low energies
- Some observables/physics only appear at **subleading topologies**
 - $U(1)_A$ Anomaly
 - Anomalous dimensions of four-quark (non-penguin) operators
 - Short-distance logarithms
 - χ PT loops (Final State Interactions)
 - Long-distance logarithms
- Successful leading-order phenomenology

Challenge: Rigorous control of subleading corrections