Bending Moment in Membrane Theory

by

G. BOUCHITTÉ (Université de Toulon et du Var)

I. FONSECA (Carnegie Mellon University)

L. MASCARENHAS (Universidade de Lisboa)

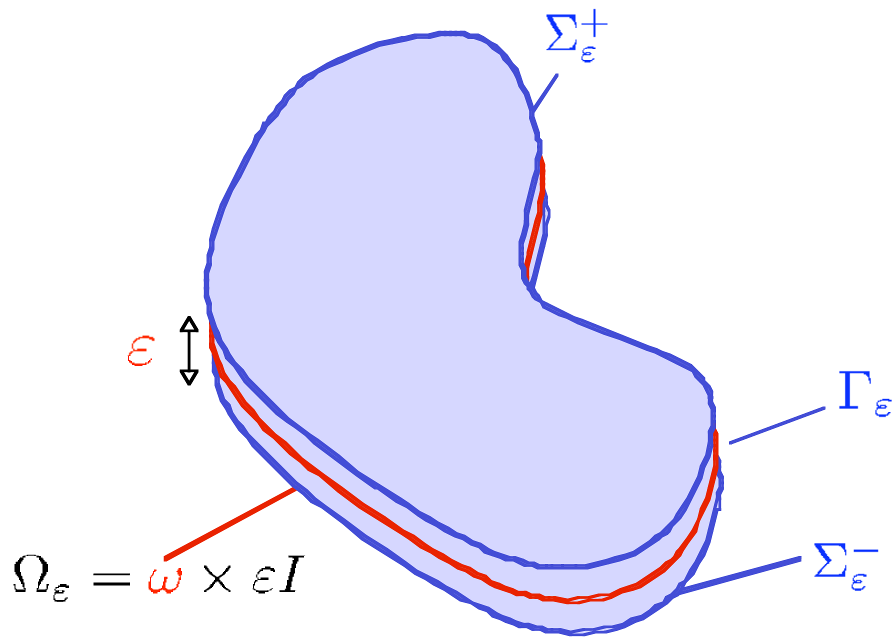
Abstract (J. Elasticity 73, no. 1-3, 75–99 (2004))

Via Γ-convergence, we deduce a 2-D membrane model from a 3-D nonlinear elasticity framework where we consider a class of surface forces generating the bending moment.

The problem

The rescaled total energy of a deformation U of Ω_ε is given by

$$\frac{1}{\varepsilon}\int_{\Omega_\varepsilon} W(DU)\, d\tilde{x} - < F_\varepsilon\,, U >$$

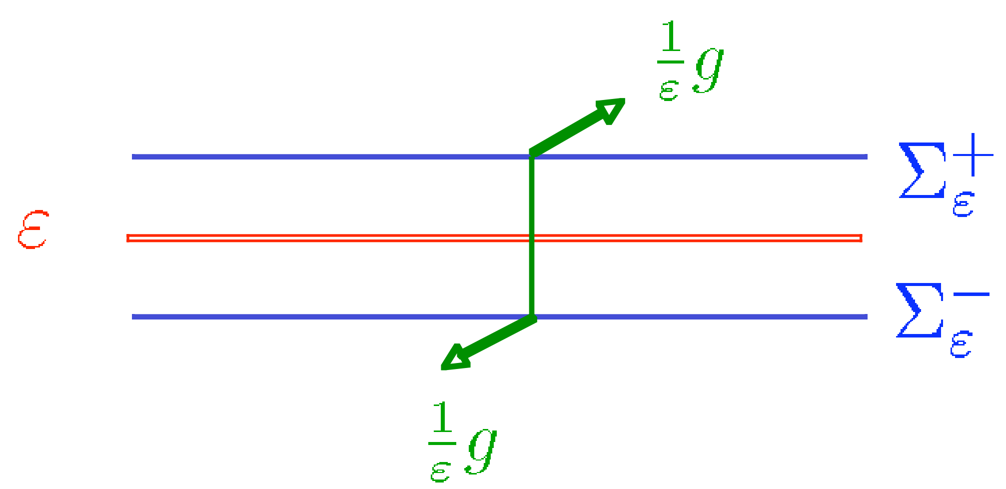


Where, for $p' = p/(p-1)$**,** $1 < p < +\infty$**,**

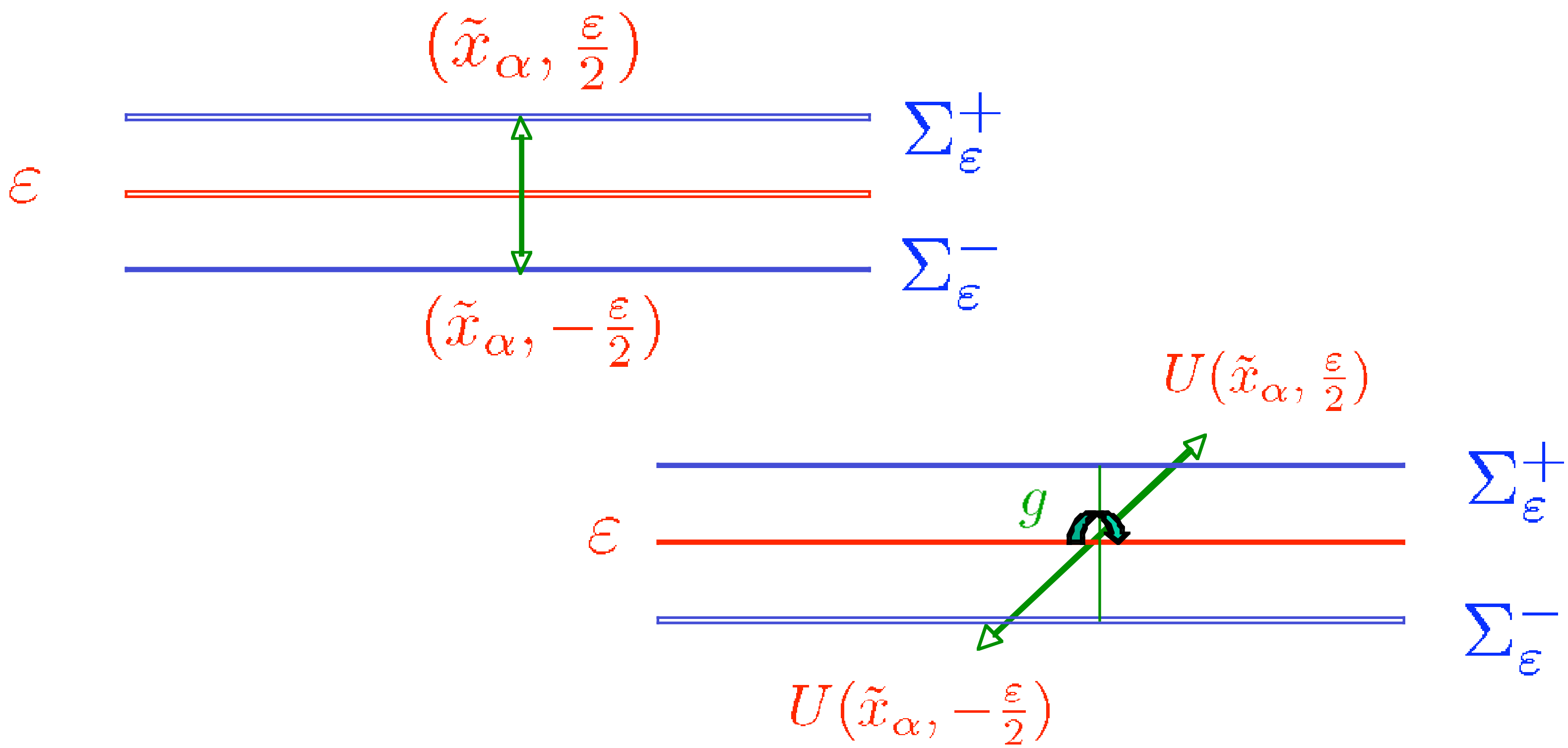
$$< F_\varepsilon\,, U >:= \int_{\Omega_\varepsilon} f_\varepsilon\ U\ d\tilde{x} + \int_\omega \Big(g_0^+\ U(\tilde{x}_\alpha, \varepsilon/2) - g_0^-\ U(\tilde{x}_\alpha, -\varepsilon/2) \Big)\ d\tilde{x}_\alpha$$

$$+ \int_\omega \frac{1}{\varepsilon} g\ \Big(U(\tilde{x}_\alpha, \varepsilon/2) -\ U(\tilde{x}_\alpha, -\varepsilon/2) \Big)\ d\tilde{x}_\alpha,$$

with $g_0^+, g_0^-, g \in L^{p'}(\omega; \mathbb{R}^3)$.



$$\int_\omega \frac{1}{\varepsilon} g \left(U(\tilde{x}_\alpha, \varepsilon/2) - U(\tilde{x}_\alpha, -\varepsilon/2) \right) d\tilde{x}_\alpha = \int_\omega g \; \frac{U(\tilde{x}_\alpha, \frac{\varepsilon}{2}) - U(\tilde{x}_\alpha, -\frac{\varepsilon}{2})}{\varepsilon} \; d\tilde{x}_\alpha$$



If the deformations U satisfy a boundary condition of place on Γ_ε, the equilibrium problem under the load F_ε is :

$$(1) \qquad \inf_{U-\tilde{x}\in W^{1,p}_{\Gamma_\varepsilon}(\Omega_\varepsilon;\mathbb{R}^3)} \left\{ \frac{1}{\varepsilon}\int_{\Omega_\varepsilon} W(DU)\, d\tilde{x} - < F_\varepsilon, U > \right\}$$

We assume that the potential W is a Borel function satisfying :

$$(\text{H1}) \qquad \frac{1}{C}|\xi|^p - C \leq W(\xi) \leq C(1+|\xi|^p)$$

Existence of a solution for problem (1) can be obtained via direct method, hypothesis (H1) and the additional hypothesis that W is quasiconvex.

In order to pass to the limit in problem (1) as $\varepsilon \to 0$ we perform the usual change of variables :

$$\Omega_\varepsilon \longrightarrow \Omega = \omega \times I\Big(:= (-1/2, 1/2)\Big)$$

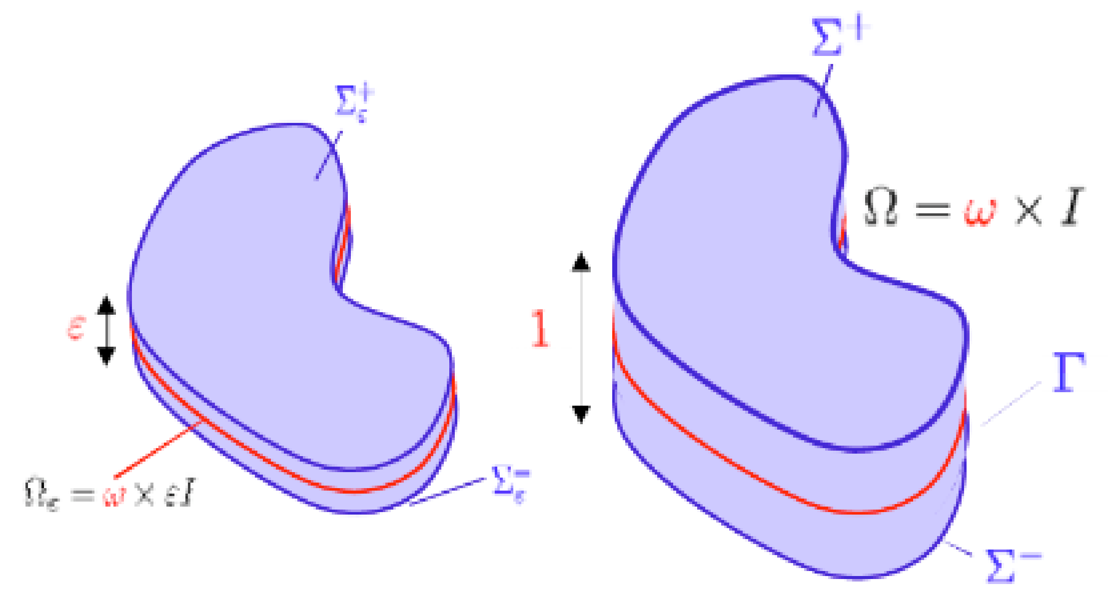
$$\tilde{x} = (\tilde{x}_\alpha, \tilde{x}_3) \in \Omega_\varepsilon \longrightarrow x = (x_\alpha, x_3) = \left(\tilde{x}_\alpha, \frac{1}{\varepsilon}\tilde{x}_3\right) \in \Omega$$

and define u, $u^\pm$, $u_{0,\varepsilon}$

$$u(x_\alpha, x_3) := U(\tilde{x}_\alpha, \tilde{x}_3)$$

$$u^\pm(x_\alpha) := u\left(x_\alpha, \pm\frac{1}{2}\right)$$

$$u_{0,\varepsilon}(x_\alpha, x_3) := (x_\alpha, \varepsilon x_3) = (\tilde{x}_\alpha, \tilde{x}_3)$$



$$f_\varepsilon := \frac{1}{\varepsilon} f\left(\tilde{x}_\alpha, \frac{\tilde{x}_3}{\varepsilon}\right), \quad f \in L^{p'}(\Omega; \mathbb{R}^3)$$

Problem (1) becomes

$$(\mathcal{P}_\varepsilon) \qquad \inf_{u-u_{0,\varepsilon}\in W^{1,p}_\Gamma(\Omega;\mathbb{R}^3)} \left\{ \int_\Omega W\left(D_\alpha u \;\Big|\; \frac{1}{\varepsilon} D_3 u \right)\, dx \; - L_\varepsilon(u) \right\},$$

with

$$L_\varepsilon(u) := \int_\Omega f\ u\ dx + \int_\omega (g_0^+\ u^+ - g_0^-\ u^-)\ dx_\alpha + \int_\omega g\ \left(\frac{u^+ - u^-}{\varepsilon}\right)\ dx_\alpha$$

where

$$\frac{U(\tilde{x}_\alpha, \frac{\varepsilon}{2}) - U(\tilde{x}_\alpha, -\frac{\varepsilon}{2})}{\varepsilon} = \frac{u^+(x_\alpha) - u^-(x_\alpha)}{\varepsilon} = \int_I \frac{1}{\varepsilon} D_3 u(x_\alpha, x_3)\ dx_3.$$

Coercivity (H1) plus b. c. $u_\varepsilon = u_{0,\varepsilon}$ **on** Γ**, imply that any diagonal infimizing sequence** $\{u_\varepsilon\}$ **satisfies :**

$$\sup_\varepsilon \left\{ \int_\Omega |D_\alpha u_\varepsilon|^p dx + \int_\Omega \frac{1}{\varepsilon^p} |D_3 u_\varepsilon|^p dx \right\} < +\infty.$$

Then $u_\varepsilon \rightharpoonup u = u(x_\alpha)$ **and** $b_\varepsilon := \dfrac{1}{\varepsilon} D_3 u_\varepsilon \rightharpoonup b$ **in** $W^{1,p}(\Omega;\mathbb{R}^3) \times L^p(\Omega;\mathbb{R}^3)$**.**

One obtains

$u = u(x_\alpha)$ **and** b **is no longer related with** u**.**

The limit problem will involve explicitly the average :

$$\bar{b}(x_\alpha) := \int_I b(x_\alpha, x_3)\ dx_3.$$

More precisely, recalling the corresponding term in $L_\varepsilon(u_\varepsilon)$ **:**

$$\int_\omega g\left(\frac{u_\varepsilon^+ - u_\varepsilon^-}{\varepsilon}\right)\ dx_\alpha = \int_\omega g\ \left(\int_I \frac{1}{\varepsilon} D_3 u_\varepsilon\ dx_3\right)\ dx_\alpha$$

$$= \int_\omega g\Big(\int_I b_\varepsilon\ dx_3\Big)\ dx_\alpha \longrightarrow \int_\omega g\Big(\int_I b\ dx_3\Big)\ dx_\alpha = \int_\omega g\ \bar{b}\ dx_\alpha.$$

Then $\lim\limits_{\varepsilon \to 0} L_\varepsilon(u_\varepsilon) = L(u, \bar{b})$ **with**

$$L(u, \bar{b}) := \int_\omega \bar{f}\ u\ dx_\alpha + \int_\omega (g_0^+ - g_0^-)\ u\ dx_\alpha + \int_\omega g\ \bar{b}\ dx_\alpha \qquad \left(\bar{f} := \int_I f\ dx_3\right)$$

In order to individualize $\bar{b}$ in the principal part of the total energy, we introduce

$$E_\varepsilon : W^{1,p}(\Omega;\mathbb{R}^3) \times L^p(\omega;\mathbb{R}^3) \to \overline{\mathbb{R}},$$

defined by

$$E_\varepsilon(u,\bar{b}) := \begin{cases} \displaystyle\int_\Omega W\left(D_\alpha u \left| \frac{1}{\varepsilon} D_3 u\right.\right)\,dx & \text{if } \displaystyle\int_I \frac{1}{\varepsilon} D_3 u\,dx_3 = \bar{b}, \\ \\ +\infty & \text{otherwise.} \end{cases}$$

The aim is to prove that

1. $E_\varepsilon(u,\bar{b})$ **Γ-converges to** $E(u,\bar{b})$ **in the weak top.** $W^{1,p}(\Omega;\mathbb{R}^3) \times L^p(\omega;\mathbb{R}^3)$**.**

2. $E(u,\bar{b})$ **has an integral representation and to characterize its density.**

Defining

$$\mathcal{V} := \{u \in W^{1,p}(\Omega;\mathbb{R}^3) \mid D_3 u(x) = 0 \text{ a.e. in } x \in \Omega\}$$

and, for $(u,\bar{b}) \in \mathcal{V} \times L^p(\omega;\mathbb{R}^3)$,

$$E(u,\bar{b}) = \int_\omega \mathcal{Q}^* W(D_\alpha u|\bar{b}),$$

where $\mathcal{Q}^* W$ **is the cross-quasiconvex envelop of** W**, introduced by H. LeDret & A. Raoult in ARMA 2000, and coincides with**

$$\mathcal{Q}^* W(F|b) := \inf_{(\varphi,\psi)} \Big\{ \int_{Q'} W(F + D_\alpha \varphi | b + \psi)\ dx_\alpha : \\ \varphi \in W^{1,p}_{\#}(Q';\mathbb{R}^3),\ \phi \in L^p_0(Q';\mathbb{R}^3) \Big\},$$

with $I := (-1/2, 1/2), Q' := I^2$**.**

Theorem

Under the hypothesis (H1), **the sequence** $\{E_\varepsilon\}$ **Γ-converges to** E, **as** $\varepsilon \to 0$, **precisely,**

i) if $u_\varepsilon \rightharpoonup u$ **in** $W^{1,p}(\Omega;\mathbb{R}^3)$ **and** $\bar{b}_\varepsilon = \left(\dfrac{u_\varepsilon^+ - u_\varepsilon^-}{\varepsilon}\right) = \displaystyle\int_I \frac{1}{\varepsilon} D_3 u_\varepsilon \ dx_3 \rightharpoonup \bar{b}$ **in** $L^p(\omega;\mathbb{R}^3)$ **then**

$$\liminf_{\varepsilon\to 0} \int_\Omega W\left(D_\alpha u_\varepsilon \,\Big|\, \frac{1}{\varepsilon} D_3 u_\varepsilon\right) \ dx \geq E(u,\bar{b}) \ ;$$

ii) for every pair $(u,\bar{b})$ **in** $\mathcal{V} \times L^p(\omega;\mathbb{R}^3)$, **there exists a sequence** $\{u_\varepsilon\}$ **such that**

$$(u_\varepsilon, \bar{b}_\varepsilon) \rightharpoonup (u,\bar{b}) \quad , \quad \lim_{\varepsilon\to 0} \int_\Omega W\left(D_\alpha u_\varepsilon \,\Big|\, \frac{1}{\varepsilon} D_3 u_\varepsilon\right) \ dx = E(u,\bar{b}) \ .$$

Corollary

Let W **satisfy (H1).**

Let $f \in L^{p'}(\Omega, \mathbb{R}^3)$, $g_0^{\pm}, g \in L^{p'}(\omega, \mathbb{R}^3)$.

Let $\{u_\varepsilon\}$ **be a diagonal infimizing sequence for** $(\mathcal{P}_\varepsilon)$.

Then the sequence $\{(u_\varepsilon\,, \bar{b}_\varepsilon)\}$ **is weakly relatively compact in** $W^{1,p}(\Omega, \mathbb{R}^3) \times L^p(\omega, \mathbb{R}^3)$.

Furthermore, any cluster point $(u, \bar{b})$ **of this sequence belongs to** $\mathcal{V} \times L^p(\omega, \mathbb{R}^3)$ **and is a solution of**

$$(\mathcal{P}) \qquad \min_{\substack{u - x_\alpha \in W_0^{1,p}(\omega;\mathbb{R}^3) \\ \bar{b} \in L^p(\omega,\mathbb{R}^3)}} \left\{ \int_\omega \mathcal{Q}^* W(D_\alpha u \mid \bar{b})\; dx_\alpha - L(u, \bar{b}) \right\}.$$

Idea of the proof

We localize the functionals E_ε :

$$E_\varepsilon(u,\bar b,A) := \begin{cases} \int_{A\times I} W(D_\alpha u \,|\, \frac{1}{\varepsilon} D_3 u)\; dx & \text{if } \frac{1}{\varepsilon}\int_I D_3 u(x_\alpha, x_3)\; dx_3 = \bar b(x_\alpha), \\ +\infty & \text{otherwise.} \end{cases}$$

To prove that $\{E_\varepsilon(\cdot,\cdot,A)\}$ Γ-converges to some functional $E_0(\cdot,\cdot,A)$ for all open $A \subset \omega$, it is enough to show that any given sequence $\{\varepsilon_k\}$ converging to 0^+ admits a subsequence $\{\varepsilon_{k_n}\}$ such that the Γ-lower limit of $E_{\varepsilon_{k_n}}$ given by

$$E^-(u,\bar b,A) := \inf\left\{\liminf_n \int_{A\times I} W\left(D_\alpha u_n \,|\lambda_n\; D_3 u_n\right) dx \;\middle|\; u_n \rightharpoonup u \;\; W^{1,p}(A\times I;\mathbb{R}^3),\right.$$
$$\left.\lambda_n \int_I D_3 u_n\; dx_3 \rightharpoonup \bar b,\;\; L^p(A;\mathbb{R}^3)\right\},$$

where $\lambda_n := (\varepsilon_{k_n})^{-1}$, coincides with $E_0(\cdot,\cdot,A)$ for all $(u,\bar b)$ in $\mathcal{V}\times L^p(\omega;\mathbb{R}^3)$.

Idea of the proof

- **We prove that** $E^-(u, \bar{b}, \cdot)$ **is the trace of a measure** $\mu << \mathcal{L}^2 \lfloor \omega$.

- **The infimum in** $E^-(u, \bar{b}, A)$ **remains unchanged if we repalce** W **by** $\mathcal{Q}W$.

- **We prove, by blow up,**

$$E^-(u, \bar{b}, A) \geq \int_A \mathcal{Q}^*(D_\alpha u|\bar{b})\ dx_\alpha.$$

- **We prove, using the fact that** $E^-(u, \bar{b}, \cdot) =: \mu << \mathcal{L}^2 \lfloor \omega$, **that**

$$E^-(u, \bar{b}, A) \leq \int_A \mathcal{Q}^*(D_\alpha u|\bar{b})\ dx_\alpha.$$

□

New problem

In the previous pb the mean condition on the bending term was imposed by the exterior forces.

So it seems natural to study the asymptotic behavior (Γ-limit) of the sequence of functionals $\mathcal{I}_\varepsilon$:

$$\mathcal{I}_\varepsilon(u, b) := \begin{cases} \int_{A\times I} W(D_\alpha u \,|\, \frac{1}{\varepsilon} D_3 u)\, dx & \text{if } \frac{1}{\varepsilon} D_3 u(x_\alpha, x_3) = b(x_\alpha, x_3), \\ +\infty & \text{otherwise,} \end{cases}$$

without imposing the mean condition.

This means to give, for any subsequence $\varepsilon_n \searrow 0$, the same integral representation to the Γ-lower limit of $\mathcal{I}_{\varepsilon_n}$ defined by

$$\mathcal{I}^-(u, b) := \inf \left\{ \liminf_n \int_{\omega\times I} W\left(D_\alpha u_n \,\Big|\, \frac{1}{\varepsilon_n} D_3 u_n\right) dx \,\Big|\, u_n \rightharpoonup u \;\; W^{1,p}(\omega \times I; \mathbb{R}^3), \right.$$
$$\left. \frac{1}{\varepsilon_n} D_3 u_n \rightharpoonup b, \;\; L^p(\omega \times I; \mathbb{R}^3) \right\},$$

for all (u, b) in $\mathcal{V} \times L^p(\Omega; \mathbb{R}^3)$.

Then

We fix a countable dense family $\{\theta_i\}_{i\in\mathbb{N}}$ **in** $L^{p'}(I;\mathbb{R}^3)$**, where** p' **is the conjugate exponent of** p**.**
For every $k\in\mathbb{N}$ **and** $(F,b)\in\mathbb{M}^{3\times 2}\times L^p(I;\mathbb{R}^3)$ **define**

$$\mathcal{Q}_k W(F|b) := \inf_{(\varphi,\lambda)} \Big\{ \int_Q W(F + D_\alpha\varphi|\ \lambda\ D_3\varphi)\ dx \ |\ \lambda\in\mathbb{R},\ \ \varphi\in W^{1,p}(Q;\mathbb{R}^3), \\ \varphi(\cdot,x_3) \textbf{ is } Q' \textbf{ periodic } \textbf{ a.e. } x_3\in I \\ \left|\int_Q \lambda D_3\varphi\ \ \theta_i\ dx - \int_I b\ \theta_i\ dx_3\right| \le \frac{1}{k},\ \ \forall\ i = 1\cdots k\Big\},$$

$$\mathcal{Q}_\infty W(F|b) := \lim_k \mathcal{Q}_k(F|b) = \sup_k \mathcal{Q}_k(F|b).$$

Proposition

The following inequality holds

$$\int_I \mathcal{Q}^* W(F|b(t))\ dt \le \mathcal{Q}_\infty W(F|b) \le \int_I W(F|b(t))\ dt,$$

for all $(F, b) \in \mathbb{M}^{3\times 2} \times L^p(I; \mathbb{R}^3)$.

Consequently, if W **is cross-quasiconvex (**$\mathcal{Q}^* W = W$ **)**

$$\mathcal{Q}_\infty W(F|b) = \int_I W(F|b(t))\ dt.$$

Theorem

Let W be locally p-Lipschitz, satisfying the p-growth condition (H1). Then the sequence $\{\mathcal{I}_\varepsilon\}$ Γ-converges to the functional defined by

$$\mathcal{I}(u,b) = \int_\omega \mathcal{Q}_\infty W(D_\alpha u(x_\alpha)|b(x_\alpha,\cdot))\ dx_\alpha$$

with $(u,b) \in \mathcal{V} \times L^p(\Omega;\mathbb{R}^3)$.

Corollary

If W is cross-quasiconvex, then

$$\mathcal{I}(u,b) = \int_{\omega\times I} W(F|b(t))\ dt\ dx_\alpha.$$

Open question

To prove (or disprove) the locallity of $\mathcal{I}(u, b)$, i.e., the existence of some density function $\tilde{W}$, s.t.

$$\mathcal{I}(u, b) = \int_\omega \mathcal{Q}_\infty W(D_\alpha u(x_\alpha)|b(x_\alpha, \cdot))\ dx_\alpha = \int_{\omega \times I} \tilde{W}(D_\alpha u(x_\alpha)|b(x_\alpha, x_3))\ dx$$

Remark

It is simple to prove that if $\mathcal{I}$ is local, then

$$\tilde{W}(F, b) = \mathcal{Q}^*(F, b)$$

for all $(F, b) \in \mathbb{M}^{3\times 2} \times \mathbb{R}^3$.