2-D incompressible flow in porous media

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THE PROBLEMS

We present the study of a nonlinear 2-D mass balance equation in porous media (Bear 1972, Nield-Bejan 1999). 2DPM is

$$\frac{D\rho}{Dt} \equiv \frac{\partial \rho}{\partial t} + v \cdot \nabla \rho = 0,$$
$$v = -\left(\nabla p + \gamma \rho\right),$$
$$\nabla \cdot v = 0.$$

Consider a stream function $\psi(x,t)$ such that $v = \nabla^{\perp}\psi \equiv \left(-\frac{\partial \psi}{\partial x_2}, \frac{\partial \psi}{\partial x_1}\right)$ and satisfies

$$-\Delta \psi = \frac{\partial \rho}{\partial x_1} \text{ and } -\Delta p = \frac{\partial \rho}{\partial x_2}$$

v can be recovered from ψ by

$$v(x,t) = -\frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{x_1 - y_1}{|x-y|^2} \nabla^{\perp}\rho(y,t)dy = \mathrm{SI}(\rho).$$

Thus, our model 2DPM can be written as

$$\frac{\partial \rho}{\partial t} + \text{ SI}(\rho) \cdot \nabla \rho = 0,$$

The problems that we study are:

1. Solutions with infinite energy

2. Solutions with finite energy

3. Interface dynamics

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SINGULARITIES WITH INFINITE ENERGY

Let the stream function ψ be defined by

$$\psi(x_1, x_2, t) = x_2 f(x_1, t) + g(x_1, t),$$

with f and g 1-periodic with respect x_1. Thus, ρ satisfies

$$\rho(t, x_1, x_2) = -x_2 \frac{\partial f}{\partial x_1}(x_1, t) - \frac{\partial g}{\partial x_1}(x_1, t),$$

and the system 2DPM under is equivalent to IPM

$$(f_x)_t = f f_{xx} - (f_x)^2,$$
$$(g_x)_t = f g_{xx} - f_x g_x.$$

Existence: Let $f^0 = f(x,0)$ and $g^0 = g(x,0)$ satisfy $f^0_x, g^0_x \in H^k_0(0,1)$ with $k \geq 1$. Then, there exists $T > 0$ such that

$$f_x, g_x \in C^1([0,T]; H^k_0(0,1))$$

are the unique solution of IPM.

Blow up results

1. Initial data satisfies $f^0_x \in H^2_0(0,1)$ and $\min_x f^0_x < 0$. Then, $\|f_x\|_{L^\infty}$ blows up in finite time $T = -1/\min_x f^0_x$.

2. Let $f^0_x \in H^1_0$ Then,

$$c(t) = \int_0^1 f_x \text{ blows up in finite time if } \int_0^1 f^0_x \leq 0$$

.

3. Let $x_1 = x_t$ be the point such that

$$f_x(x_t, t) = \min_x f_x(x, t),$$

and consider $x_2 = 1 - \frac{g_x(x_t,t)}{f_x(x_t,t)}$. Then, since

$$\rho(x_1, x_2, t) = -f_x(x_t, t)$$

blows up in finite time

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FINITE ENERGY

Local existence

Reformulate the system as an integro-differential equation for the particle trajectories

$$\frac{d\Phi}{dt}(\alpha, t) = v(\Phi(\alpha, t), t), \qquad \Phi(\alpha, t)|_{t=0} = \alpha.$$

Using Picard Theorem the local in time existence follows. Let $\nabla^{\perp}\rho_0 \in C^{\delta}(\mathbb{R}^2)$, $\delta \in (0, 1)$ and $\mathbf{B} = \{\Phi : \quad |\Phi(0)| + |\nabla_{\alpha}\Phi|_0 + |\nabla_{\alpha}\Phi|_{\delta} < \infty\}$. The mapping $v(\Phi)$ satisfies the assumptions of the Picard theorem. As a consequence, for any bounded open $\mathcal{O} \subset \mathbf{B}$ there exists $T(\mathcal{O}) > 0$ and a unique solution

$$\Phi \in C^1((-T(\mathcal{O}), T(\mathcal{O})); \mathcal{O})$$

The 2DPM has quantities conserved in time, the L^p norm of ρ for $1 \leq p \leq \infty$, i.e.,

$$\|\rho(t)\|_p = \|\rho_0\|_p, \qquad \forall t > 0, \quad 1 \leq p \leq \infty.$$

The operators associated with v are singular integrals with Calderón-Zygmund kernels. Then for $1 < p < \infty$ the L^p norm of the velocity is bounded for any time $t > 0$.

Blow-up criterion

Let ρ be the solution of equation 2DPM with initial data $\rho_0 \in H^s(\mathbb{R}^2)$ with $s > 2$. Then, the following are equivalent:

(A) The interval $[0, \infty)$ is the maximal interval of H^s existence for ρ.

(B) The quantity

$$\int_0^T \|\nabla \rho\|_{BMO}(t)\, dt < \infty \qquad \forall T > 0$$

Beale, Kato, Majda 84.

Using that

$$\|\nabla\rho\|_{BMO} \leq C\|\nabla\rho\|_{L^\infty},$$

we get a blow-up characterization for numerical simulations.

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Geometric constrains on singular solutions

The infinitesimal length of a level set for ρ is $|\nabla^{\perp}\rho|$ and from

$$\frac{D\nabla^{\perp}\rho}{Dt} = (\nabla v)\nabla^{\perp}\rho,$$

the evolution of $|\nabla^{\perp}\rho|$ is given by

$$\frac{D|\nabla^{\perp}\rho|}{Dt} = \mathcal{L}\,|\nabla^{\perp}\rho|.$$

The factor $\mathcal{L}(x,t)$ is defined through by

$$\mathcal{L}(x,t) = \begin{cases} \mathcal{D}\eta \cdot \eta, & \eta \neq 0, \\ 0, & \eta = 0. \end{cases}$$

where the direction of $\nabla^{\perp}\rho$ is denoted by

$$\eta = \frac{\nabla^{\perp}\rho}{|\nabla^{\perp}\rho|}$$

and $\mathcal{D}(x,t)$ is the deformation matrix defined by

$$\mathcal{D} = (\mathcal{D}_{ij}) = \left[\frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)\right].$$

1. Analogous to 3D Euler (Constantin, Fefferman, Majda 1996) and to 2DQG (Constantin, Majda, Tabak 1994). Let Ω such that there exists $\delta > 0$

$$\sup_{x\in\overline{\Omega}} \int_0^T \|\nabla\eta(\cdot,t)\|^2_{L^\infty(B_\delta(\Phi(x,t)))} dt < \infty,$$

where $\overline{\Omega} = \{x \in \Omega;\ |\nabla \rho_0(x)| \neq 0\}$, and

$$\int_0^T \|R_j \rho\|_{L^\infty}(t)dt < \infty, \quad j = 1, 2, \quad \forall T > 0,$$

where R_j denotes the Riesz transform in the direction x_j. Then

$$\sup_{\mathcal{O}_T(\Omega)} |\nabla \rho(x,t)| < \infty.$$

2. Now, we present a geometric conserved quantity that relates the curvature of the level sets and $|\nabla^\perp \rho|$ in a similar way as (Constantin 94). We define the curvature of the level sets by

$$\kappa(x,t) = (\eta \cdot \nabla \eta) \cdot \eta^\perp(x,t),$$

The following identity is satisfied

$$\frac{D(\kappa|\nabla^\perp \rho|)}{Dt} = \nabla^\perp \rho \cdot \nabla \beta$$

with

$$\beta(x,t) = (\eta \cdot \nabla v) \cdot \eta^{\perp}(x,t).$$

The integral of the quantity $\kappa|\nabla^{\perp}\rho|$ over a region given by two different level sets is conserved along the time, i.e.

$$\frac{d}{dt}\left(\int_{\{x:\, C_1 \leq \rho(x,t) \leq C_2\}} \kappa|\nabla^{\perp}\rho|\, dx\right) = 0.$$

This can be showed using the equation and integrating by parts. Thus, in the case that $|\nabla^{\perp}\rho|$ is large by the formula the curvature κ is small if the level sets do not oscillate.

In all of our numerical experiments, we find no evidence of level set oscillations. On the contrary, we observe that the level sets are flattering where the gradient of ρ is growing.

Numerical simulations

We present two examples of numerical simulations for solutions of the 2DPM with initial data in a period-cell $[0, 2\pi]^2$.

This numerical method is similar to the scheme developed by E and Shu 1994.

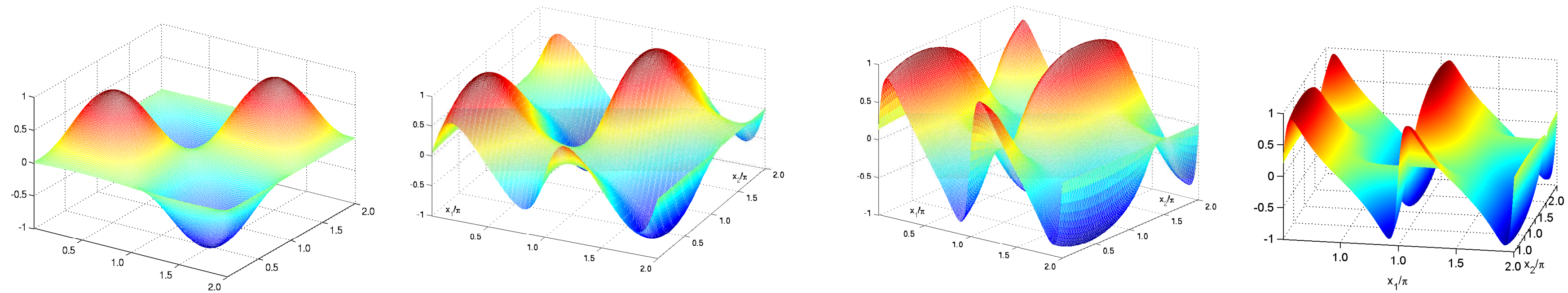
This algorithm is the standard Fourier-collocation method.

We smooth the gradients adding filters to the spectral method.

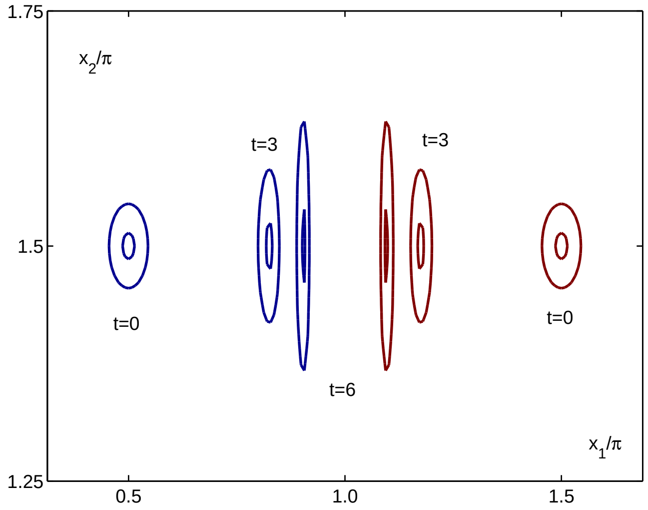
For the temporal discretization, we use Runge-Kutta methods of various order.

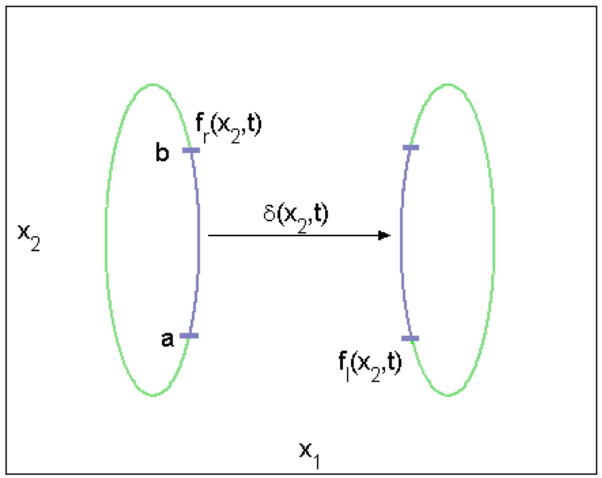
Initial resolution of $(256)^2$ Fourier modes. Preserving the relation space-time, we conclude our numerical simulations with a resolution of $(8192)^2$ Fourier modes.

Evolution of ρ in Case 1: $\rho_0(x_1, x_2) = \sin(x_1)\sin(x_2)$, for $t = 0, 3, 6, 8.5$.



Evolution of the level sets -0.999, -0.99 (on the left) and 0.99, 0.999 (on the right) of ρ in Case 1 ($t = 0, 3, 6$).





In order for the two graphs f_l, f_r to collapse at time T in any interval $x_2 \in [a, b]$, i.e., $\lim_{t \to T_-} [f_r(x_2, t) - f_l(x_2, t)] = 0 \, \forall x_2 \in [a, b]$, it is necessary that

$$\int_0^T \|v\|_\infty(s)ds = \infty.$$

Assume that the minimum and maximum of δ are comparable:

$$\min \delta(x_2,t) \leq c \max \delta(x_2,t) \qquad \forall x_2 \in [a,b].$$

Then we can obtain an evolution equation for the area

$$A(t) = \frac{1}{b-a}\int_a^b [f_r(x_2,t) - f_l(x_2,t)]dx_2$$

satisfies (Córdoba-Fefferman02)

$$\left|\frac{dA}{dt}(t)\right| \leq \frac{C}{b-a} \sup_{a\leq x_2\leq b} |\psi(f_r(x_2,t),x_2,t) - \psi(f_l(x_2,t),x_2,,t)|$$

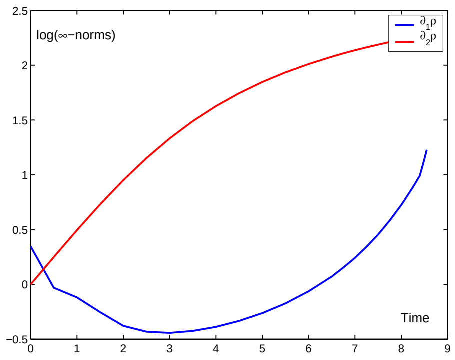
Using that for any $x, y \in \mathbb{R}^2$

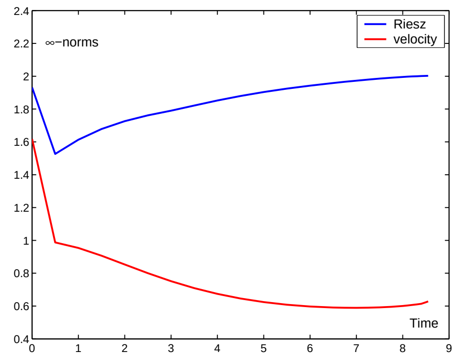
$$|\psi(x,t) - \psi(y,t)| \leq C(\|\rho_0\|_\infty, \|\rho_0\|_{L^2})|x-y|(1 - ln|x-y|),$$

we get that the area $A(t)$ is bounded by

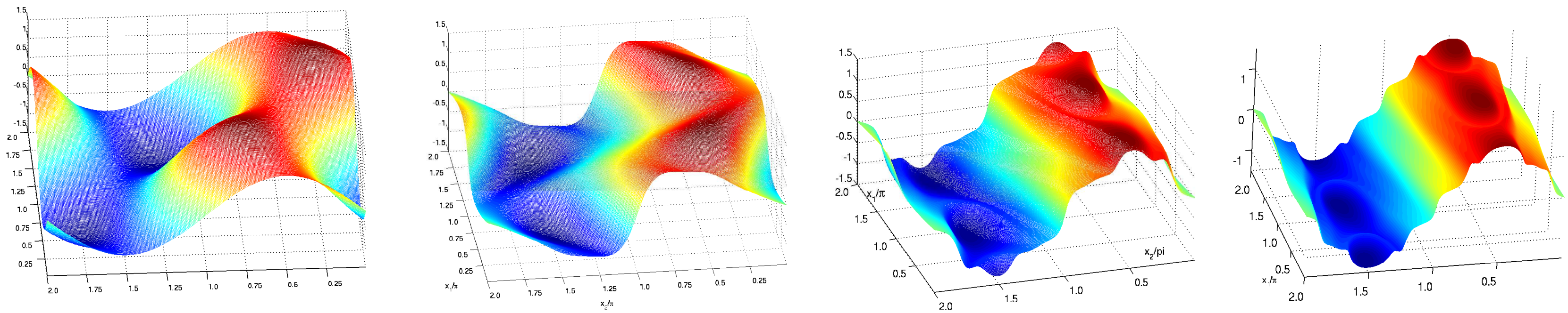
$$A(t) \geq A_0 e^{-Ce^t}.$$

Evolution of the L^∞-norms of $\nabla\rho$, v and the Riesz transforms $(R_1\rho, R_2\rho)$ for Case 1

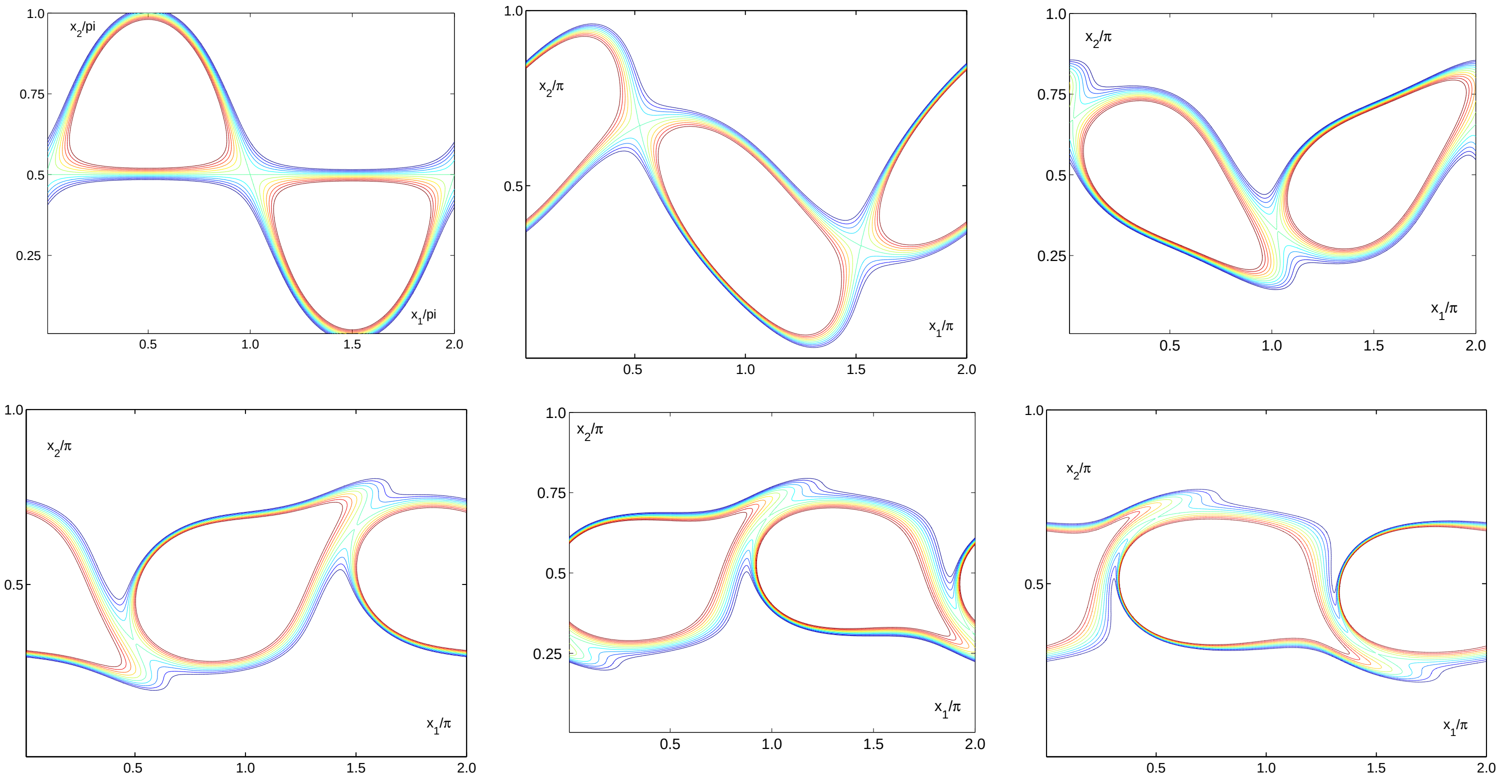




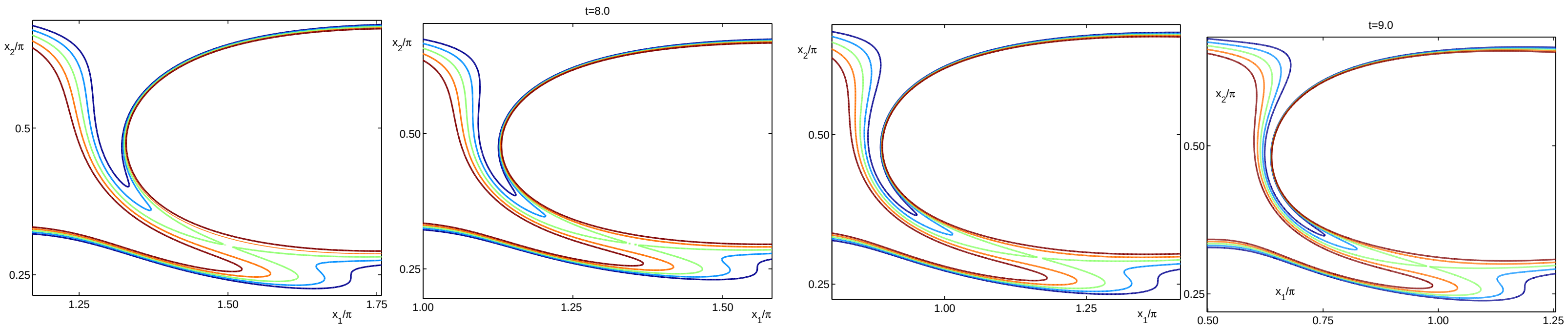
Evolution of the density in Case 2 $\rho(x_1, x_2) = \sin(x_1)\cos(x_2) + \cos(x_1)$ for $t = 0, 3, 6, 9$



Evolution of around $\rho(x_1, x_2) = 1$ in Case 2 for $t = 0, 1.5, 3, 4.5, 6, 7.5$.



Level contour (0.98,0.99,1.0,1.01,1.02) of ρ in Case 2 for times $t = 7.5, 8, 8.5, 9$.



INTERFACE DYNAMICS

The model: two fluids with different densities, so ρ is given by

$$\rho(x,y,t) = \begin{cases} \rho_1, & \{y > f(x,t)\} = \Omega_1(t), \\ \rho_2, & \{y < f(x,t)\} = \Omega_2(t), \end{cases}$$

where $f(x,t)$ is the interface. This is known as Muskat problem (1946), worked by Saffman-Taylor 1958, Escher-Simonett 1997, Siegel-Caflish-Howison 2004, Ambrose 2004.

We consider $x \in \mathbb{R}$ with $f(x,t)$ vanishing at infinity, or $f(x,t)$ periodic with $x \in \mathbb{T}$. We have

$$\nabla^{\perp}\rho = (\rho_2 - \rho_1)(1, \partial_x f(x,t))\delta(y - f(x,t)).$$

and the velocity reads

$$v = -\partial_x \Delta^{-1} \nabla^{\perp} \rho,$$

the velocity can be written as

$$v(x,y,t) = -\frac{\bar{\rho}}{\pi} PV \int_{\mathbb{R}} \frac{(1, \partial_x f(x-\alpha,t))\alpha d\alpha}{\alpha^2 + (y - f(x-\alpha,t))^2},$$

where $\bar{\rho}$ is denoted by

$$\bar{\rho} = \frac{\rho_2 - \rho_1}{2},$$

and the principal value is taken at infinity.

When y approaches $f(x,t)$, the velocity presents a discontinuity in the tangential direction, as the vorticity is concentrated on the interface.

The velocity in the tangential direction does not affect the shape of the interface because only moves the particles on the curve $f(x,t)$ (Hou, Lowengrub, Shelley 94)

We introduce a term of the form $\lambda(x,t)(1,\partial_x f)$ which yields

$$v_1(x, f(x)) = 0,$$

and we obtain the contour equation (CE)

$$\begin{cases} f_t(x,t) = & \dfrac{\bar{\rho}}{\pi} PV \displaystyle\int_{\mathbb{R}} \dfrac{(\partial_x f(x,t) - \partial_x f(x-\alpha,t))\alpha}{\alpha^2 + (f(x,t) - f(x-\alpha,t))^2} d\alpha, \\ \\ f(x,0) = & f_0(x). \end{cases}$$

The unstable case

Let $s > 3/2$, then for any $f_0 \in H^s$, (CE) is ill-possed for $\rho_2 < \rho_1$; i.e. for any $\varepsilon > 0$ there exists a solution f of (CE) and $0 < \delta < \varepsilon$ such that $\|f_0\|_{H^s} \leq \varepsilon$ and $\|f\|_{H^s}(\delta) = \infty$.

Idea: Ignoring the terms of order two in (CE), we have that for small solutions $f(x,t)$ it is satisfied

$$\begin{aligned} f_t(x,t) &= -\bar{\rho}\Lambda f(x,t), \\ f(x,0) &= f_0(x), \end{aligned}$$

where $\Lambda = (-\Delta)^{1/2}$. By taking the Fourier transform, we get

$$\hat{f}(\xi) = \hat{f}_0(\xi) e^{-\bar{\rho}|\xi| t},$$

and we obtain an ill-posed problem for $\bar{\rho} < 0$ with general initial data in Sobolev spaces. Dombre-Pumir-Siggia 1994.

The stable case

Local existence: Let $f_0 \in H^s$ for $s \geq 3$ and $\bar{\rho} > 0$. Then there exists a time $T > 0$ so that there is a unique solution f of (CE) in $C^1([0,T]; H^s)$ such that $f(x,0) = f_0(x)$.

Using an approximation of the equation (CE) given by

$$\begin{aligned} f_t(x,t) = &\ -\bar{\rho}\frac{\Lambda f(x,t)}{1+(\partial_x f(x,t))^2}, \\ f(x,0) = &\ f_0(x). \end{aligned}$$

We have that

$$\frac{d}{dt}\|f\|^2_{H^3}(t) \leq C\|f\|^k_{H^3}(t)$$

for C and $k > 2$ fixed constants. Integrating in time we obtain existence.

Mass conservation: Region U equal to $\mathbb{T}$ or $\mathbb{R}$. By using the equation (CE), we have

$$\begin{aligned}
&\int_U f_t(x,t)\,dx = \\
&= \frac{\bar{\rho}}{\pi}\int_U PV\!\!\int_{\mathbb{R}} \frac{(\partial_x f(x,t)-\partial_x f(x-\alpha,t))\alpha}{\alpha^2+(f(x,t)-f(x-\alpha,t))^2}d\alpha dx \\
&= \frac{\bar{\rho}}{\pi}PV\!\!\int_{\mathbb{R}}\int_U \partial_x \arctan\left(\frac{f(x,t)-f(x-\alpha,t)}{\alpha}\right)dx d\alpha = 0
\end{aligned}$$

Maximum principle: Let $f_0 \in H^s$ with $s \geq 3$, then the solution of the system (CE) satisfies

$$\|f\|_{L^\infty}(t) \leq \|f_0\|_{L^\infty}.$$

Global solution for small initial data:

1. Let $f_0 \in H^s$ with $s \geq 3$, and $\|\partial_x f_0\|_{L^\infty} < 1$. Then the solution of the system (CE) satisfies

$$\|\partial_x f\|_{L^\infty}(t) < 1.$$

2. If the following norm of the initial data is small:

$$\sum |\xi||\hat{f}_0(\xi)|,$$

then there exist global-in-time solution.

Numerical simulations

We study the periodic case. We consider the following contour equation (CEM)

$$f_t(x,t) = \int_{-\pi}^{\pi} \frac{(\partial_x f(x,t) - \partial_x f(x-\alpha,t))2\tan(\frac{\alpha}{2})}{4\tan^2(\frac{\alpha}{2}) + (f(x,t) - f(x-\alpha,t))^2} d\alpha.$$

This equation is not exactly (CE) because we modify the integral equation introducing the tangent function.

An equivalent equation to (CE) in the periodic case is given by

$$\begin{aligned} f_t(x,t) =& \frac{\bar{\rho}}{\pi}\int_{\mathbb{T}} \frac{(\partial_x f(x,t) - \partial_x f(x-\alpha,t))\alpha}{\alpha^2 - (f(x,t) - f(x-\alpha,t))^2} P_f(x,\alpha) d\alpha \\ &+ \frac{\bar{\rho}}{\pi}\int_{\mathbb{T}} (\partial_x f(x,t) - \partial_x f(x-\alpha,t)) Q_f(x,\alpha) d\alpha, \end{aligned}$$

with

$$P_f(x,\alpha) = P(\alpha, f(x,t) - f(x-\alpha,t)),$$
$$Q_f(x,\alpha) = Q(\alpha, f(x,t) - f(x-\alpha,t)).$$

This formula is obtained making a similar analysis as in Stein-Weiss 1971, and using the kernel of the integral operator $-\partial_x \Delta^{-1}$ in the periodic setting

$$K_p(x,y) = -\frac{1}{2\pi}\left(\frac{x}{x^2+y^2}P(x,y) + Q(x,y)\right).$$

We can choose the function P satisfying that

$$\begin{array}{l} P(x,y) \in C_c^\infty(\mathbb{T}\times\mathbb{R}), \quad P \geq 0, \\ \text{supp}\, P \subset \{x^2+y^2 << 1\}, \\ \text{and} \quad P(-x,-y) = P(x,y), \end{array}$$

and the function $Q(x,y)$ belongs to $C_b^\infty(\mathbb{T}\times\mathbb{R})$ and $Q(0,0) = 0$.

Due to the fact that L^∞ norm of the solutions decreases (see principle maximum), and for $\alpha << 1$ we have $2\tan(\alpha/2) \sim \alpha$, then the singular part of the equation (CE) can be approximated by the one given in (CEM). The modification is natural to work with periodic interfaces (see Baker-Meiron-Orzasg 80, H-L-S 94).

The Case 1 is defined by the following initial data

$$f_1(x) = \begin{cases} \sigma(x) & 0 < x < (0.8)\pi \\ \sigma(x) - h(x) & (0.8)\pi < x < (1.2)\pi \\ \sigma(x) & (1.2)\pi < x < 2\pi \end{cases}$$

with

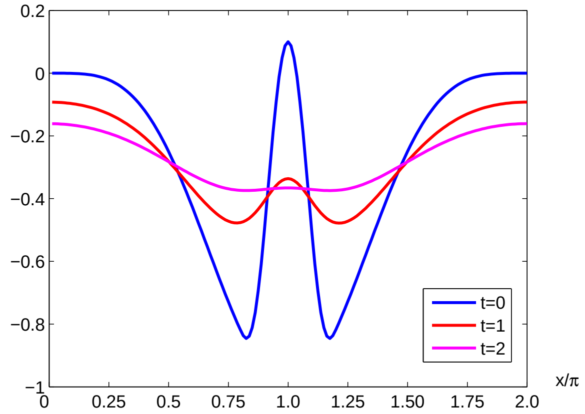
$$\sigma(x) = -\sin^4(x) \text{ and } h(x) = \frac{11}{10}\Big(1 - 5(\frac{x}{\pi} - 1)^2\Big)^3.$$

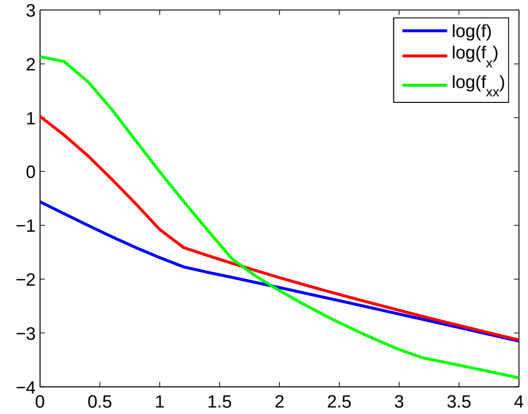
The initial data of Case 2 is given by

$$f_2(x) = \begin{cases} \sin^3(\frac{x-x_0}{2h_0}) & x_0 < x < x_1 \\ 1 & x_1 < x < x_2 \\ \sin^3(\frac{x_3-x}{2h_0}) & x_2 < x < x_3 \\ 0 & \text{otherwise} \end{cases}$$

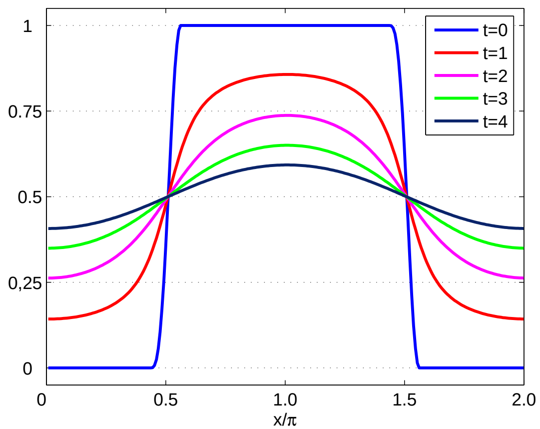
where $x_0 = \pi(1-h_0)/2$, $x_1 = \pi(1+h_0)/2$, $x_2 = \pi(3-h_0)/2$ and $x_3 = \pi(3+h_0)/2$ with $h_0 = 1/16$.

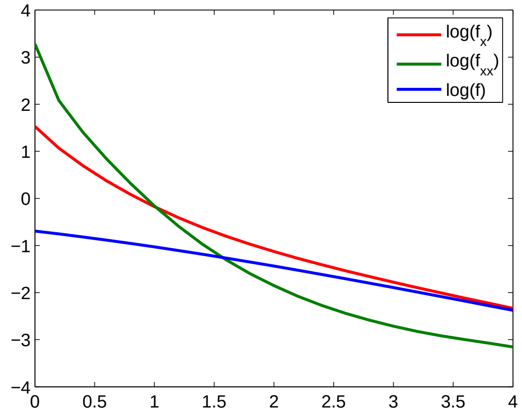
Evolution of f and of the logarithm of L^{∞}-norms in Case1.





Evolution of f and of the logarithm of L^∞-norms in Case 2.





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