Equations with nonlinear boundary conditions in domains with rapidly varying boundaries

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“Partial differential equations, optimal design and numerics”

Benasque Center for Science
September 2007

General problem

Let us cosider the following evolution problem in space X_0

$$\begin{cases} x' + A_0 x = F_0(x), \quad t > 0 \\ x(0) = x_0 \in X_0 \end{cases}$$

Let us assume that appropriate conditions are satisfied so that we have global existence of solutions and contiuous dependence with respect to initial data. Hence, the equation generates a dynamical system in X_0 (phase space):

$$\begin{aligned} T_0(t) : X_0 &\to X_0 \\ x_0 &\to x(t, x_0) \end{aligned}$$

Under certain conditions on the equation we guarantee that T_0 is dissipative and asymptotically compact

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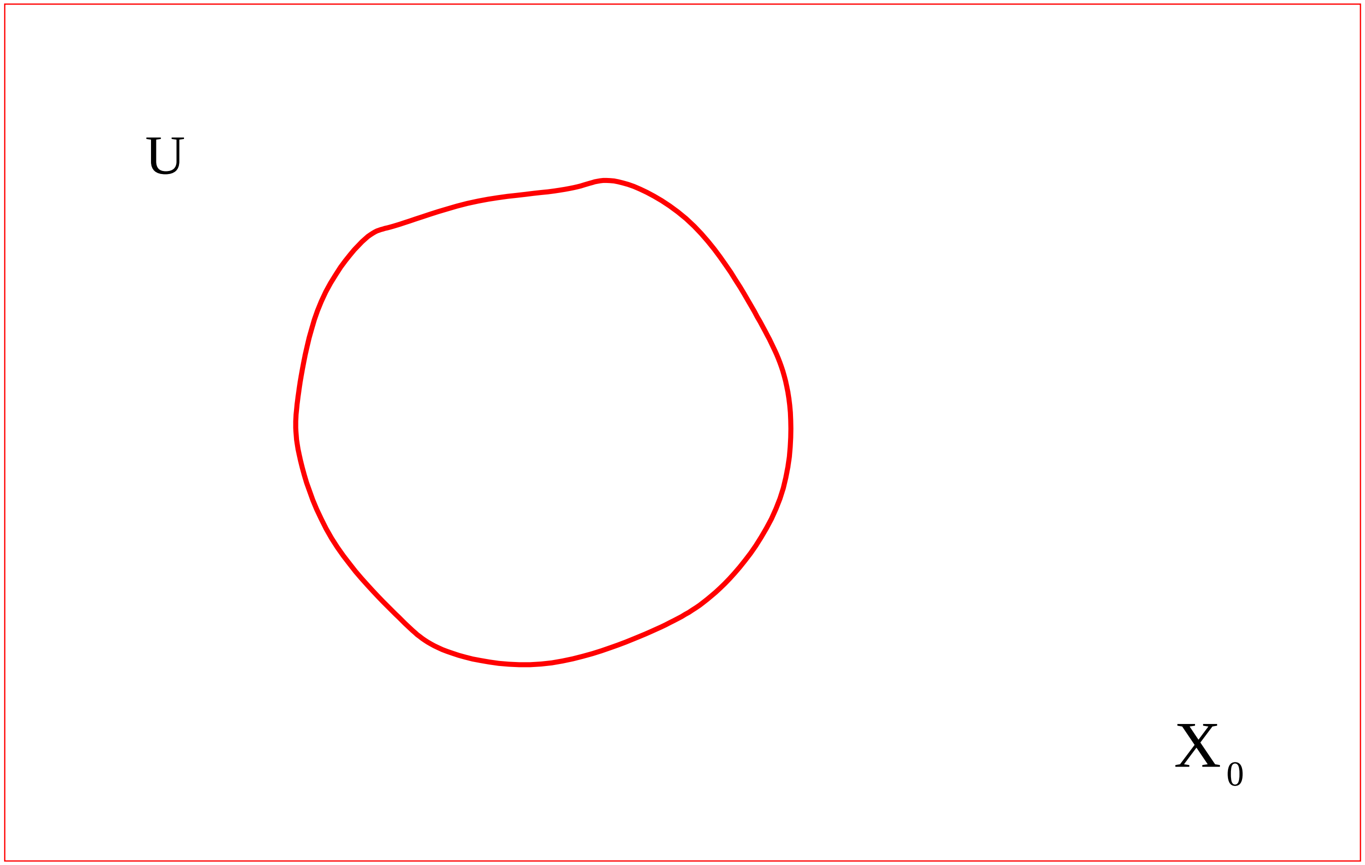
These two conditions guarantee the existence of the attractor of the equation, $\mathcal{A}_0 \subset X_0$.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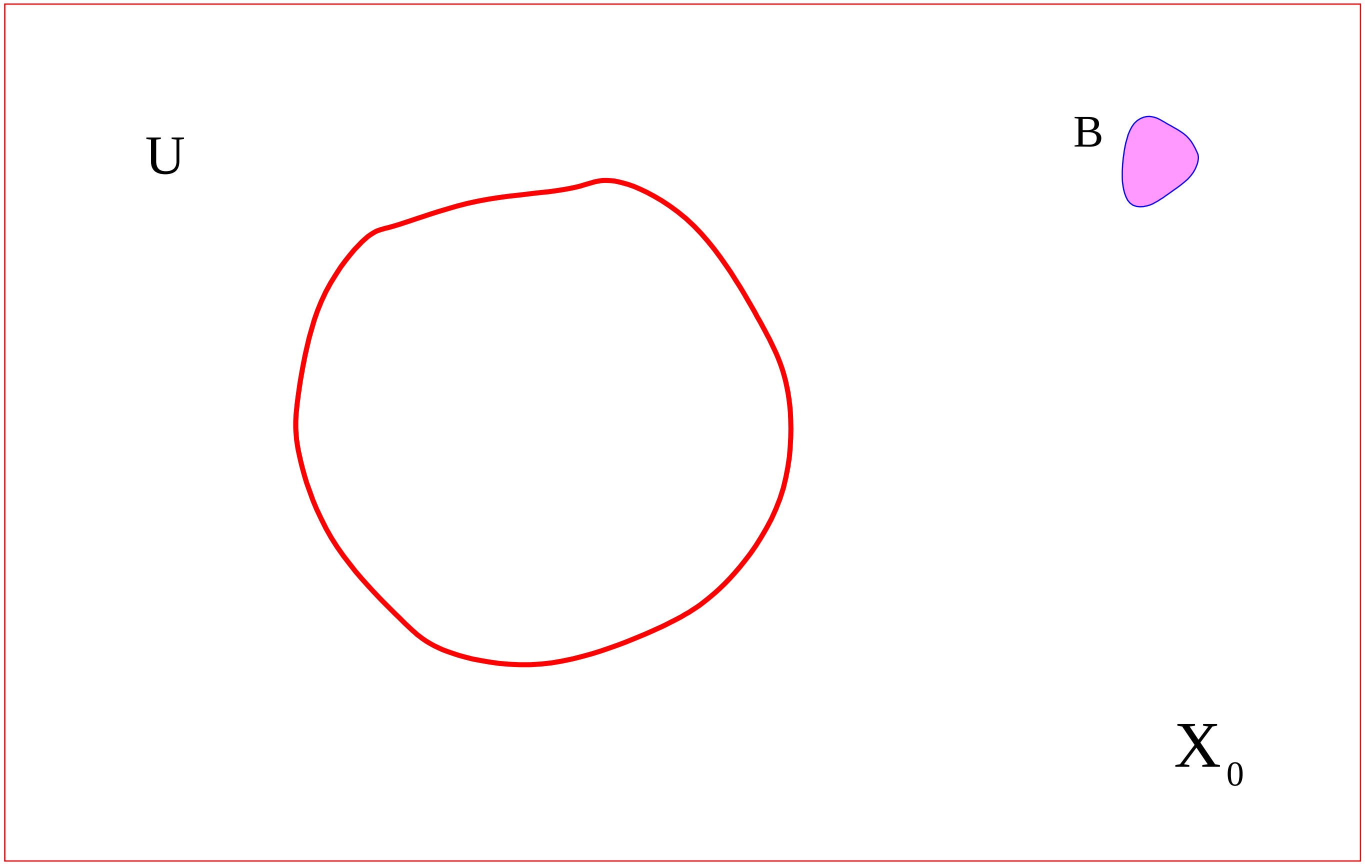
X_0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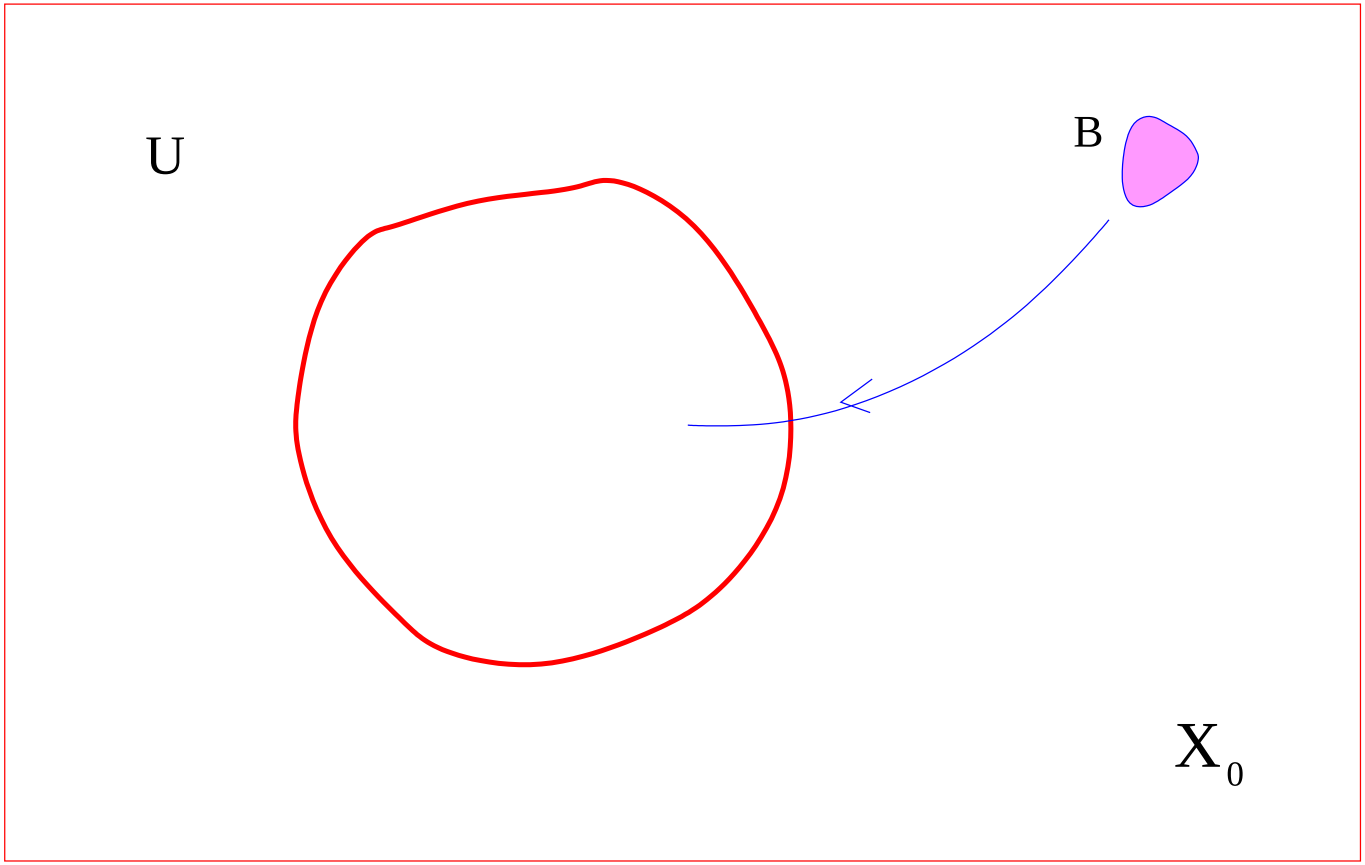
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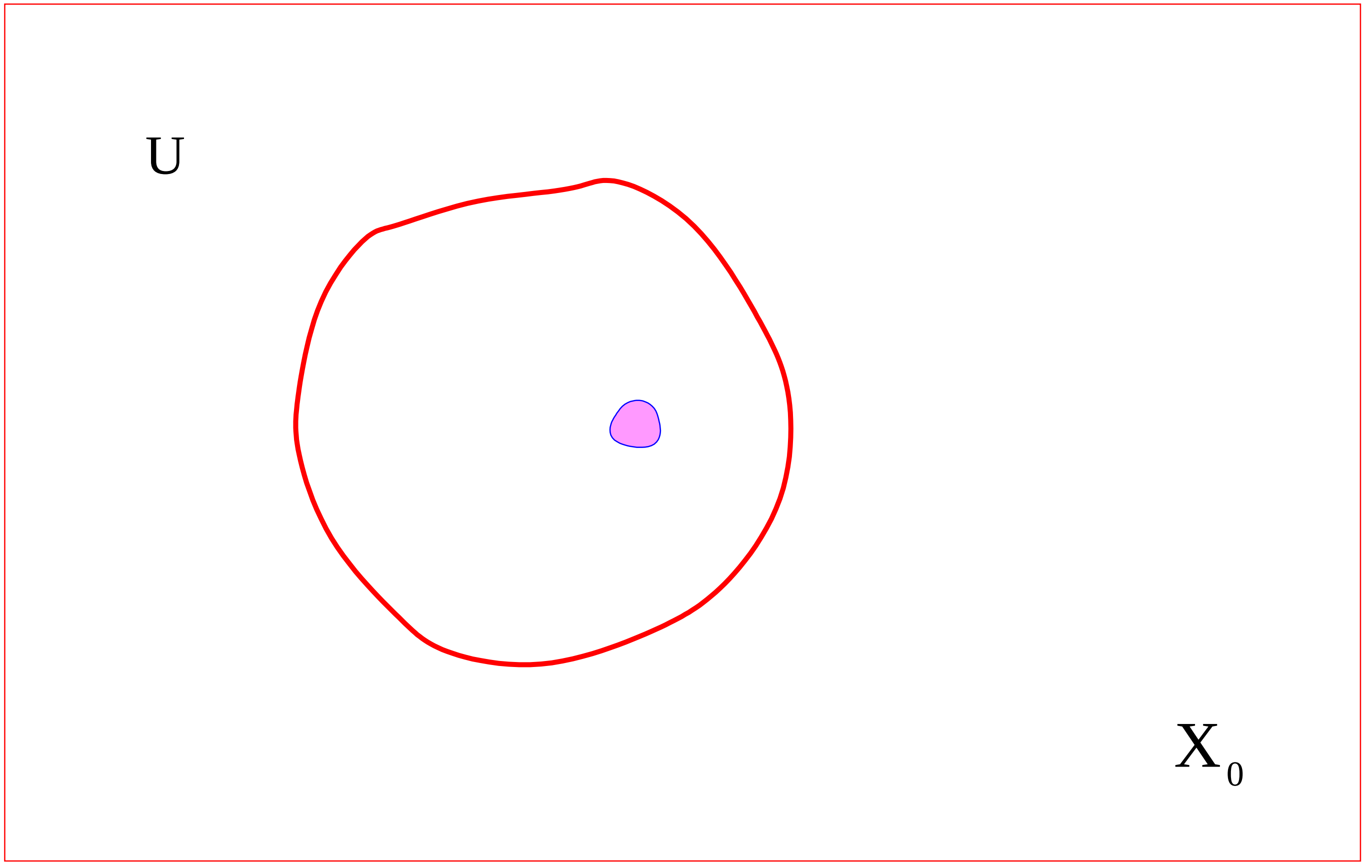
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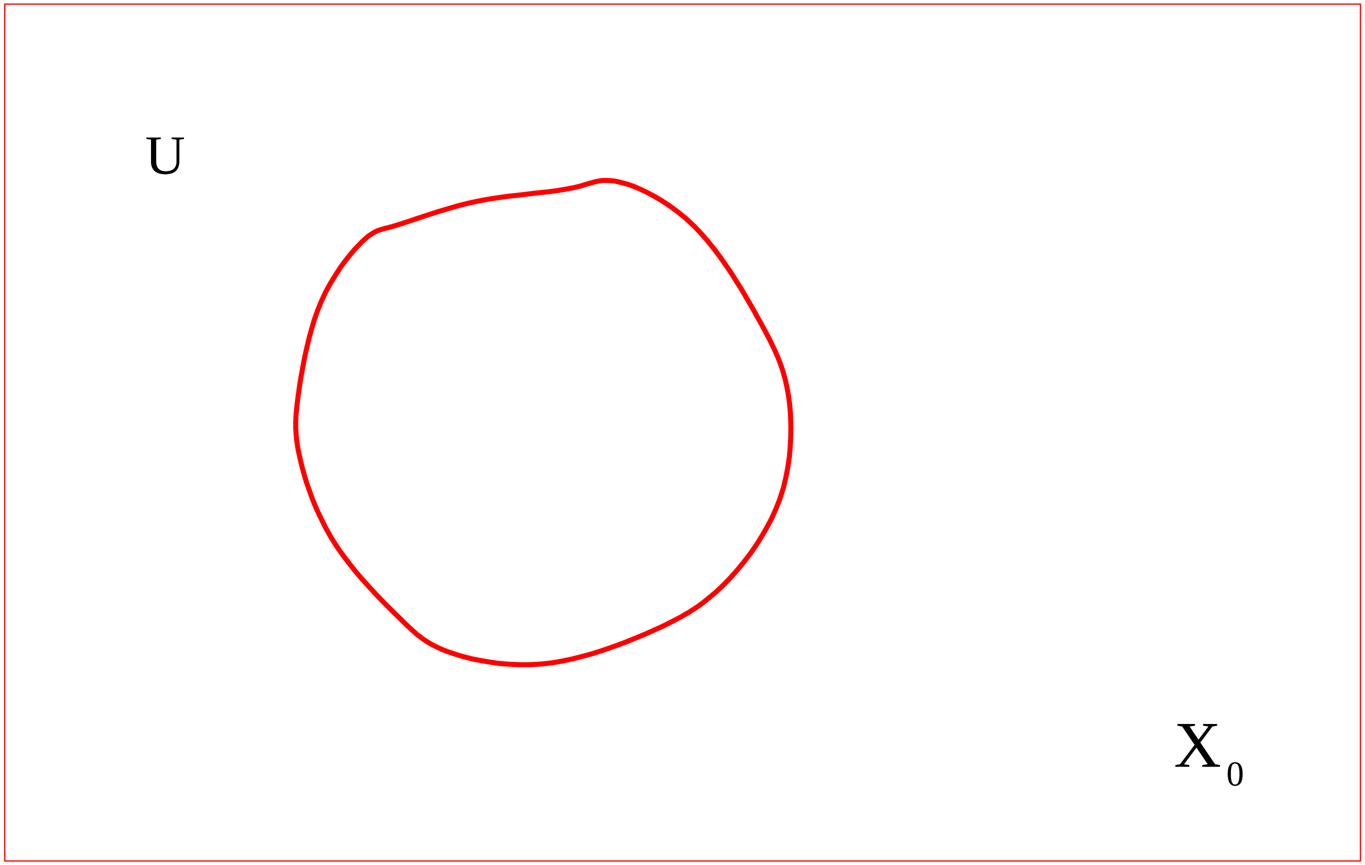
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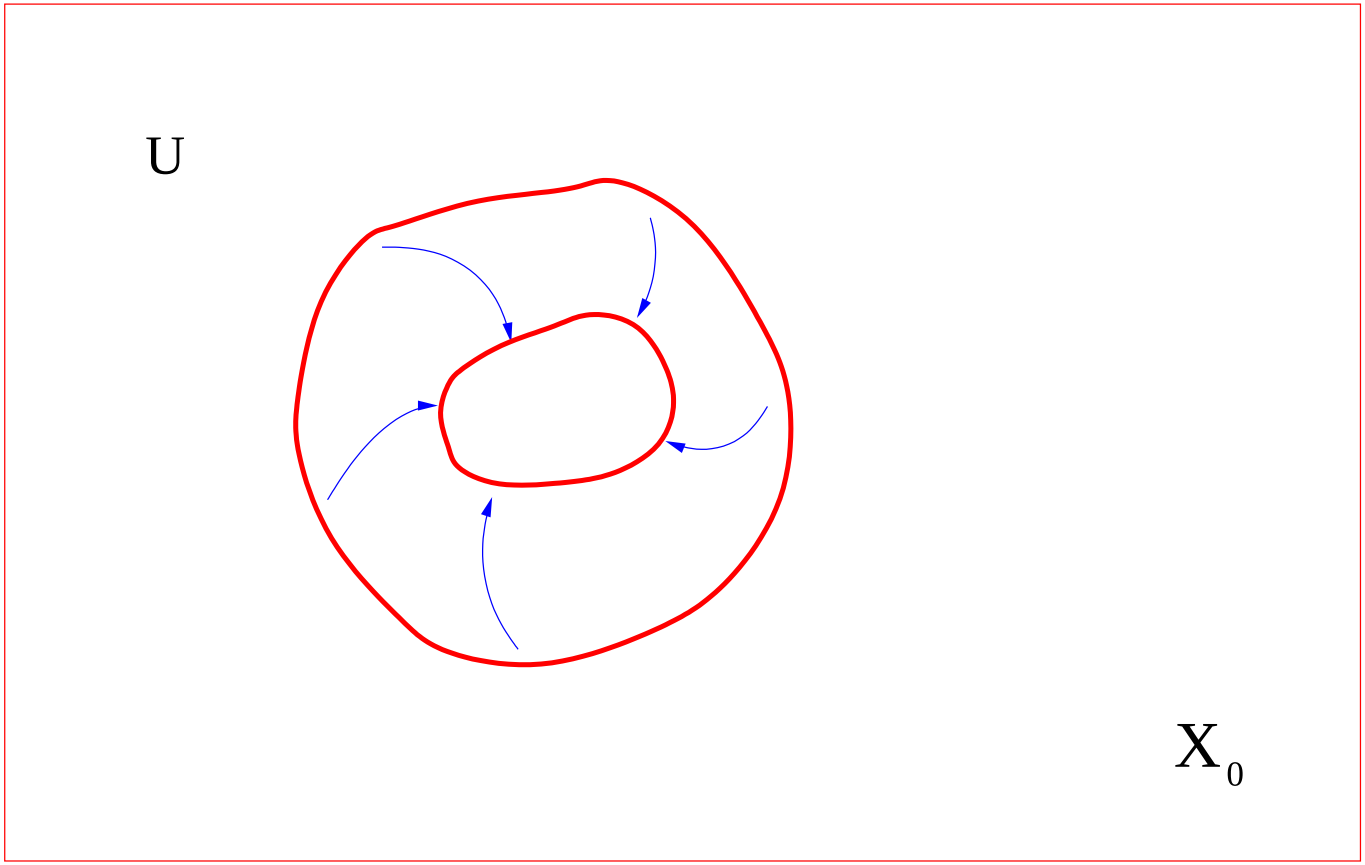
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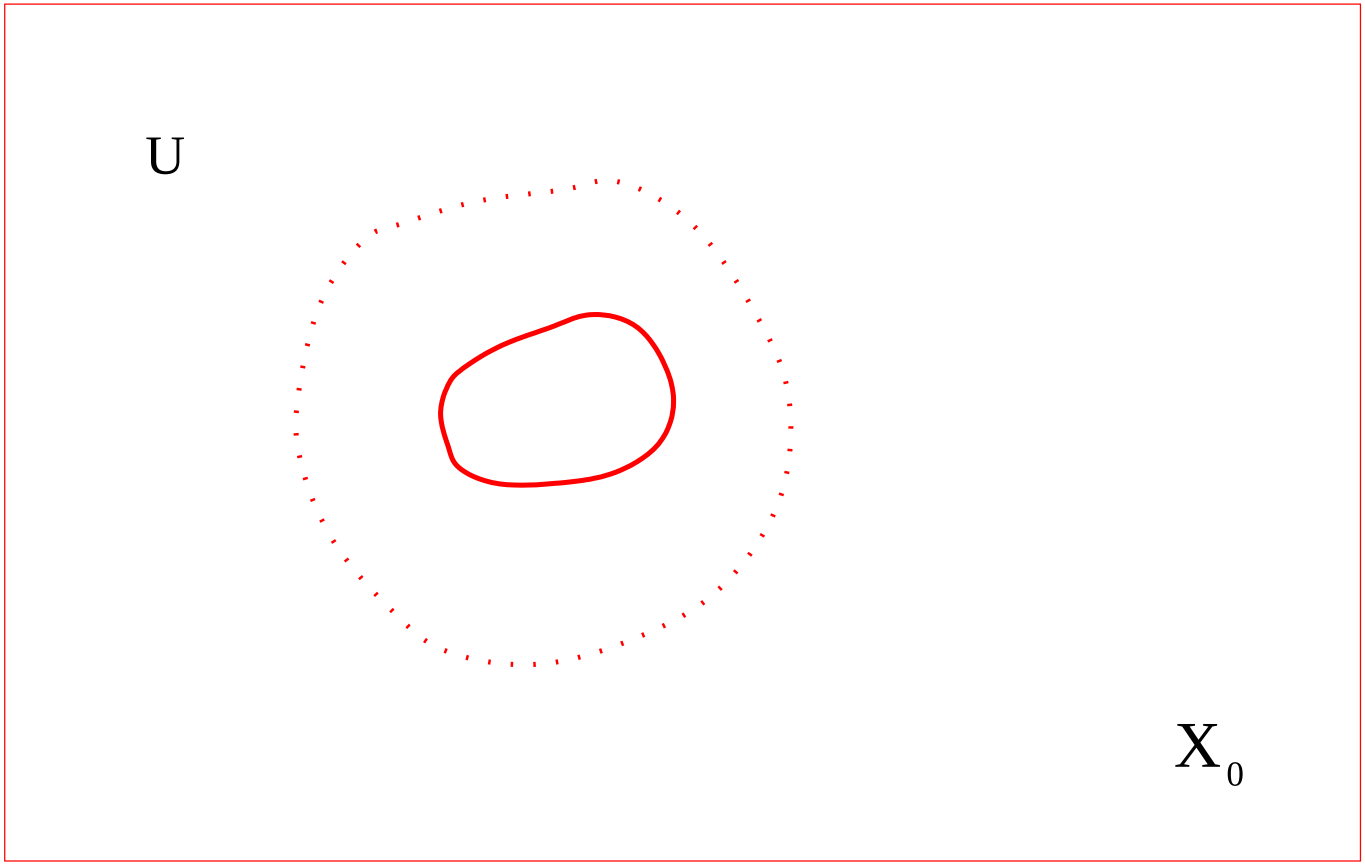
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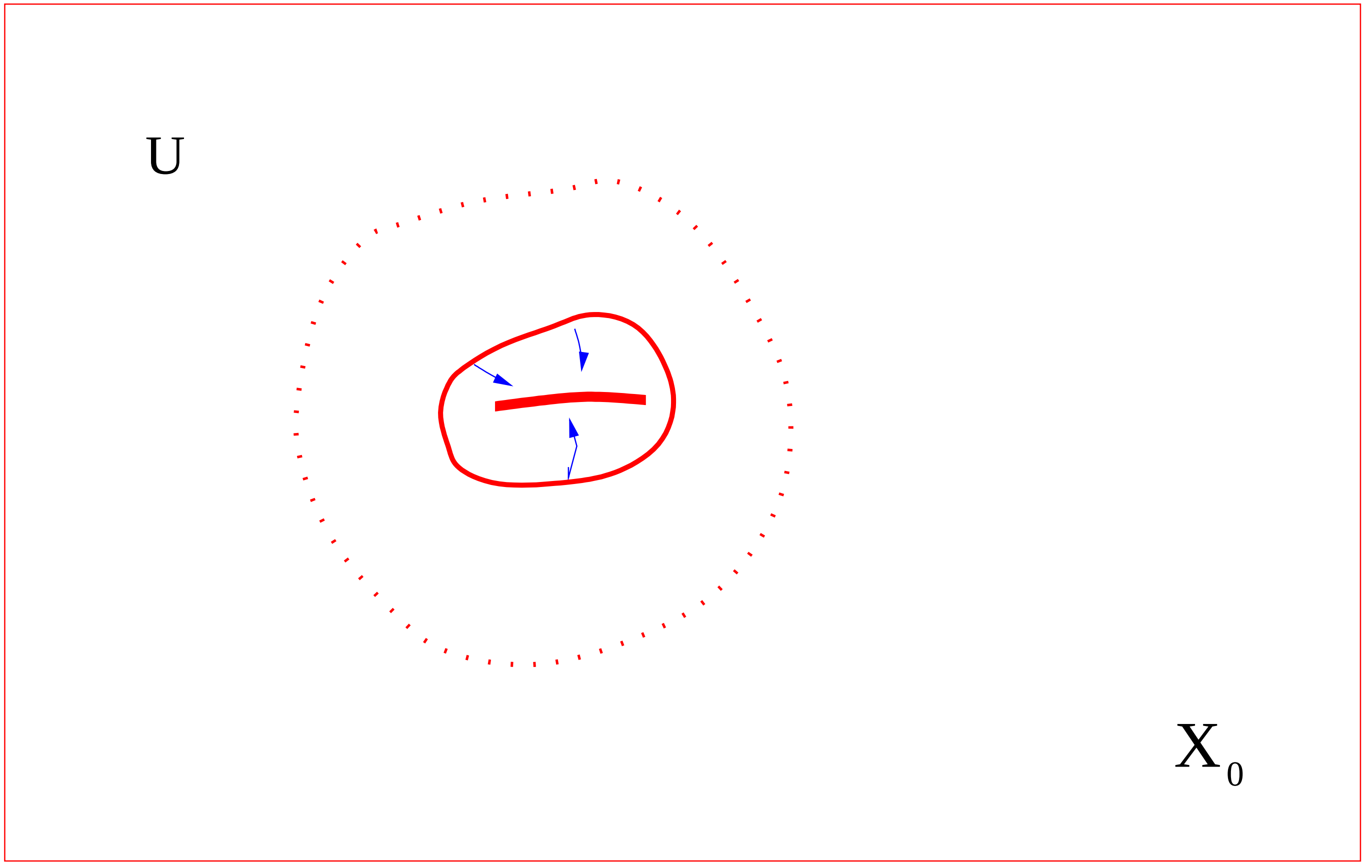
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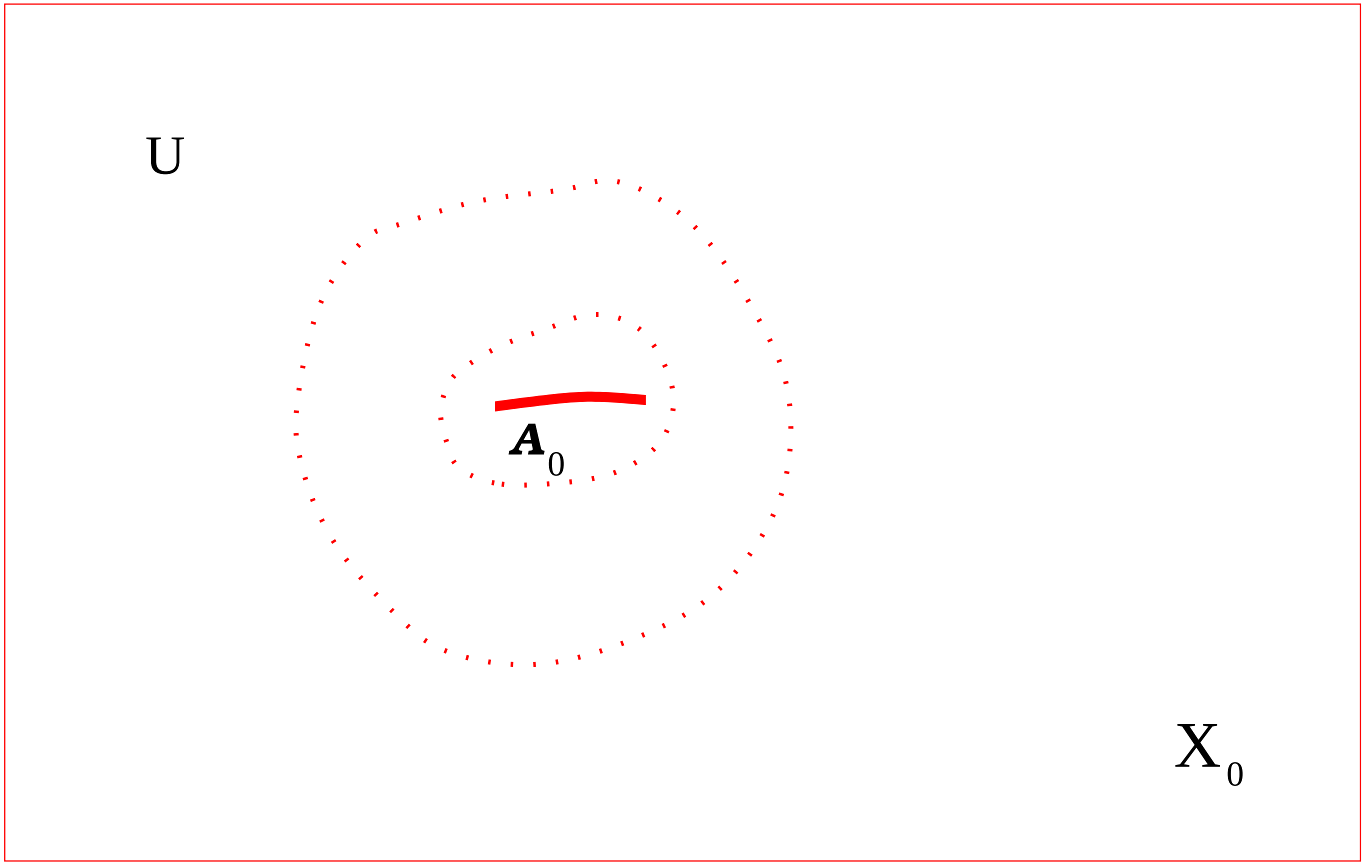
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A_0
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Attractor: largest compact, invariant set which attracts every bounded set of the phase space.

- It contains all global and bounded orbits: equilibria, periodic orbits, conecting orbits, etc ..
- The dynamics in the attractor contains all the asymptotic dynamics.

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J.K. Hale "Asymptotic behavior of dissipative systems" Mathematical Surveys and Monographs **25** American Mathematical Society, Providence 1988.

A. Babin, M.I. Vishik "Attractors of evolution equations" North Holland, 1992

R. Temam "Infinite dimensional dynamical systems in mechanics and physics", Springer 1988

We consider the following problem

$$\begin{cases} x' + A_0 x = F_0(x), \quad t > 0 \\ x(0) = x_0 \in X_0 \end{cases}$$

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Let us consider a perturbed problem ($0 < \epsilon \leq \epsilon_0$)

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Questions:

- What is the relation between attractors $\mathcal{A}_0$ and $\mathcal{A}_\epsilon$?.
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Since we are comparing elements of X_0 with elements of X_ϵ, we need a concept of "closeness" or "convergence" for elements living in different spaces.

If for instance there exists an space Y so that $X_\epsilon \hookrightarrow Y$, $0 \leq \epsilon \leq \epsilon_0$, then we can talk of convergence in Y.

In each case we need to define this concept in a very precise way.

The attractor is a global entity of the dynamical system. Therefore, understanding its structure and its behavior under perturbations is a global problem, which is far away from being resolved in this generality.

The attractor may have a very complicated structure and it is not easy to analyze its behavior under perturbations.

Nevertheless, if the dynamical system is gradient, the attractor structure is simpler. It is made of

- Equilibria.

- Conections among equilibria.

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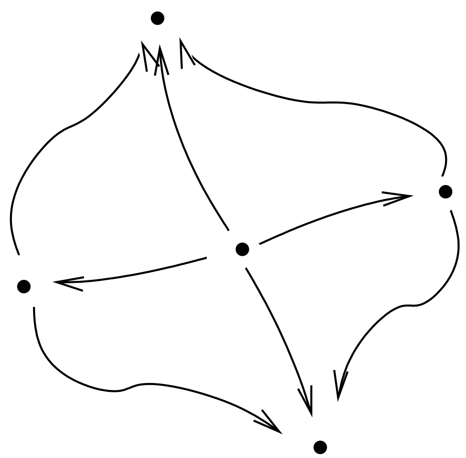

Domain Perturbation

Case 1. General domain perturbation and Neumann boundary conditions.

$$\begin{cases} u_t - \Delta u = f(x,u) & \text{in } \Omega_\epsilon \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega_\epsilon. \end{cases}$$

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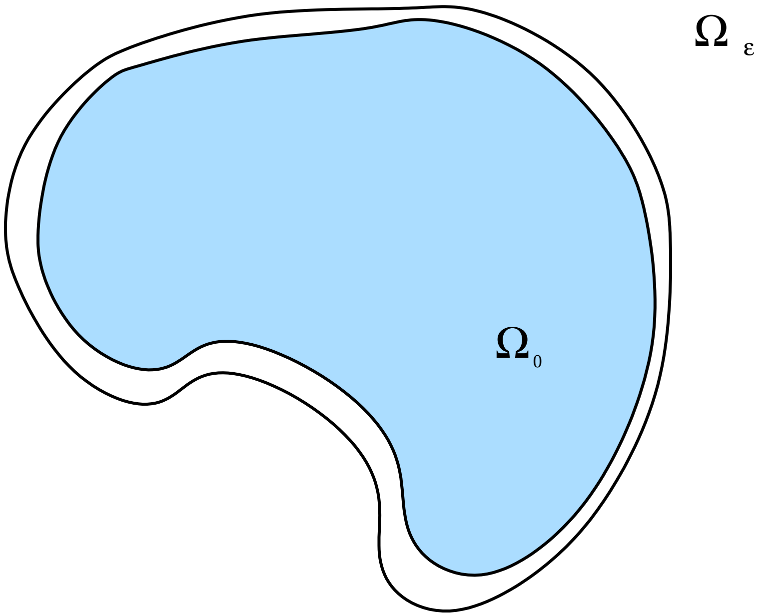

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J.A., A.N. Carvalho " Spectral Convergence and Nonlinear Dynamics of Reaction-Diffusion Equations under Perturbations of the Domain " Journal of Differential Equations 199 (2004) pp 143-178

Case 2. Dumbbell type domain

$$\begin{cases} u_t - \Delta u = f(u) & \text{in } \Omega_\epsilon \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega_\epsilon. \end{cases}$$

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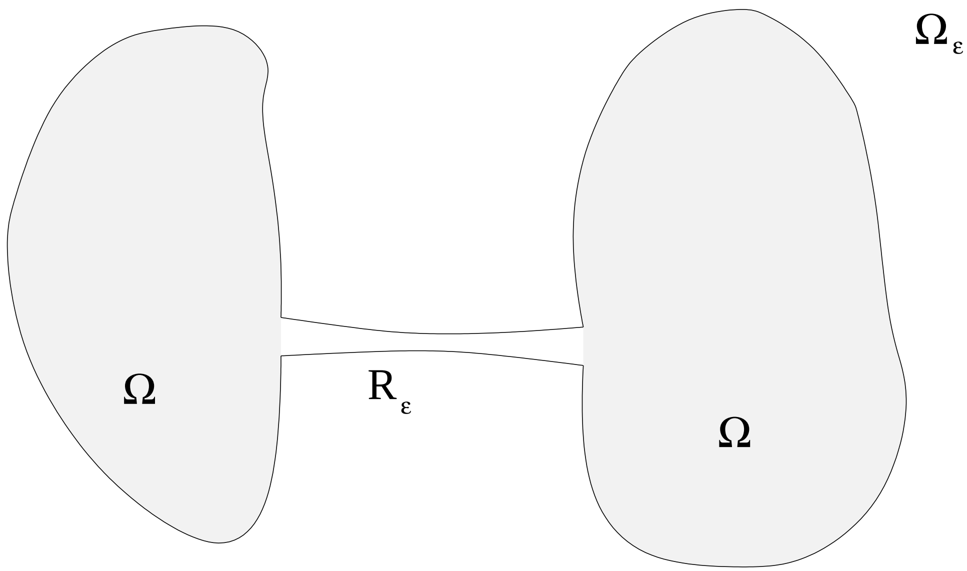
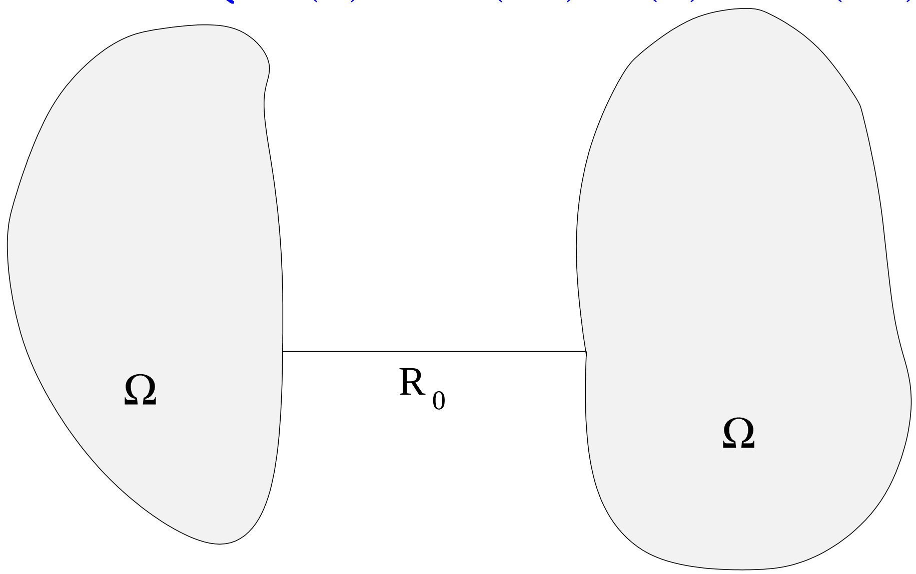
$\Omega_\epsilon = \Omega \cup R_\epsilon$



The limit problem and limit “domain” are

$$
\left\{
\begin{array}{l}
w_t - \Delta w = f(w) \quad \text{in } \Omega \\
\frac{\partial w}{\partial n} = 0 \quad \text{on } \partial\Omega \\
\\
v_t - \frac{1}{g}(g v_x)_x = f(v), \quad x \in (0,1) \\
v(0) = w(P_0), v(1) = w(P_1)
\end{array}
\right.
$$

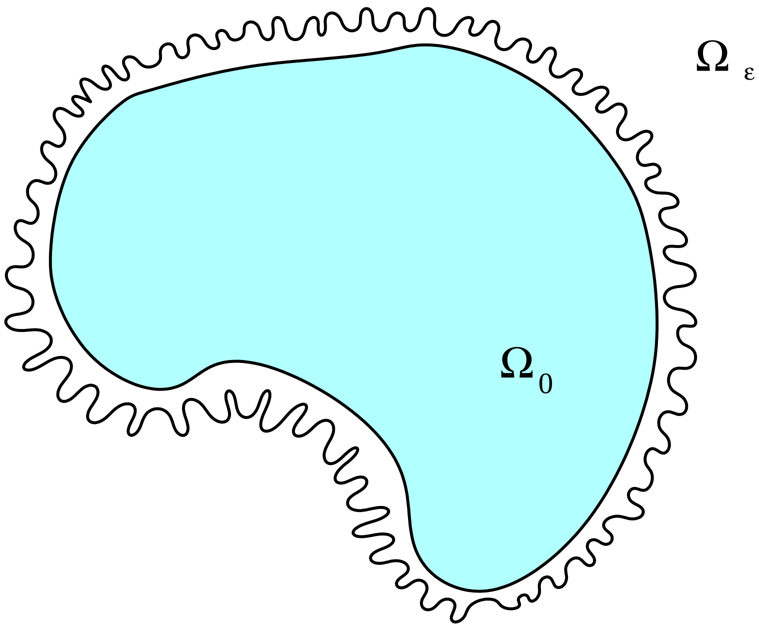


J.A., A.N. Carvalho, G. Lozada-Cruz “Dynamics in Dumbbell Domains I. Continuity of the set of equilibria”, *Journal of Differential Equations*, 231, Issue 2, pp. 551-597, (2006),

J.A., A.N. Carvalho, G. Lozada-Cruz “Dynamics in Dumbbell Domains II. Continuity of attractors”, In preparation

Case 3. Nonlinear boundary conditions and boundary oscillations

$$\begin{cases} u_t - \Delta u + u = f(x,u) & \text{in } \Omega_\epsilon \\ \frac{\partial u}{\partial n} + g(x,u) = 0 & \text{on } \partial\Omega_\epsilon. \end{cases}$$



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J.A., S.M. Bruschi “Boundary oscillations and nonlinear boundary conditions´´, *C. R. Acad. Sci. Paris*, t. 343, Series I, pp. 99-104 (2006)

J.A., S.M. Bruschi “Rapidly varying boundaries in equations with nonlinear boundary conditions. The case of a Lipschitz deformation´´, *Math. Methods and Models in Applied Science* (2007). To appear.

J.A., S.M. Bruschi “Very rapidly varying boundaries in equations with nonlinear boundary conditions.´´, *In preparation*

Boundary oscillations

Joint work with Simone M. Bruschi, UNESP, Brazil

$$\begin{cases} u_t - \Delta u + u = f(x,u) & \text{in } \Omega \\ \frac{\partial u}{\partial n} + g(x,u) = 0 & \text{en } \partial\Omega. \end{cases}$$

i) $\Omega \subset \mathbb{R}^N$ bounded smooth domain

ii) $f, g : \mathbb{R}^N \times \mathbb{R} \longrightarrow \mathbb{R}$, regular enough and satisfying the dissipativeness conditions: $\exists M > 0$, such that

$$f(x,u)u \leq 0, \qquad -g(x,u)u \leq 0, \quad \forall |u| \geq M, \quad x \in \mathbb{R}^N.$$

To simplify, let us assume that both $f(x, \cdot)$ and $g(x, \cdot)$ are globally Lipschitz functions, uniformly in $x \in \mathbb{R}^N$, that is:

$$|f(x, s) - f(x, r)| \leq L|s - r|$$

$$|g(x, s) - g(x, r)| \leq L|s - r|$$

This problem is well posed in spaces $L^2(\Omega)$ and $H^1(\Omega)$, it generates a dynamical system (semiflow, nonlinear semigroup) in both spaces.

If we denote $X(\Omega) = L^2(\Omega)$ or $H^1(\Omega)$:

$$
\begin{aligned}
T_\Omega(t) : X(\Omega) &\to X(\Omega) \\
\xi &\to u(t, \cdot, \xi)
\end{aligned}
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$$\begin{aligned} T_\Omega(t) : X(\Omega) &\to X(\Omega) \\ \xi &\to u(t, \cdot, \xi) \end{aligned}$$

where $u(t, \cdot, \xi)$ is the solution at time t, with initial data $\xi \in X(\Omega)$, that is

$$\begin{cases} u_t - \Delta u + u = f(x, u) \quad \text{en } \Omega \\ \frac{\partial u}{\partial n} + g(x, u) = 0 \quad \text{en } \partial\Omega, \\ u(0, \cdot) = \xi \end{cases}$$

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▶ it has bounds in $L^\infty(\Omega)$, independent of Ω. Actually, by the maximum principle, we have

$$\|\varphi\|_{L^\infty(\Omega)} \leq M, \quad \forall \varphi \in \mathcal{A}_\Omega,$$

where M is such that $f(x,u)u \leq 0$ and $-g(x,u)u \leq 0$, for $|u| \geq M$.

The dynamical system is C^1 and gradient.

$$V(u) = \frac{1}{2}\int_\Omega (|\nabla u(x)|^2 + |u|^2)dx - \int_\Omega \int_0^{u(x)} f(x,s)ds dx +$$

$$+ \int_{\partial\Omega} \int_0^{u(x)} g(x,s)ds \; dx$$

is a Lyapunov function.

As a matter of fact, we have

$$\frac{d}{dt}(V(u(t))) = -\int_\Omega |u_t(t,x)|^2 dx$$

Hence, if $\mathcal{E}_\Omega$, is the set of equilibria, that is, all solutions of

$$\begin{cases} -\Delta u + u = f(x,u) & \text{en } \Omega \\ \frac{\partial u}{\partial n} + g(x,u) = 0 & \text{en } \partial\Omega. \end{cases}$$

then $\mathcal{A}_\Omega$ is formed by

- $\mathcal{E}_\Omega$

- Connections among elements of $\mathcal{E}_\Omega$.

Hence, if we want to analyze the dependence of attractor $\mathcal{A}_\Omega \subset H^1(\Omega)$ as a function of the domain Ω, we better start understanding the dependence of the set of equilibria, $\mathcal{E}_\Omega$, with respect to the domain.

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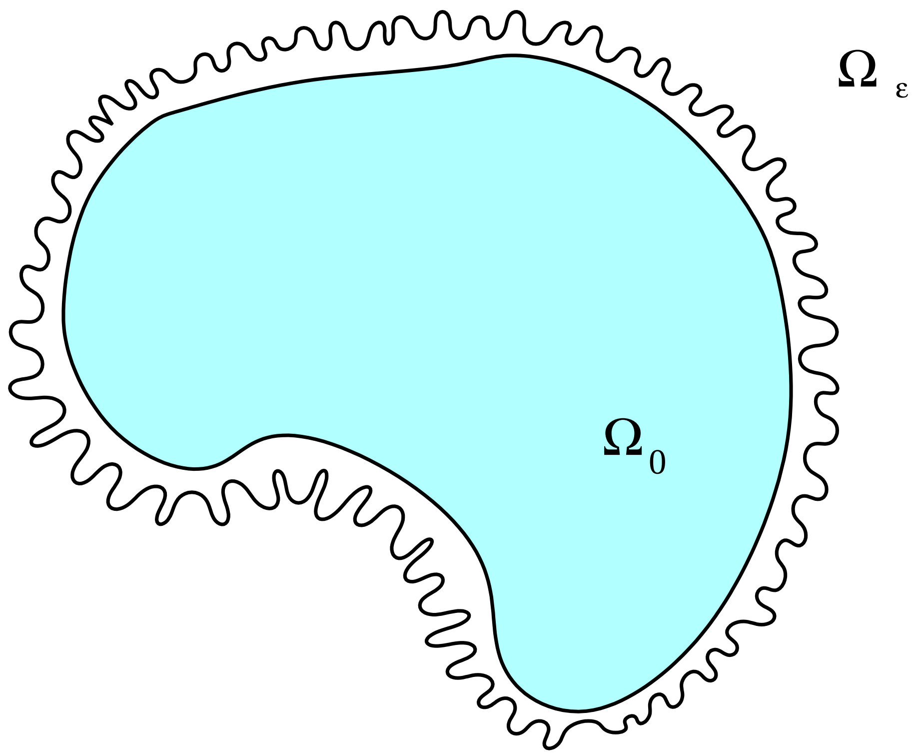
Hence, we consider a family of domains Ω_ϵ converging in certain sense to Ω_0 as $\epsilon \longrightarrow 0$. We want to understand the behavior of $\mathcal{E}_{\Omega_\epsilon}$ as $\epsilon \longrightarrow 0$.


Ω_ε
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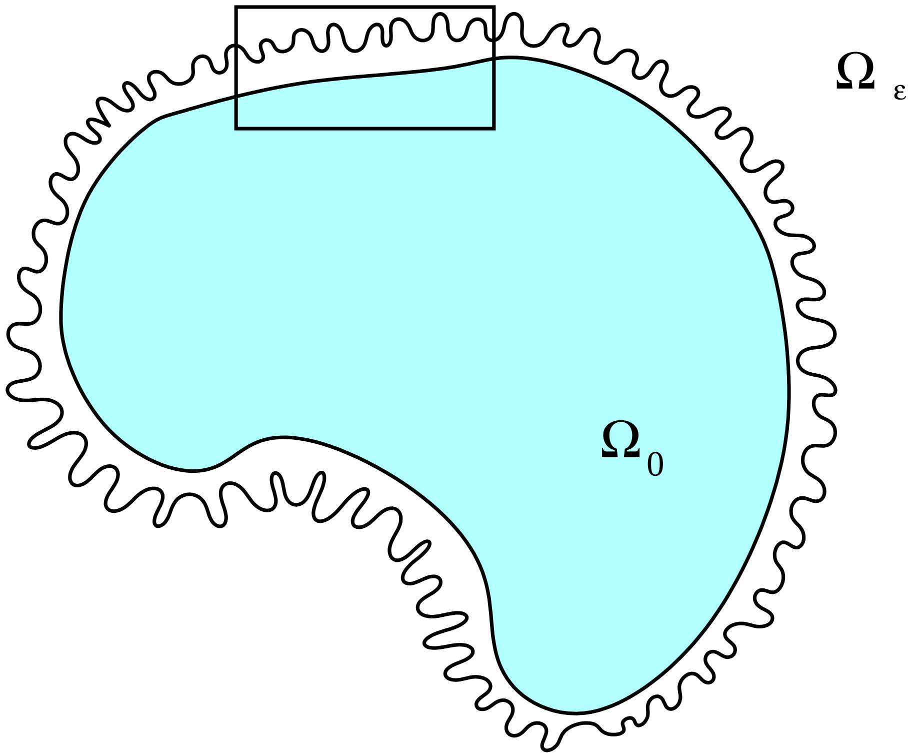
Ω_ε
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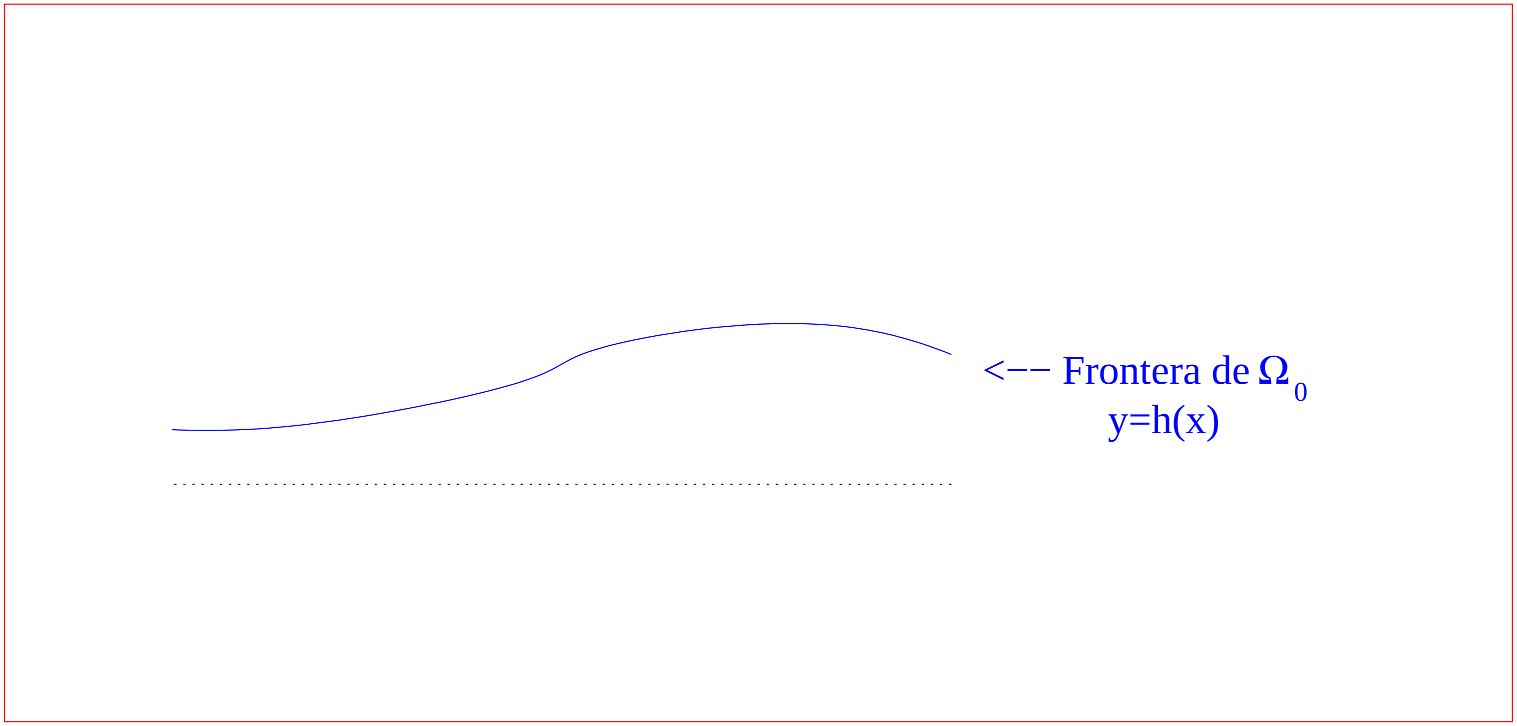
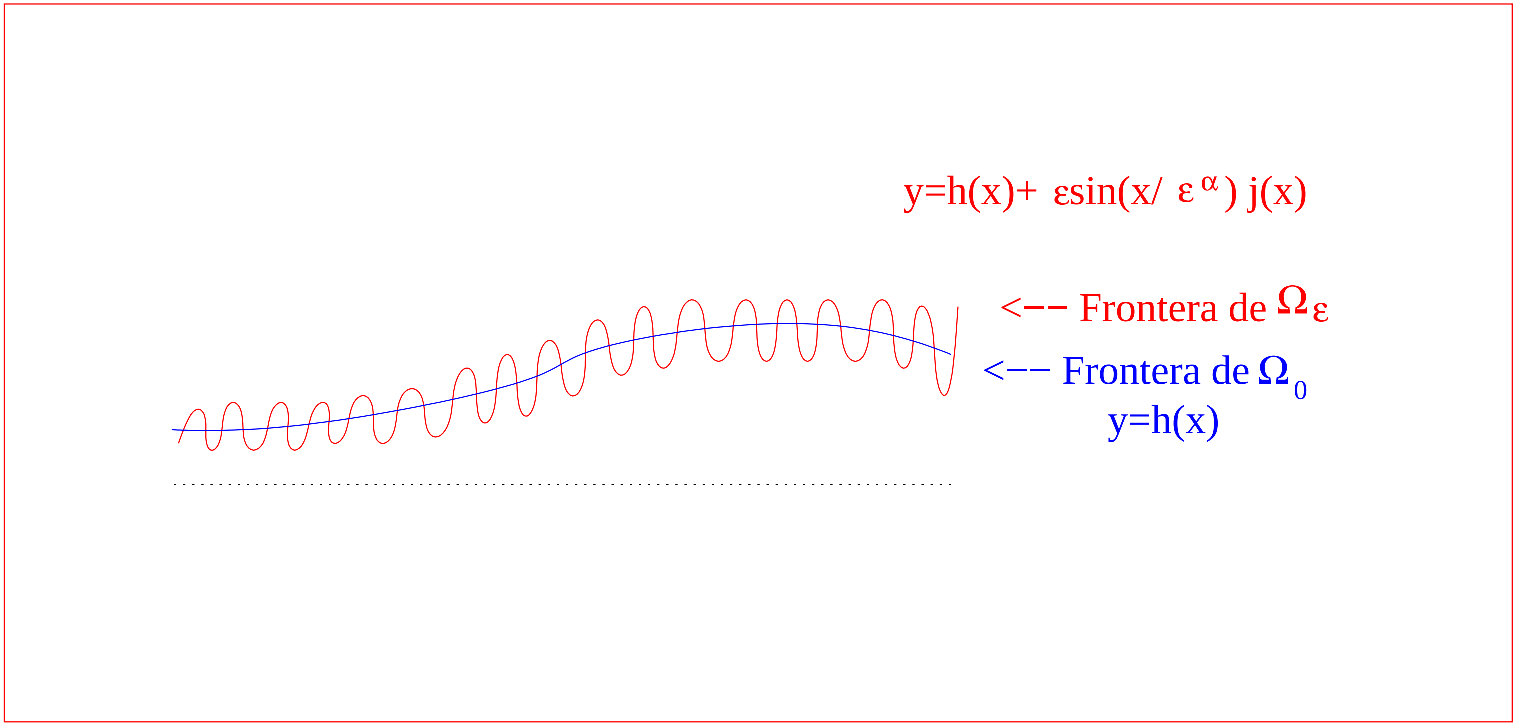

<-- Frontera de Ω0
y=h(x)


y=h(x)+ εsin(x/ ε^α) j(x)
<-- Frontera de Ω_ε
<-- Frontera de Ω_0
y=h(x)

What is the limit problem?.

What is the limit problem?. Let us consider a simpler case:

$$\begin{cases} -\Delta u_\epsilon + u_\epsilon = 0 & \text{en } \Omega_\epsilon \\ \frac{\partial u_\epsilon}{\partial n} + g(x) = 0 & \text{en } \partial\Omega_\epsilon. \end{cases}$$

($g \in C^0(\mathbb{R}^N)$).

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($g \in C^0(\mathbb{R}^N)$).

It is equivalent to,

$$\int_{\Omega_\epsilon} (\nabla u_\epsilon \nabla \varphi + u_\epsilon \varphi) = -\int_{\partial\Omega_\epsilon} g\varphi, \quad \forall \varphi \in C^\infty(\mathbb{R}^N)$$

With apriori estimates, weak limits, etc... we have that there exist $u_0 \in H^1(\Omega_0)$ such that

$$\int_{\Omega_\epsilon} (\nabla u_\epsilon \nabla \varphi + u_\epsilon \varphi) \to \int_{\Omega_0} (\nabla u_0 \nabla \varphi + u_0 \varphi), \quad \forall \varphi \in C^\infty(\mathbb{R}^N)$$

Moreover,

$$\int_{\partial\Omega_\epsilon} g(x)\varphi(x) \to \int_{\partial\Omega_0} \gamma(x) g(x)\varphi(x)$$

where the function γ satisfies $1 \leq \gamma(x) \leq +\infty$.

Moreover,

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So that the limit problem is given by

$$\begin{cases} -\Delta u + u = 0 & \text{in } \Omega_0 \\ \frac{\partial u}{\partial n} + \gamma(x) g(x) = 0 & \text{on } \partial\Omega_\epsilon. \end{cases}$$

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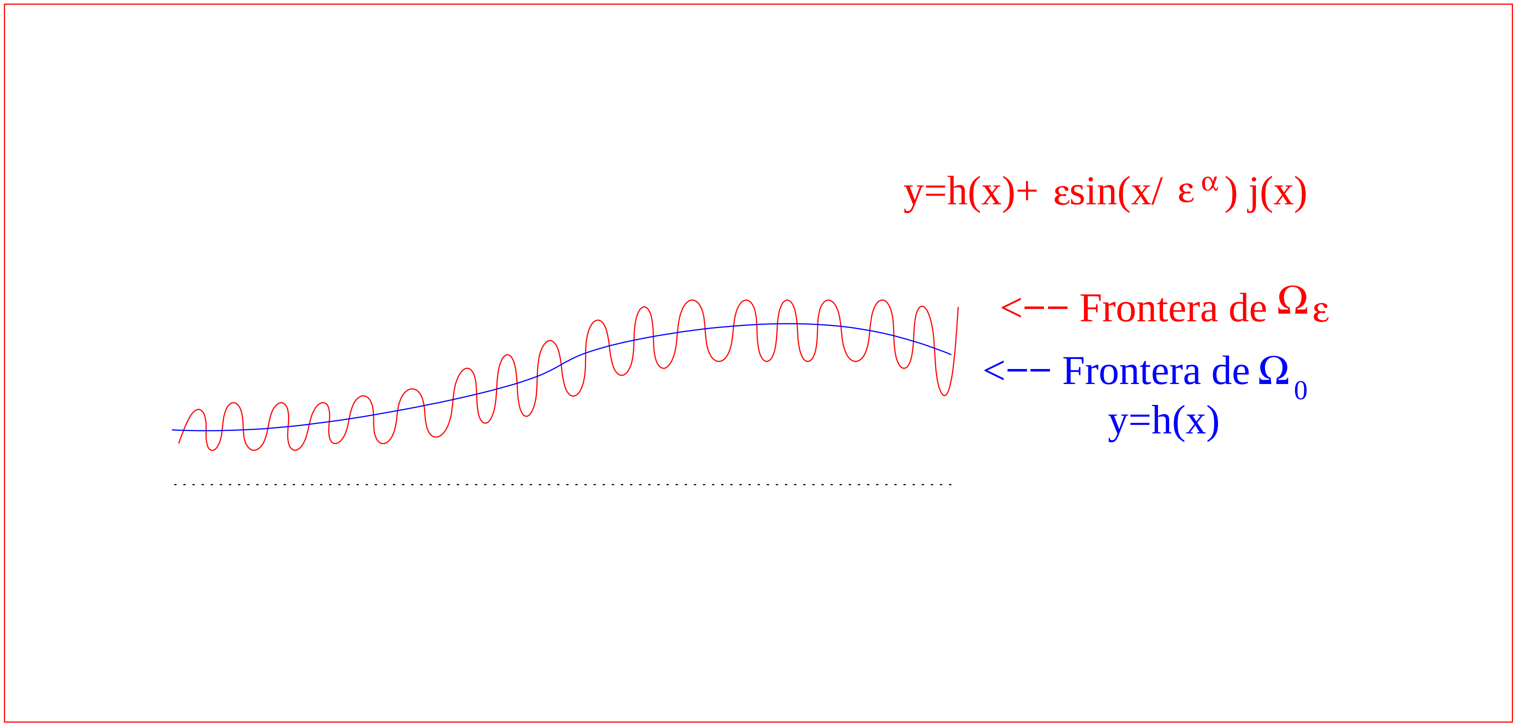
For $x_0 \in \partial\Omega_0$, the value $\gamma(x_0)$ represents the relative measure of $\partial\Omega_\epsilon$ with respect to $\partial\Omega_0$ locally around x_0. That is,

$$\gamma(x_0) \approx \frac{|\partial\Omega_\epsilon \cap B(x_0, r)|}{|\partial\Omega_0 \cap B(x_0, r)|}, \quad \text{as } \epsilon, r \to 0$$

For the case



- If $0 \le \alpha < 1$, then $\gamma(x) \equiv 1$.
- If $\alpha > 1$, then $\gamma = +\infty$
- If $\alpha = 1$, then $1 \le \gamma(x) \le C$. For instance, if $h \equiv 0$, then

$$\gamma(x) = \frac{1}{2\pi} \int_0^{2\pi} \sqrt{1 + (j(x)\cos(z))^2}\, dz$$

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We will restrict our exposition to the case $\alpha \leq 1$.

In particular, domains Ω_ϵ satisfy

- $\bigcup_{0\le\epsilon\le\epsilon_0} \bar{\Omega}_\epsilon \subset U$, where U is a bounded domain.
- $\Omega_\epsilon \to \Omega_0$, $\partial\Omega_\epsilon \to \partial\Omega_0$, in the sense of Hausdorff
- Ω_ϵ are smooth domains and uniformly Lipschitz in ϵ.
  - $\exists$ extension operators $P_\epsilon : H^1(\Omega_\epsilon) \to H^1(U)$, with norms uniformly bounded in ϵ. Same for $W^{1,p}(\Omega_\epsilon)$, $C^\beta(\Omega_\epsilon)$.
  - The norms of Sobolev embeddings $W^{1,p}(\Omega_\epsilon) \hookrightarrow L^q(\Omega_\epsilon)$ and trace operators $W^{1,p}(\Omega_\epsilon) \to L^r(\partial\Omega_\epsilon)$ are uniformly bounded in ϵ.
- There exists a function γ defined on $\partial\Omega_0$, with $1 \le \gamma \le M$, such that, for all $f \in W^{1,1}(U)$,

$$\int_{\partial\Omega_\epsilon} f \to \int_{\partial\Omega_0} \gamma f$$

Hence, for the nonlinear problem

$$(P)_\epsilon \quad \begin{cases} -\Delta u + u = f(x,u) & \text{in } \Omega_\epsilon \\ \frac{\partial u}{\partial n} + g(x,u) = 0 & \text{on } \partial\Omega_\epsilon. \end{cases}$$

the limit problem is given by

$$(P)_0 \quad \begin{cases} -\Delta u + u = f(x,u) & \text{in } \Omega_0 \\ \frac{\partial u}{\partial n} + \gamma(x) g(x,u) = 0 & \text{on } \partial\Omega_0. \end{cases}$$

E.N. Dancer & D. Daners, “Domain Perturbations for Elliptic Equations subject to Robin Boundary Conditions”, *J. of Diff. Equations* 138, (1997) 86-132.

They treat the case $g(x, u) = b(x)u$, with $b(x) \geq b_0 > 0$

How do we compare functions in Ω_0 with functions in Ω_ϵ?

In what sense can we say that a sequence of functions u_ϵ, each of them defined in Ω_ϵ converges to a function u_0 defined in Ω_0?

We have an extension operator

$$E_\epsilon : H^1(\Omega_0) \to H^1(\Omega_\epsilon)$$

which is obtained as $E_\epsilon = R_\epsilon \circ E$, where $E : H^1(\Omega_0) \to H^1(\mathbb{R}^N)$ and R_ϵ is the restriction operator of functions defined in $\mathbb{R}^N$ to functions defined in Ω_ϵ.

(E_ϵ is also an extension operator for functions L^p; $W^{1,p}$, C^β).

We define,

Definition. *A sequence of elements* $u_\epsilon \in H^1(\Omega_\epsilon)$ *is said to be* E*-convergent to* $u \in H^1(\Omega_0)$ *if* $\|u_\epsilon - E_\epsilon u\|_{H^1(\Omega_\epsilon)} \to 0$ *as* $\epsilon \to 0$*. We write this as* $u_\epsilon \xrightarrow{E} u$*.*

Theorem. (Continuity of the set of solutions)

i) If u_ϵ^*, $0 < \epsilon \leq \epsilon_0$, *is a family of solutions of problem* $(P)_\epsilon$ *then there exists a subsequence, still denoted by* u_ϵ^*, *and a solution of problem* $(P)_0$, $u_0^* \in H^1(\Omega_0)$, *with the property that, for some* $0 < \beta < 1$

$$\|u_\epsilon^* - E_\epsilon u_0^*\|_{H^1(\Omega_\epsilon)} + \|u_\epsilon^* - E_\epsilon u_0^*\|_{C^\beta(\bar{\Omega}_\epsilon)} \to 0, \text{ as } \epsilon \to 0.$$

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i) If u_ϵ^*, $0 < \epsilon \leq \epsilon_0$, *is a family of solutions of problem* $(P)_\epsilon$ *then there exists a subsequence, still denoted by* u_ϵ^*, *and a solution of problem* $(P)_0$, $u_0^* \in H^1(\Omega_0)$, *with the property that, for some* $0 < \beta < 1$

$$\|u_\epsilon^* - E_\epsilon u_0^*\|_{H^1(\Omega_\epsilon)} + \|u_\epsilon^* - E_\epsilon u_0^*\|_{C^\beta(\bar{\Omega}_\epsilon)} \to 0, \text{ as } \epsilon \to 0.$$

ii) If $u_0^* \in H^1(\Omega_0)$ *is a solution of* $(P)_0$, *which is hyperbolic (*$\lambda = 0$ *is not an eigenvalue of the linearized problem of* $(P)_0$ *around* u_0^**), then, there exists* $\delta > 0$ *small such that problem* $(P)_\epsilon$ *has one and only one solution* u_ϵ^*, *satisfying* $\|u_\epsilon^* - E_\epsilon u_0^*\|_{H^1(\Omega_\epsilon)} \leq \delta$ *for* ϵ *small enough.*

u_0^* hyperbolic

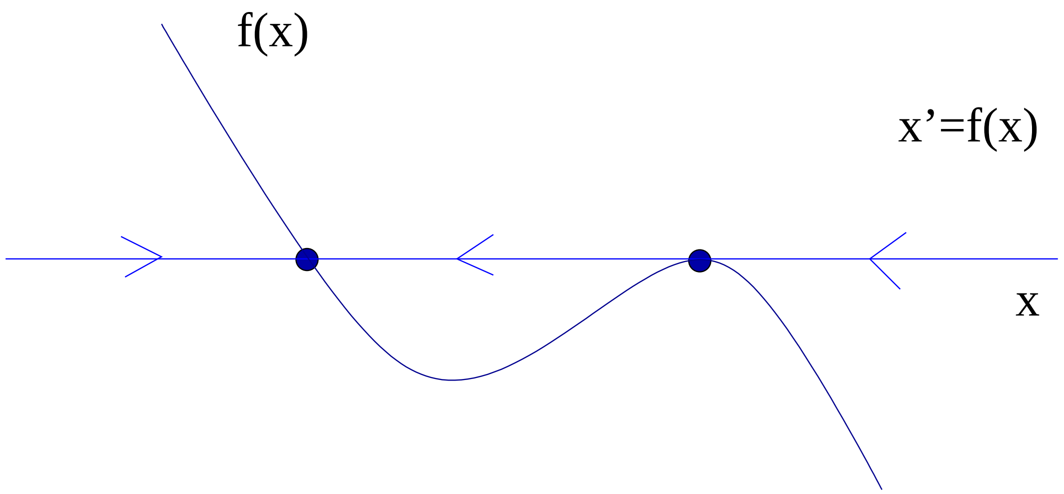
u_0^* hyperbolic $\quad\Longleftrightarrow\quad$ $\lambda = 0$ is not an eigenvalue of

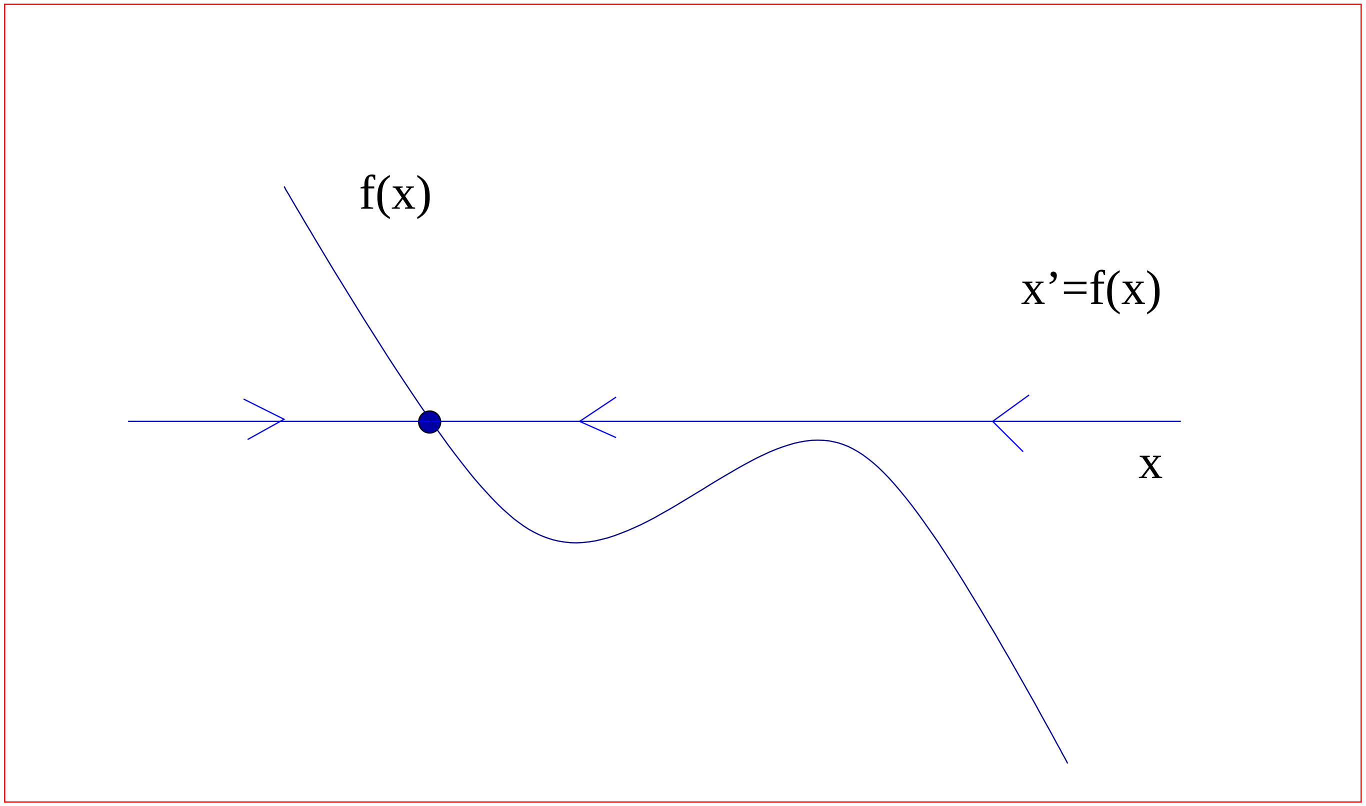
$$\begin{cases} -\Delta w + w - \partial_u f(x, u_0^*)w = \lambda w & \text{in } \Omega_0, \\ \dfrac{\partial w}{\partial n} + \gamma \partial_u g(x, u_0^*)w = 0 & \text{on } \partial\Omega_0. \end{cases} \tag{1}$$

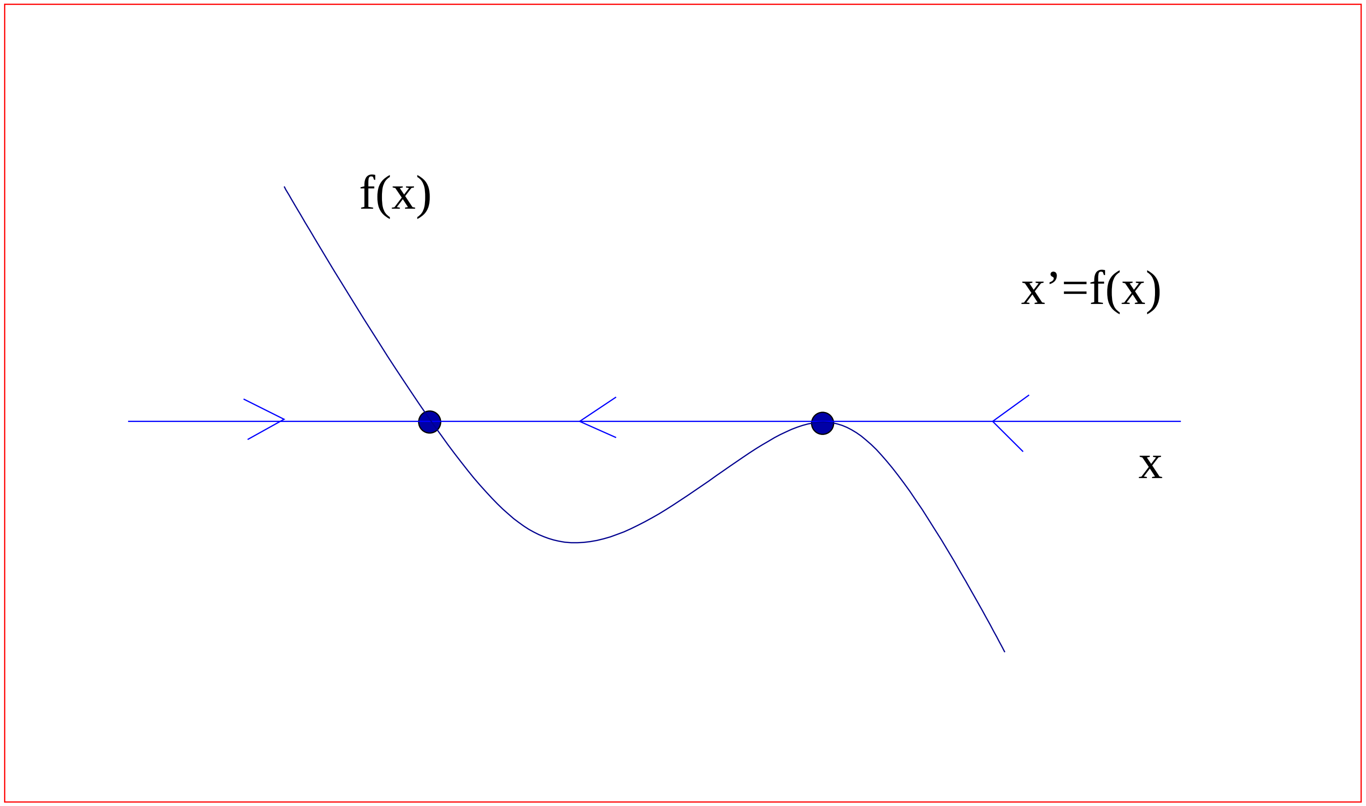
u_0^* hyperbolic $\quad\Longleftrightarrow\quad$ $\lambda = 0$ is not an eigenvalue of

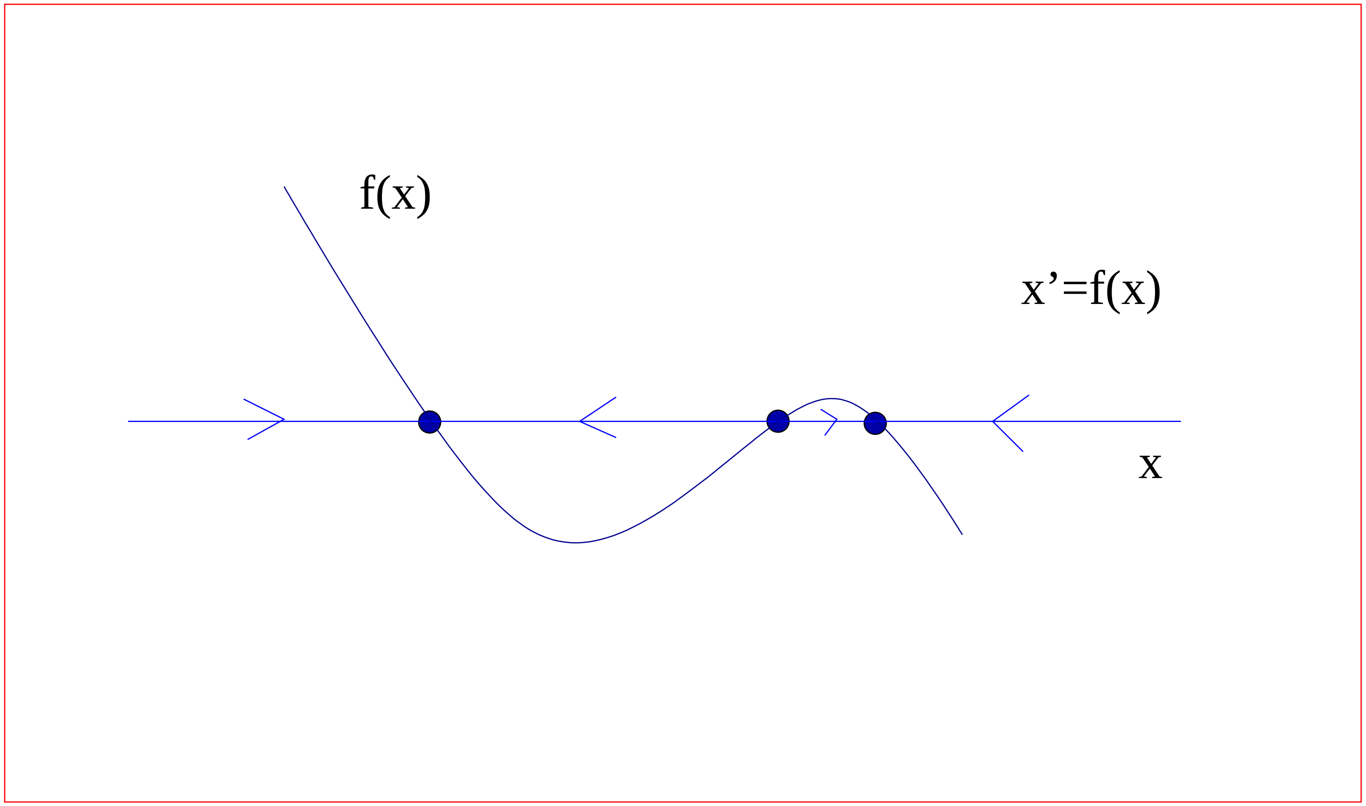
$$\begin{cases} -\Delta w + w - \partial_u f(x, u_0^*)w = \lambda w & \text{in } \Omega_0, \\ \dfrac{\partial w}{\partial n} + \gamma \partial_u g(x, u_0^*)w = 0 & \text{on } \partial\Omega_0. \end{cases} \tag{2}$$

This condition is not technical.


f(x)
x’=f(x)
x


f(x)
x’=f(x)
x


f(x)
x’=f(x)
x


f(x)
x'=f(x)
x

Idea of the proof.

• We use a functional analysis for operators defined in different spaces.

Let H_ϵ, $0 < \epsilon \le \epsilon_0$ and H be Hilbert spaces. Let also

$$E_\epsilon : H \longrightarrow H_\epsilon$$

be a bounded linear operator, such that $\|E_\epsilon u\|_{H_\epsilon} \to \|u\|_H$.

For instance, $H_\epsilon = H^1(\Omega_\epsilon)$, $H = H^1(\Omega_0)$ and $E_\epsilon : H \longrightarrow H_\epsilon$ is the constructed extension operator.

Definition. *A sequence of elements* $u_\epsilon \in H_\epsilon$ *is said to be* E*-convergent to* $u \in H$ *if* $\|u_\epsilon - E_\epsilon u\|_{H_\epsilon} \to 0$ *as* $\epsilon \to 0$. *We write this as* $u_\epsilon \xrightarrow{E} u$.

Definition. *A sequence of elements* $u_\epsilon \in H_\epsilon$ *is said to be* E*-precompact if for any subsequence* $\{u_{\epsilon_n}\}$ *there exist a subsequence* $\{u_{\epsilon_{n'}}\}$ *and* $u \in H$ *such that* $u_{\epsilon_{n'}} \xrightarrow{E} u$, *as* $n' \to \infty$.

Definition. *A family of operators* $T_\epsilon : H_\epsilon \to H_\epsilon$, $\epsilon \in (0, 1]$, E*-converges to* $T : H \to H$, *if* $T_\epsilon u_\epsilon \xrightarrow{E} Tu$, *whenever* $u_\epsilon \xrightarrow{E} u$. *We denote this by* $T_\epsilon \xrightarrow{EE} T$.

Definition. *A family of compact operators* $T_\epsilon : H_\epsilon \to H_\epsilon$, $\epsilon \in (0, 1]$ *converges compactly to a compact* $T : H \to H$ *if for any family* u_ϵ *with* $\|u_\epsilon\|_{H_\epsilon}$ *bounded, the family* $\{T_\epsilon u_\epsilon\}$ *is* E*-precompact and* $T_\epsilon \xrightarrow{EE} T$. *We write* $T_\epsilon \xrightarrow{CC} T$.

Theorem. *Let* $T_\epsilon : H_\epsilon \to H_\epsilon$ *be a family of compact operators such that* $T_\epsilon \xrightarrow{CC} T$*. Let* u_ϵ *be a fixed point of* T_ϵ *such that* $\|u_\epsilon\|_{H_\epsilon}$ *is uniformly bounded. Then, there exists a subsequence* u_{ϵ_k} *and* $u \in H$ *a fixed point of* T*, such that* $u_{\epsilon_k} \xrightarrow{E} u$*.*

Proof. If $u_\epsilon = T_\epsilon u_\epsilon$ and $\|u_\epsilon\|_{H_\epsilon} \leq M$, by compact convergence $T_\epsilon \xrightarrow{CC} T$, there exists a subsequence $\epsilon_k \to 0$, and a $u \in H$ such that $T_{\epsilon_k} u_{\epsilon_k} \xrightarrow{E} u$. Hence $u_{\epsilon_k} = T_{\epsilon_k} u_{\epsilon_k} \xrightarrow{E} u$ and therefore, $T_{\epsilon_k} u_{\epsilon_k} \xrightarrow{E} Tu$. This implies $u = Tu$.

G. Vainikko, Approximative methods for nonlinear equations (two approaches to the convergence problem), *Nonlinear Analysis, Theory, Methods & Applications*, vol2, 6 (1978) 647-687.

A. Carvalho, S. Piskarev, A general approximation scheme for attractors of abstract parabolic problems, *Numer. Funct. Anal. Optim,* 27 (2006), no. 7-8, 785–829.

For our case, we have $H_\epsilon = H^1(\Omega_\epsilon)$, $H = H^1(\Omega_0)$, E_ϵ the extension operator.

$T_\epsilon : H^1(\Omega_\epsilon) \to H^1(\Omega_\epsilon)$ is given by $T_\epsilon(z_\epsilon) = u_\epsilon$, where u_ϵ is the unique solution of

$$\begin{cases} -\Delta u_\epsilon + u_\epsilon = f(x, z_\epsilon) & \text{in } \Omega_\epsilon \\ \frac{\partial u_\epsilon}{\partial n} + g(x, z_\epsilon) = 0 & \text{on } \partial\Omega_\epsilon. \end{cases}$$

$T : H^1(\Omega_0) \to H^1(\Omega_0)$ is given by $T(z) = u$, where u is the unique solution of

$$\begin{cases} -\Delta u + u = f(x, z) & \text{in } \Omega_0 \\ \frac{\partial u}{\partial n} + \gamma(x) g(x, z) = 0 & \text{on } \partial\Omega_0. \end{cases}$$

That is,

$$T_\epsilon = A_\epsilon^{-1} \circ h_\epsilon$$

where $A_\epsilon = -\Delta + I$ with homogeneous Neumann boundary conditions and $h_\epsilon : H^1(\Omega_\epsilon) \to H^{-\alpha}(\Omega_\epsilon)$ is defined as

$$< h_\epsilon(z_\epsilon), \varphi_\epsilon >= \int_{\Omega_\epsilon} f(x, z_\epsilon)\varphi_\epsilon(x)dx - \int_{\partial\Omega_\epsilon} g(x, z_\epsilon)\varphi_\epsilon(x)d\sigma_\epsilon$$

Similarly,

$$T = A^{-1} \circ h$$

where $A = -\Delta + I$ with homogeneous Neumann boundary conditions and $h : H^1(\Omega_0) \to H^{-\alpha}(\Omega_0)$ is defined as

$$< h(z), \varphi >= \int_{\Omega_0} f(x, z)\varphi(x)dx - \int_{\partial\Omega_0} \gamma(x)g(x, z)\varphi(x)d\sigma$$

To show that $T_\epsilon \xrightarrow{CC} T$, we get a sequence of $z_\epsilon \in H^1(\Omega_\epsilon)$, $\|z_\epsilon\|_{H^1(\Omega_\epsilon)} \leq M$ and let $u_\epsilon = T_\epsilon(z_\epsilon)$, that is,

$$\begin{cases} -\Delta u_\epsilon + u_\epsilon = f(x, z_\epsilon) & \text{in } \Omega_\epsilon \\ \frac{\partial u_\epsilon}{\partial n} + g(x, z_\epsilon) = 0 & \text{on } \partial\Omega_\epsilon. \end{cases}$$

We have $\|u_\epsilon\|_{H^1(\Omega_\epsilon)} \leq M'$. Using the extension operators P_ϵ and getting subsequences, we obtain z and u, such that

$$P_{\epsilon_k} z_{\epsilon_k} \rightharpoonup z \in H^1(U), \quad P_{\epsilon_k} u_{\epsilon_k} \rightharpoonup u \in H^1(U)$$

Using the weak formulation of the equation,

$$\int_{\Omega_\epsilon} \nabla u_\epsilon \nabla \varphi + u_\epsilon \varphi = \int_{\Omega_\epsilon} f(x, z_\epsilon)\varphi - \int_{\partial\Omega_\epsilon} g(x, z_\epsilon)\varphi$$

We can easily pass to the limit in the first three terms. For the boundary term, we have

$$\int_{\partial\Omega_\epsilon} g(x, z_\epsilon)\varphi = \int_{\partial\Omega_\epsilon} (g(x, z_\epsilon) - g(x, z))\varphi + \int_{\partial\Omega_\epsilon} g(x, z)\varphi$$

and

$$|\int_{\partial\Omega_\epsilon} (g(x, z_\epsilon) - g(x, z))\varphi| \leq C \int_{\partial\Omega_\epsilon} |z_\epsilon - z| \to 0$$

Moreover,

$$\int_{\partial\Omega_\epsilon} g(x, z)\varphi(x) \to \int_{\partial\Omega_0} \gamma(x) g(x, z)\varphi(x)$$

Hence, passing to the limit

$$\int_{\Omega_0} \nabla u \nabla \varphi + u\varphi = \int_{\Omega_0} f(x,z)\varphi - \int_{\partial\Omega_0} \gamma(x) g(x,z)\varphi$$

which means that u is solution of

$$\begin{cases} -\Delta u + u = f(x,z) & \text{in } \Omega_0 \\ \frac{\partial u}{\partial n} + \gamma(x) g(x,z) = 0 & \text{on } \partial\Omega_0. \end{cases}$$

To show that actually

$$\|u_\epsilon - E_\epsilon u\|_{H^1(\Omega_\epsilon)} \to 0,$$

we show the convergence of the norms, that is $\|u_\epsilon\|_{H^1(\Omega_\epsilon)} \to \|u\|_{H^1(\Omega_0)}$.

Convergence in C^β-norms.

We are able to obtain Hölder estimates of u_ϵ^* independent of ϵ.

Uniqueness of solutions near a hyperbolic solution.

Let u_0^* be a hyperbolic solution of the limit problem. That, is, $\lambda = 0$ is not an eigenvalue of

$$\begin{cases} -\Delta\phi + \phi - \partial_u f(x, u_0^*)\phi = \lambda\phi & \text{in } \Omega_0 \\ \frac{\partial\phi}{\partial n} + \gamma(x)\partial_u g(x, u_0^*)\phi = 0 & \text{on } \partial\Omega_0. \end{cases}$$

In particular,

- u_0^* is an isolated solution of $(P)_0$.

- With index-degree theory it is possible to show that for all $0 < \epsilon \leq \epsilon_0$ there exists at least one solution u_ϵ^* of $(P)_\epsilon$ near u_0^*. In particular, there exists a sequence $u_\epsilon^* \xrightarrow{E} u_0^*$.

Let us see that there exists a $\delta > 0$, such that no other function $w_\epsilon \in H^1(\Omega_\epsilon)$ with $\|w_\epsilon - u_\epsilon^*\|_{H^1(\Omega_\epsilon)} \leq \delta$ is a fixed point of $T_\epsilon = A_\epsilon^{-1} \circ h_\epsilon$.

$$\|w_\epsilon - T_\epsilon(w_\epsilon)\|_{H^1(\Omega_\epsilon)} = \|w_\epsilon - u_\epsilon^* - (T_\epsilon(w_\epsilon) - T_\epsilon(u_\epsilon^*))\|_{H^1(\Omega_\epsilon)}$$

$$\geq \|w_\epsilon - u_\epsilon^* - B_\epsilon(w_\epsilon - u_\epsilon^*) + (T_\epsilon(w_\epsilon) - T_\epsilon(u_\epsilon^*) - B_\epsilon(w_\epsilon - u_\epsilon^*))\|_{H^1(\Omega_\epsilon)}$$
$$\geq \|(I - B_\epsilon)(w_\epsilon - u_\epsilon^*)\|_{H^1(\Omega_\epsilon)} -$$
$$-\|T_\epsilon(w_\epsilon) - T_\epsilon(u_\epsilon^*) - B_\epsilon(w_\epsilon - u_\epsilon^*)\|_{H^1(\Omega_\epsilon)}$$

The linear operator B_ϵ, is defined by $v_\epsilon = B_\epsilon(z_\epsilon)$ where

$$\begin{cases} -\Delta v_\epsilon + v_\epsilon = \partial_u f(x, u_\epsilon^*) z_\epsilon & \text{in } \Omega_\epsilon \\ \frac{\partial v_\epsilon}{\partial n} + \partial_u g(x, u_\epsilon^*) z_\epsilon = 0 & \text{on } \partial\Omega_\epsilon. \end{cases}$$

We can show $B_\epsilon \xrightarrow{CC} B_0$, where $v = B_0(z)$ where

$$\begin{cases} -\Delta v + v = \partial_u f(x, u_0^*) z & \text{in } \Omega_0 \\ \frac{\partial v}{\partial n} + \gamma(x) \partial_u g(x, u_0^*) z = 0 & \text{on } \partial\Omega_0. \end{cases}$$

The fact that u_0^* is a hyperbolic solution, is equivalent to say that $1 \notin \sigma(B_0)$ and therefore $I - B_0$ is invertible, that is $\|(I - B_0)^{-1}\| \leq C$.

The compact convergence of B_ϵ to B_0 implies that for ϵ small $1 \notin \sigma(B_\epsilon)$ and that $\|(I - B_\epsilon)^{-1}\| \leq C'$. Hence, if $\eta = 1/C'$,

$$\|(I - B_\epsilon)\chi\|_{H^1(\Omega_\epsilon)} \geq \eta \|\chi\|_{H^1(\Omega_\epsilon)}$$

In particular,

$$\|(I - B_\epsilon)(w_\epsilon - u_\epsilon^*)\|_{H^1(\Omega_\epsilon)} \geq \eta \|w_\epsilon - u_\epsilon^*\|_{H^1(\Omega_\epsilon)}$$

On the other hand, if we denote by

$$v_\epsilon = T_\epsilon(w_\epsilon) - T_\epsilon(u_\epsilon^*) - B_\epsilon(w_\epsilon - u_\epsilon^*)$$

we have

$$\begin{cases} -\Delta v_\epsilon + v_\epsilon = f(x, w_\epsilon) - f(x, u_\epsilon^*) - \partial_u f(x, u_\epsilon^*)(w_\epsilon - u_\epsilon^*) & \text{in } \Omega_\epsilon \\ \frac{\partial v_\epsilon}{\partial n} + g(x, w_\epsilon) - g(x, u_\epsilon^*) - \partial_u g(x, u_\epsilon^*)(w_\epsilon - u_\epsilon^*) = 0 & \text{on } \partial\Omega_\epsilon. \end{cases}$$

$$\|v_\epsilon\|^2_{H^1(\Omega_\epsilon)} = \int_{\Omega_\epsilon} (f(x, w_\epsilon) - f(x, u^*_\epsilon) - \partial_u f(x, u^*_\epsilon)(w_\epsilon - u^*_\epsilon))v_\epsilon$$

$$- \int_{\partial\Omega_\epsilon} (g(x, w_\epsilon) - g(x, u^*_\epsilon) - \partial_u g(x, u^*_\epsilon)(w_\epsilon - u^*_\epsilon))v_\epsilon$$

$$\|v_\epsilon\|^2_{H^1(\Omega_\epsilon)} = \int_{\Omega_\epsilon} (f(x,w_\epsilon) - f(x,u^*_\epsilon) - \partial_u f(x,u^*_\epsilon)(w_\epsilon - u^*_\epsilon))v_\epsilon$$

$$- \int_{\partial\Omega_\epsilon} (g(x,w_\epsilon) - g(x,u^*_\epsilon) - \partial_u g(x,u^*_\epsilon)(w_\epsilon - u^*_\epsilon))v_\epsilon$$

$$\leq C\|w_\epsilon - u^*_\epsilon\|^{2+\frac{2}{N-1}}_{H^1(\Omega_\epsilon)} + \frac{1}{2}\|v_\epsilon\|^2_{H^1(\Omega_\epsilon)}$$

$$\|v_\epsilon\|^2_{H^1(\Omega_\epsilon)} = \int_{\Omega_\epsilon} (f(x, w_\epsilon) - f(x, u^*_\epsilon) - \partial_u f(x, u^*_\epsilon)(w_\epsilon - u^*_\epsilon))v_\epsilon$$

$$- \int_{\partial\Omega_\epsilon} (g(x, w_\epsilon) - g(x, u^*_\epsilon) - \partial_u g(x, u^*_\epsilon)(w_\epsilon - u^*_\epsilon))v_\epsilon$$

$$\leq C\|w_\epsilon - u^*_\epsilon\|^{2+\frac{2}{N-1}}_{H^1(\Omega_\epsilon)} + \frac{1}{2}\|v_\epsilon\|^2_{H^1(\Omega_\epsilon)}$$

which implies

$$\|v_\epsilon\|_{H^1(\Omega_\epsilon)} \leq C\|w_\epsilon - u^*_\epsilon\|^{1+\frac{1}{N-1}}_{H^1(\Omega_\epsilon)}$$

$$\|v_\epsilon\|^2_{H^1(\Omega_\epsilon)} = \int_{\Omega_\epsilon} (f(x, w_\epsilon) - f(x, u^*_\epsilon) - \partial_u f(x, u^*_\epsilon)(w_\epsilon - u^*_\epsilon))v_\epsilon$$

$$- \int_{\partial\Omega_\epsilon} (g(x, w_\epsilon) - g(x, u^*_\epsilon) - \partial_u g(x, u^*_\epsilon)(w_\epsilon - u^*_\epsilon))v_\epsilon$$

$$\leq C\|w_\epsilon - u^*_\epsilon\|^{2+\frac{2}{N-1}}_{H^1(\Omega_\epsilon)} + \frac{1}{2}\|v_\epsilon\|^2_{H^1(\Omega_\epsilon)}$$

which implies

$$\|v_\epsilon\|_{H^1(\Omega_\epsilon)} \leq C\|w_\epsilon - u^*_\epsilon\|^{1+\frac{1}{N-1}}_{H^1(\Omega_\epsilon)}$$

Equivalently,

$$\|T_\epsilon(w_\epsilon) - T_\epsilon(u^*_\epsilon) - B_\epsilon(w_\epsilon - u^*_\epsilon)\|_{H^1(\Omega_\epsilon)} \leq C\|w_\epsilon - u^*_\epsilon\|^{1+\frac{1}{N-1}}_{H^1(\Omega_\epsilon)}$$

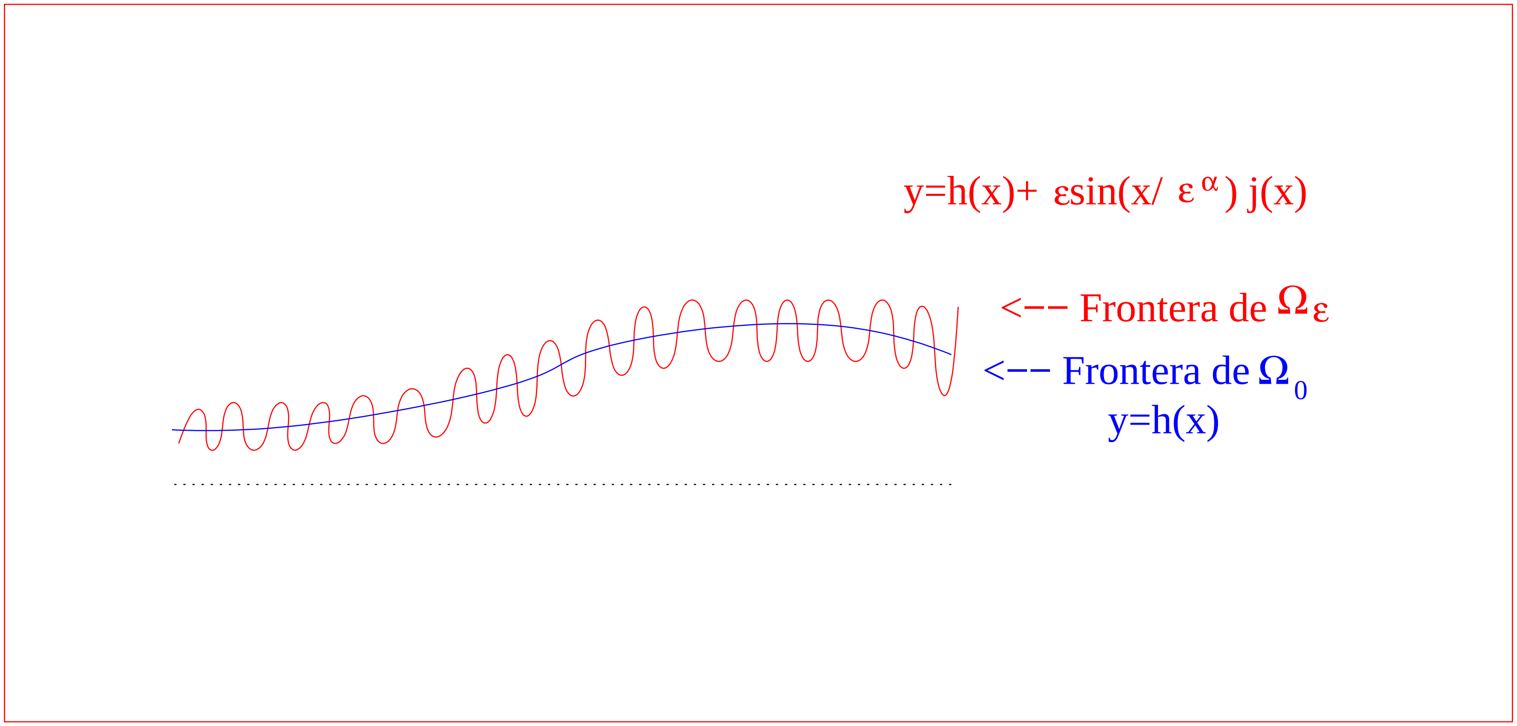
Therefore,

$$\|w_\epsilon - T_\epsilon(w_\epsilon)\|_{H^1(\Omega_\epsilon)} \geq \eta \|w_\epsilon - u_\epsilon^*\|_{H^1(\Omega_\epsilon)} - C\|w_\epsilon - u_\epsilon^*\|_{H^1(\Omega_\epsilon)}^{1+\frac{1}{N-1}}$$

$$\geq \frac{\eta}{2}\|w_\epsilon - u_\epsilon^*\|_{H^1(\Omega_\epsilon)}, \quad \text{if } \|w_\epsilon - u_\epsilon^*\|_{H^1(\Omega_\epsilon)} \leq |\eta/2C|^{N-1}$$

Hence, the unique fixed point of T_ϵ in $B_{H^1(\Omega_\epsilon)}(u_\epsilon^*, |\eta/2C|^{N-1})$ is u_ϵ^*.

Case $\alpha > 1$. Very rapid oscillations.



Formally we have $\gamma = +\infty$.

For the nonlinear problem

$$(P)_\epsilon \quad \begin{cases} -\Delta u + u = f(x,u) & \text{in } \Omega_\epsilon \\ \frac{\partial u}{\partial n} + g(x,u) = 0 & \text{on } \partial\Omega_\epsilon. \end{cases}$$

When $\alpha \leq 1$, the limit problem was given by

$$(P)_0 \quad \begin{cases} -\Delta u + u = f(x,u) & \text{in } \Omega_0 \\ \frac{\partial u}{\partial n} + \gamma(x) g(x,u) = 0 & \text{on } \partial\Omega_0. \end{cases}$$

For the nonlinear problem

$$(P)_\epsilon \quad \begin{cases} -\Delta u + u = f(x,u) & \text{in } \Omega_\epsilon \\ \frac{\partial u}{\partial n} + g(x,u) = 0 & \text{on } \partial\Omega_\epsilon. \end{cases}$$

When $\alpha \leq 1$, the limit problem was given by

$$(P)_0 \quad \begin{cases} -\Delta u + u = f(x,u) & \text{in } \Omega_0 \\ \frac{\partial u}{\partial n} + \gamma(x) g(x,u) = 0 & \text{on } \partial\Omega_0. \end{cases}$$

The factor γ amplifies the dissipativity properties of the boundary condition.

Hence, if $\alpha > 1$ so that $\gamma = +\infty$ and the boundary condition is dissipative: for instance $g(x,u)u \geq \eta|u|^2$ for some $\eta > 0$, then the limit problem is:

$$(P)_0 \quad \begin{cases} -\Delta u + u = f(x,u) & \text{in } \Omega_0 \\ u = 0 & \text{on } \partial\Omega_0. \end{cases}$$

J.A. & S. Bruschi, “Boundary oscillations and nonlinear boundary conditions”, *C.R.A.S.* 343 (2006)

J.A. & S. Bruschi, ”Very rapidly varying boundaries in equations with nonlinear boundary conditions.”(In preparation)

If $\alpha > 1$ and the boundary condition is not dissipative: say $g(x,u) = -u$

$$(P)_\epsilon \quad \begin{cases} -\Delta u + u = f(x,u) & \text{in } \Omega_\epsilon \\ \frac{\partial u}{\partial n} - u = 0 & \text{on } \partial\Omega_\epsilon. \end{cases}$$

The behavior as $\epsilon \longrightarrow 0$ is not clear and it may be very complicated.

For instance, if $f(x,u) = f(u)$ and $f(0) = 0$ and if we denote by $f'(0) = a$, then the eigenvalue problem of the linearized equation around the trivial solution is given by:

$$(EP)_\epsilon \quad \begin{cases} -\Delta \phi + \phi - a\phi = \lambda\phi & \text{in } \Omega_\epsilon \\ \frac{\partial \phi}{\partial n} - \phi = 0 & \text{on } \partial\Omega_\epsilon. \end{cases}$$

The Raleigh quotient is:

$$J(\phi) = \frac{\displaystyle\int_{\Omega_\epsilon} |\nabla\phi|^2 + (1-a)|\phi|^2 - \int_{\partial\Omega_\epsilon} |\phi|^2}{\displaystyle\int_{\Omega_\epsilon} |\phi|^2}$$

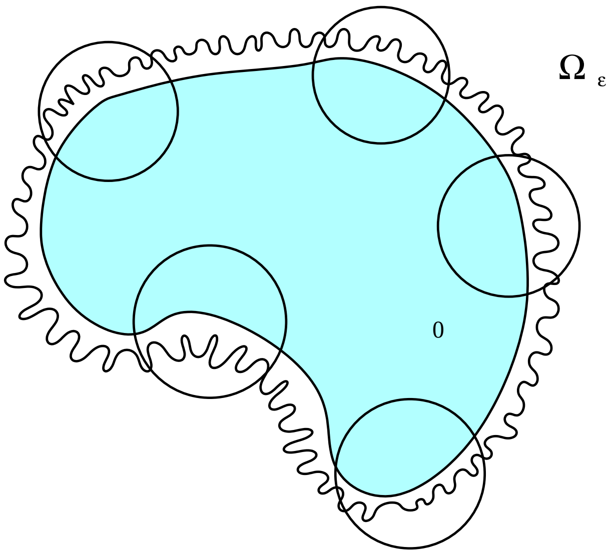
It is not difficult to prove that if λ_n^ϵ is the n-th eigenvalue, we have

$$\lambda_n^\epsilon \to -\infty \text{ as } \epsilon \to 0$$

for all $n = 1, 2, \ldots$.

The Raleigh quotient is:

$$J(\phi) = \frac{\displaystyle\int_{\Omega_\epsilon} |\nabla\phi|^2 + (1-a)|\phi|^2 - \int_{\partial\Omega_\epsilon} |\phi|^2}{\displaystyle\int_{\Omega_\epsilon} |\phi|^2}$$



This is an indication that as $\epsilon \to 0$, the trivial solution becomes more and more unstable, it undergoes a sequence of bifurcations and the dynamics complicates tremendously.