On a weighted total variation minimization problem

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Benasque, Partial differential equations, optimal design and numerics, September 2007.

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Introduction

Given Ω, an open bounded subset of $\mathbb{R}^N$ and nonnegative functions f and g (more precise assumptions on the data Ω, f and g will be given later on), consider

$$\mu := \inf_{u \in BV_0} \mathcal{R}(u) \tag{1}$$

where

$$BV_0 := \{u \in BV(\mathbb{R}^N),\ u \equiv 0 \text{ on } \mathbb{R}^N \setminus \overline{\Omega}\}, \tag{2}$$

and for $u \in BV_0$ such that $\int_\Omega fu \neq 0$,

$$\mathcal{R}(u) := \frac{\displaystyle\int_{\mathbb{R}^N} g(x)\,\mathrm{d}|Du(x)|}{\displaystyle\int_\Omega f(x)|u(x)|\,\mathrm{d}x}. \tag{3}$$

When $g = f = 1$, it is well-known that the infimum in (1) coincides with the infimum of $\mathcal{R}$ over characteristic functions of sets of finite perimeter. In this case, (1) appears as a natural relaxation of:

$$\lambda(\Omega) := \inf_{A \subset \overline{\Omega},\ \chi_A \in BV} \frac{\|D\chi_A\|(\mathbb{R}^N)}{|A|} \tag{4}$$

where $|A|$ and $\|D\chi_A\|(\mathbb{R}^N)$ denote respectively the Lebesgue measure of A and the total variation of $D\chi_A$. Problem (4) is known as Cheeger's problem, its value $\lambda(\Omega)$ is called the Cheeger constant of Ω and its minimizers are called Cheeger sets of Ω. Note also that $\lambda(\Omega)$ is the first eigenvalue of the 1-Laplacian on Ω.

Throughout the paper, we will assume that

- Ω is a nonempty open bounded subset of $\mathbb{R}^N$ with a Lipschitz boundary,
- $f \in L^\infty(\Omega)$, $f \geq f_0$ for a positive constant f_0,
- $g \in C^0(\overline{\Omega})$, $g \geq g_0$ for a positive constant g_0.

In what follows, every $u \in BV(\Omega)$ will be extended by 0 outside $\overline{\Omega}$, and thus will also be considered as an element of $BV(\mathbb{R}^N)$, still denoted u. Set, for every u in BV_0:

$$\mathcal{G}(u) := \int_{\mathbb{R}^N} g(x)\, \mathrm{d}\, |Du(x)|\,. \tag{5}$$

Since $\partial\Omega$ is Lipschitz, note that, for $u \in BV(\Omega)$:

$$\mathcal{G}(u) = \int_{\Omega} g(x)\, \mathrm{d}\, |Du(x)| + \int_{\partial\Omega} g(x)\, |u(x)|\, \mathrm{d}\, \mathcal{H}^{N-1}(x)$$

Taking advantage of the homogeneity of (1), it is convenient to reformulate (1) as the convex minimization problem

$$\mu = \inf_{u \in BV_f} \mathcal{G}(u) \tag{6}$$

where

$$BV_f := \left\{ u \in BV(\mathbb{R}^N),\ u \geq 0,\ u \equiv 0 \text{ on } \mathbb{R}^N \setminus \overline{\Omega},\ \int_\Omega fu = 1 \right\}. \tag{7}$$

In analogy with the case $g = f = 1$, it is natural to consider the generalized Cheeger problem:

$$\lambda := \inf_{A \in \mathcal{E}} \frac{\displaystyle\int_{\mathbb{R}^N} g(x)\, \mathrm{d}|D\chi_A(x)|}{\displaystyle\int_A f(x)\, \mathrm{d}x} = \inf_{A \in \mathcal{E}} \mathcal{R}(\chi_A) \tag{8}$$

where

$$\mathcal{E} := \{A \subset \overline{\Omega} \quad \text{with} \quad \int_A f(x)\,\mathrm{d}x > 0 \quad \text{and} \quad \chi_A \in BV(\mathbb{R}^N)\}. \tag{9}$$

Again (1) can be interpreted as a relaxed formulation of (8) and one aim of the talk is to investigate the precise connections between (1) and (8). Solutions of (8) : Generalized Cheeger sets.

Some recent results:

- convergence of the first eigenvalue of the p-laplacian to the Cheeger constant as p tends to 1, convexity results (Kawohl and Fridman, 2003)
- Uniqueness of the Cheeger set when Ω is convex (Alter and Caselles, 2007 and Caselles, Chambolle, Novaga, 2006 in the smooth uniformly convex case),
- Full characterization of the Cheeger set of a convex subset of the plane (Lachand-Robert and Kawohl, 2006),
- extension to the case of a Finsler metric (Kawohl and Novaga, 2006).

Rich and hot topic: constrained isoperimetric problem, shape optimization, Faber-Krahn type inequality, first eigenvalue and eigenfunction of the 1-laplacian, motion by mean curvature.

Recently related to landslide modelling (Ionescu and Lachand-Robert), motivation for general weights f and g that respectively represent the body forces and the (inhomogoneous) yield limit distribution of the geomaterial.

Existence

$$\mu := \inf_{u \in BV_f} \mathcal{G}(u) = \int_{\mathbb{R}^N} g(x) \, \mathrm{d} \, |Du(x)| \, .$$

where

$$BV_f := \Big\{ u \in BV(\mathbb{R}^N), \; u \geq 0, \; u \equiv 0 \text{ on } \mathbb{R}^N \setminus \overline{\Omega}, \int_\Omega f u = 1 \Big\}.$$

and Cheeger's problem:

$$\lambda := \inf_{A \in \mathcal{E}} \frac{\displaystyle\int_{\mathbb{R}^N} g(x) \, \mathrm{d}|D\chi_A(x)|}{\displaystyle\int_A f(x) \, \mathrm{d}x} = \inf_{A \in \mathcal{E}} \mathcal{R}(\chi_A)$$

The direct method of the calculus of variations yields:

Proposition 1 *The infimum of (6) is achieved in BV_f and the infimum of (8) is achieved in $\mathcal{E}$.*

Outline

① Invariance

② Generalized Cheeger sets

③ Qualitative properties of solutions

④ Maximal Cheeger sets

⑤ Selection of Maximal Cheeger sets

Invariance

Proposition 2 *Let* $H \in W^{1,\infty}(\mathbb{R},\mathbb{R}) \cap C^{\infty}(\mathbb{R},\mathbb{R})$ *be such that* $H(0) = 0$ *and* $H' > 0$ *on* $\mathbb{R}$*. If* u *is a solution of (6) then so is* $T_H(u)$ *defined by*

$$T_H(u) := \frac{H \circ u}{\displaystyle\int_\Omega f(x) H(u(x))\,\mathrm{d}x}. \tag{10}$$

Idea of the proof: Consider $X_t(.)$ the flow of the ordinary differential equation

$$\dot{v} = -H(v).$$

and set $u_t(x) := X_t(u(x))$.

Let us also define

$$h(t) := \int_{\mathbb{R}^N} g(x)\,\mathrm{d}|Du_t(x)| - \mu \int_\Omega f(x)u_t(x)dx.$$

so that:

$$\frac{h(t)-h(0)}{t} \geq 0. \tag{11}$$

Using the chain rule for BV functions and letting $t \to 0^+$, we get:

$$0 \geq \int_{\mathbb{R}^N} g(x)\,\mathrm{d}|D(H \circ u)(x)| - \mu \int_\Omega f(x)H(u(x))\,\mathrm{d}x$$

which proves that $H \circ u$ minimizes $\mathcal{R}$, or equivalently, that $T_H(u)$ solves (6).

Corollary 1 *Let u be a solution of (6) and $H \in W^{1,\infty}(\mathbb{R}, \mathbb{R})$ be a nondecreasing function such that $H(0) = 0$. If $H \circ u \neq 0$ then $T_H(u)$ defined by (10) also solves (6).*

Remark Note that corollary 1 applies in particular to $H(v) = (v - t_0)_+$ and $H(v) = \min(v, t_0)$.

Remark Taking H bounded shows the existence of bounded solutions to (6). We shall see in Theorem 3 that in fact *every* solution (6) is in fact L^∞.

Generalized Cheeger sets

Theorem 1 *Let u be a solution of (6) and for every $t \geq 0$, define $E_t := \{x \in \mathbb{R}^N \,:\, u(x) > t\}$. For every $t \geq 0$ such that E_t has positive Lebesgue measure* $\dfrac{1}{\int_{E_t} f}\,\chi_{E_t}$ *solves (6). In particular,* $\dfrac{1}{\int_{\{u>0\}} f}\,\chi_{\{u>0\}}$ *solves (6).*

Proof:

$E_0 := \{u > 0\}$. Define for every $n \in \mathbb{N}^*$ and $v \in \mathbb{R}$:

$$H_n(v) := \begin{cases} 0 & \text{if } v \leq 0 \\ nv & \text{if } v \in [0, \frac{1}{n}] \\ 1 & \text{if } v \geq \frac{1}{n}. \end{cases}$$

Corollary 1 implies that $T_{H_n}(u)$ solves (6). Since $T_{H_n}(u)$ converges in $L^1(\mathbb{R}^N)$ to $\dfrac{1}{\int_{\{u>0\}} f}\,\chi_{\{u>0\}}$, we conclude that $E_0 = \{u > 0\}$ is a Cheeger set.

Let $t \geq 0$ be such that E_t has positive Lebesgue measure. From corollary 1, $v := \dfrac{(u-t)_+}{\int f(u-t)_+}$ solves (6), hence so does

$$\frac{1}{\int_{\{v>0\}} f}\,\chi_{\{v>0\}} = \frac{1}{\int_{E_t} f}\,\chi_{E_t}.$$

Converse:

Proposition 3 *Let* $u \in BV_0$, $u \geq 0$. *If for every* $t \geq 0$ *such that* $E_t := \{x \in \mathbb{R}^N \ : \ u(x) > t\}$ *has positive Lebesgue measure,* χ_{E_t} *solves (1) then* u *solves (1).*

Straightforward application: relaxation

Corollary 2 *The values of problems (6) and (8) coincide:*

$$\mu = \inf_{u \in BV_0} \mathcal{R}(u) = \lambda = \inf_{A \in \mathcal{E}} \mathcal{R}(\chi_A).$$

Remark One can obtain the relaxation result of corollary 2 as a direct consequence of the coarea and Cavalieri's formulae. Obviously has $\mu \leq \lambda$ and if $u \in BV_0$, $u \geq 0$, setting $E_t := \{u > t\}$, the coarea and Cavalieri's formulae yield:

$$\int_{\mathbb{R}^N} g \, \mathrm{d}|Du(x)| - \lambda \int_{\mathbb{R}^N} fu$$
$$= \int_0^\infty \left(\int_{\partial^* E_t} g \, \mathrm{d}\mathcal{H}^{N-1} - \lambda \int_{E_t} f(x) \, \mathrm{d}x \right) dt \geq 0$$

which proves that $\mu \geq \lambda$. From the previous argument, in fact, we see that the converse also holds: u solves (6) if and only if $E_t := \{u > t\}$ (which has finite perimeter for a.e. t) solves (8) for a.e. $t \geq 0$. Note that in Theorem 1, we have proved that E_t solves (8) for *all* t (and we have not used the coarea formula).

Of course, theorem 1 and its proof contain much more information than corollary 2. A more precise consequence of theorem 1 is the following

Corollary 3 *$A \in \mathcal{E}$ solves (8) if and only if there exists u solving (6) such that $A = \{u > 0\}$.*

A straightforward application:

Theorem 2 *Let $(A_n)_n$ be a sequence of solutions of (8) then $\bigcup_n A_n$ is also a solution of (8).*

Qualitative properties of solutions

Theorem 3 *Let u be a solution of (6). Then u belongs to $L^\infty(\Omega)$.*

Idea of the proof: Use the invariance with powers of $\min(u, M)$ and a bootstrap argument.

Combining the L^∞ estimate with the invariance property also yields:

Theorem 4 *Let u be a solution of (6), then the set $\{u = \|u\|_\infty\}$ has positive Lebesgue measure.*

Maximal Cheeger sets

In general (with weights and or in a nonconvex Ω) the Cheeger set is not unique, there are even known examples where there are infinitely many (even a continuum)! However, nonuniqueness is rather rare in the sense of Baire:

Proposition 4 *Let $g \in C^0(\overline{\Omega})$ with $g \geq g_0$ for a positive constant g_0. Then there exists a G_δ dense subset X of $C^0(\overline{\Omega}, \mathbb{R}^+)$ such that for every $f \in X$,* (6) *admits a unique solution (equivalently $\mathcal{C}$ is a singleton).*

When uniqueness fails: maximal Cheeger set.

Let us denote by $\mathcal{C}$ the set of Cheeger sets. We have seen that a (countable) union of Cheeger sets still is a Cheeger set, one then easily deduces the following:

Proposition 5 *There exists a unique maximal Cheeger set, i.e. a unique $C_0 \in \mathcal{C}$ such that for every $C \in \mathcal{C}$, C is included in C_0 up to a Lebesgue negligible set.*

Selection of maximal Cheeger sets

Natural questions:

- Do natural approximation schemes (e.g. p-laplacian) select the maximal Cheeger set at the limit?
- Do solutions of approximated problems converge to (a multiple of) the characteristic function of the maximal Cheeger set?
- Does at least their support identify the maximal Cheeger set at the limit?

Two approximation schemes:

1) p-laplacian approximation:

$$\mu_p := \sup\left\{ \int_\Omega f u \, dx \ : \ \int_\Omega g|Du|^p \, dx \leq 1, \ u \in W_0^{1,p}(\Omega) \right\}. \quad (12)$$

The unique (nonnegative) maximizer u_p of (12) is of course the solution of the PDE

$$-\operatorname{div}\left(g|Du|^{p-2}Du\right) = \lambda_p f, \ u \in W_0^{1,p}(\Omega), \ \text{with } \lambda_p := \frac{1}{\mu_p}. \quad (13)$$

2) Concave penalization:

$$\sup\left\{ \int_\Omega f\left(u - \varepsilon\Phi(u)\right) dx \ : \ \int_{\mathbb{R}^d} g \, d|Du| \leq 1, \ u \in BV_0(\Omega) \right\} \quad (14)$$

with Φ strictly convex and $\Phi(0) = 0$

p-laplacian approximation

Limit problem

$$\mu_1 = \sup\left\{\int_\Omega f u \, dx \; : \; u \in BV_0(\Omega), \; \int_{\mathbb{R}^d} g \, d|Du| \leq 1\right\}, \qquad (15)$$

(inverse of the Cheeger constant).

Convergence: (see also Kawohl and Fridman)

$$\lim_{p \to 1^+} \mu_p = \mu_1$$

and (up to some subsequence) u_p converges in L^1 to u a solution of (15).

- Is u (a multiple of) the characteristic function of the maximal Cheeger set (of course it is when there is uniqueness)?
- Is $\{u > 0\}$ the maximal Cheeger set?

Answer is NO to both questions (1d counter-examples).

Concave penalization

$$\sup\left\{\int_\Omega f\big(u - \varepsilon\Phi(u)\big)\,dx \;:\; \int_{\mathbb{R}^d} g\,d|Du| \le 1,\ u \in BV_0(\Omega)\right\} \quad (16)$$

where $\varepsilon > 0$ is a perturbation parameter and Φ is a strictly convex nonnegative function that satisfies:

$$\Phi(0) = 0, \qquad 0 \le \Phi(t) < +\infty \quad \forall t \in \mathbb{R}^+. \qquad (17)$$

Theorem 5 *Let u_ε be the solution of* (16)*; then the following holds:*

- $(u_\varepsilon)_\varepsilon$ *converges in* $L^1(\Omega)$, *as* $\varepsilon \to 0^+$, *to the solution* $\overline{u}$ *of*

$$\inf\left\{\int_\Omega f\Phi(u)\,dx \;:\; u \in Q\right\}, \tag{18}$$

- $\overline{u} = \alpha\chi_{C_0}$ *for some* $\alpha > 0$ *and* $C_0 \subset \overline{\Omega}$,
- C_0 *is the maximal Cheeger set, i.e.* $C_0 \in \mathcal{C}$ *and* C_0 *contains every other Cheeger set (up to a Lebesgue negligible set).*