Structural optimization and control for Partial Differential Equations on networks

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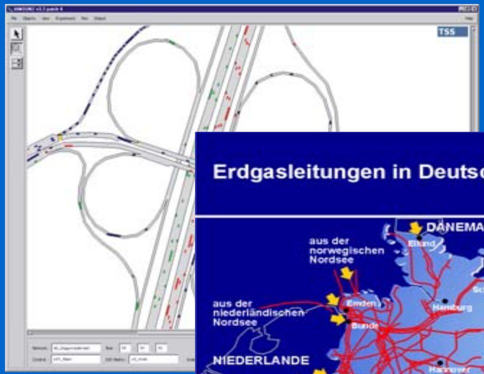
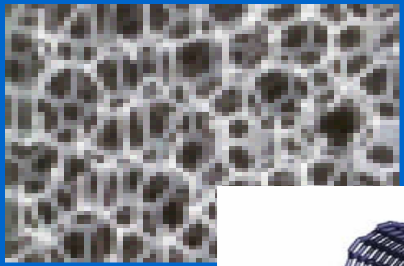
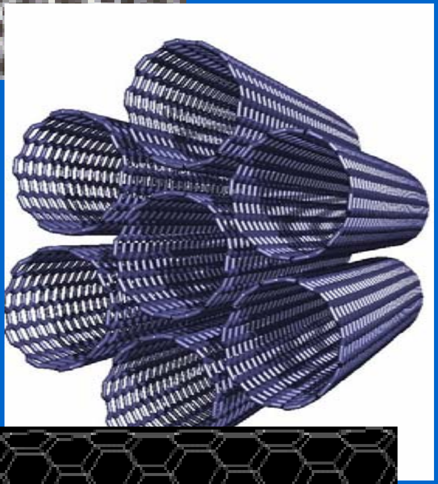
Benasque, 27.08.-07.09.2007

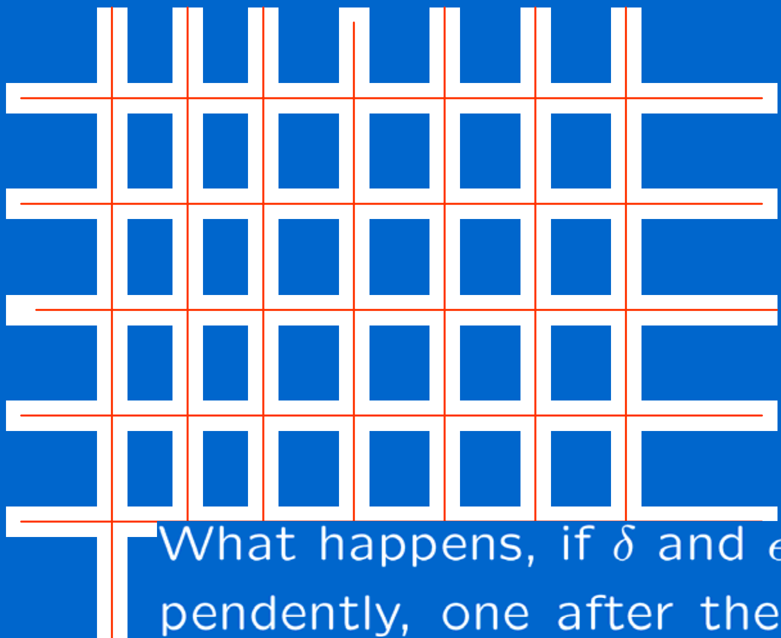
Opimization of transportation/propagation-networks (macro-, meso- and nanoscale)







Fattened graphs and skeletons



skeleton

fattened graph

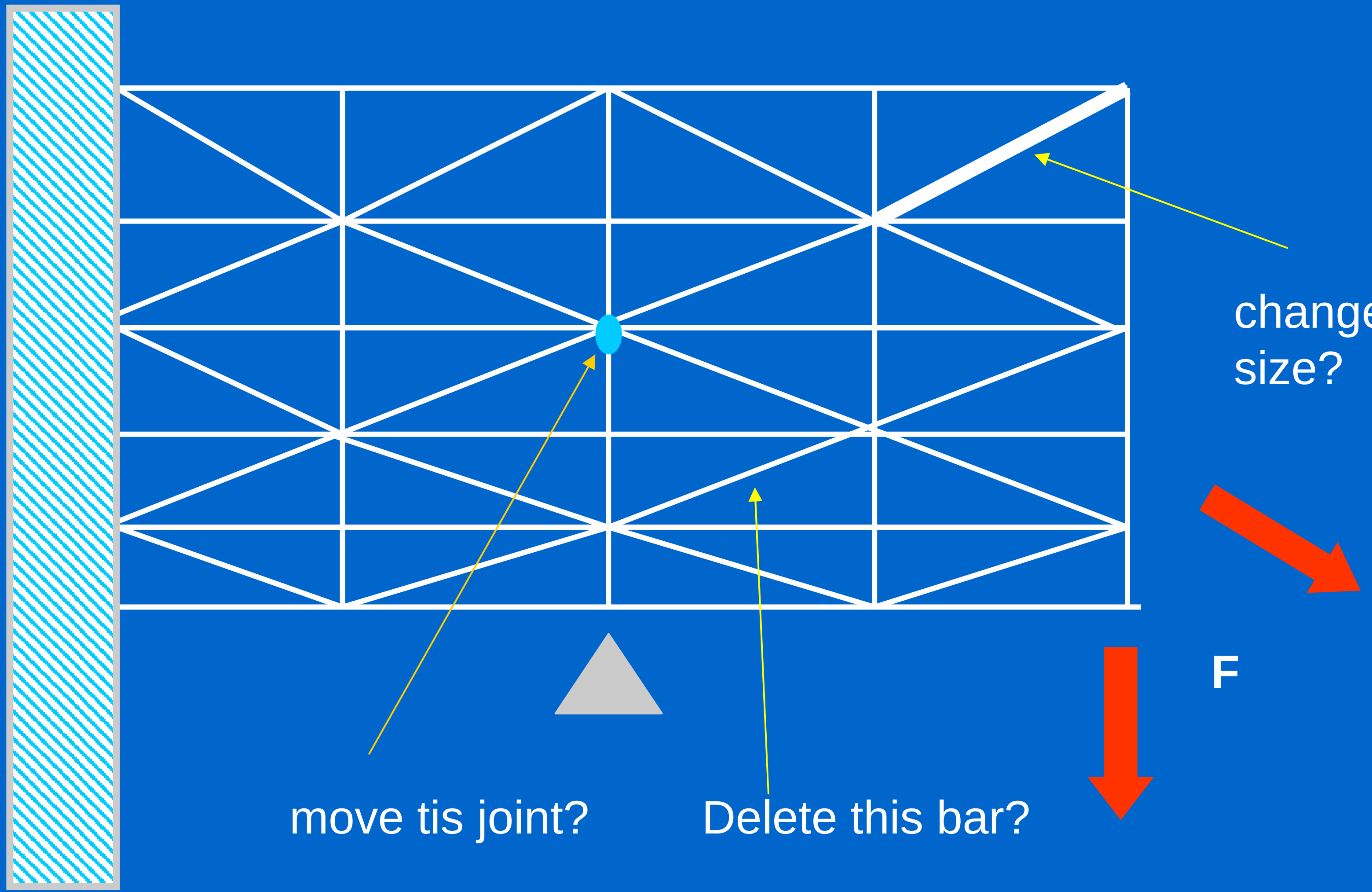
δ thickness

ϵ cell size

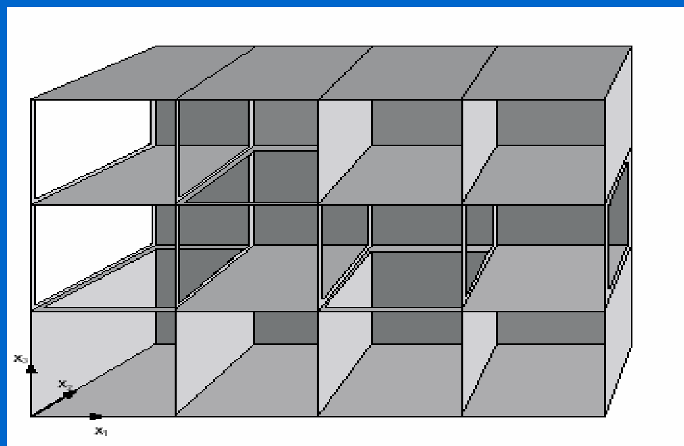
What happens, if δ and ϵ tend to zero? (independently, one after the other.....$\rightarrow$ homogenization of optimization problems for PDEs on reticulated domains

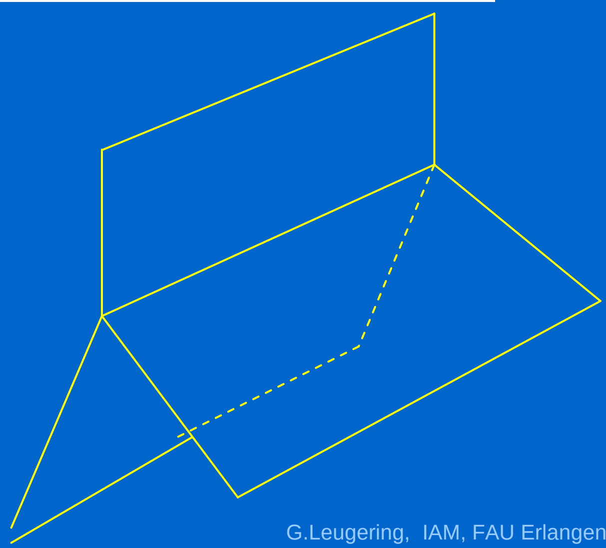
See Peter Kogut's talk right after this

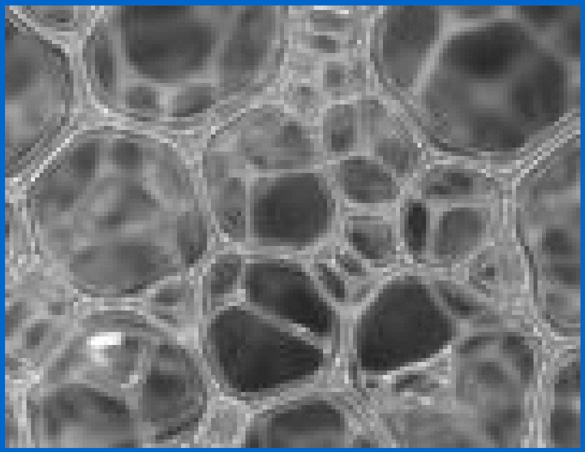
Industrial application: topology optimization for problems defined on graphs (thin domains)



Higher dimensional networks (Lagnese and G.L. 04, Nicaise 97)







Optimization/control on networks

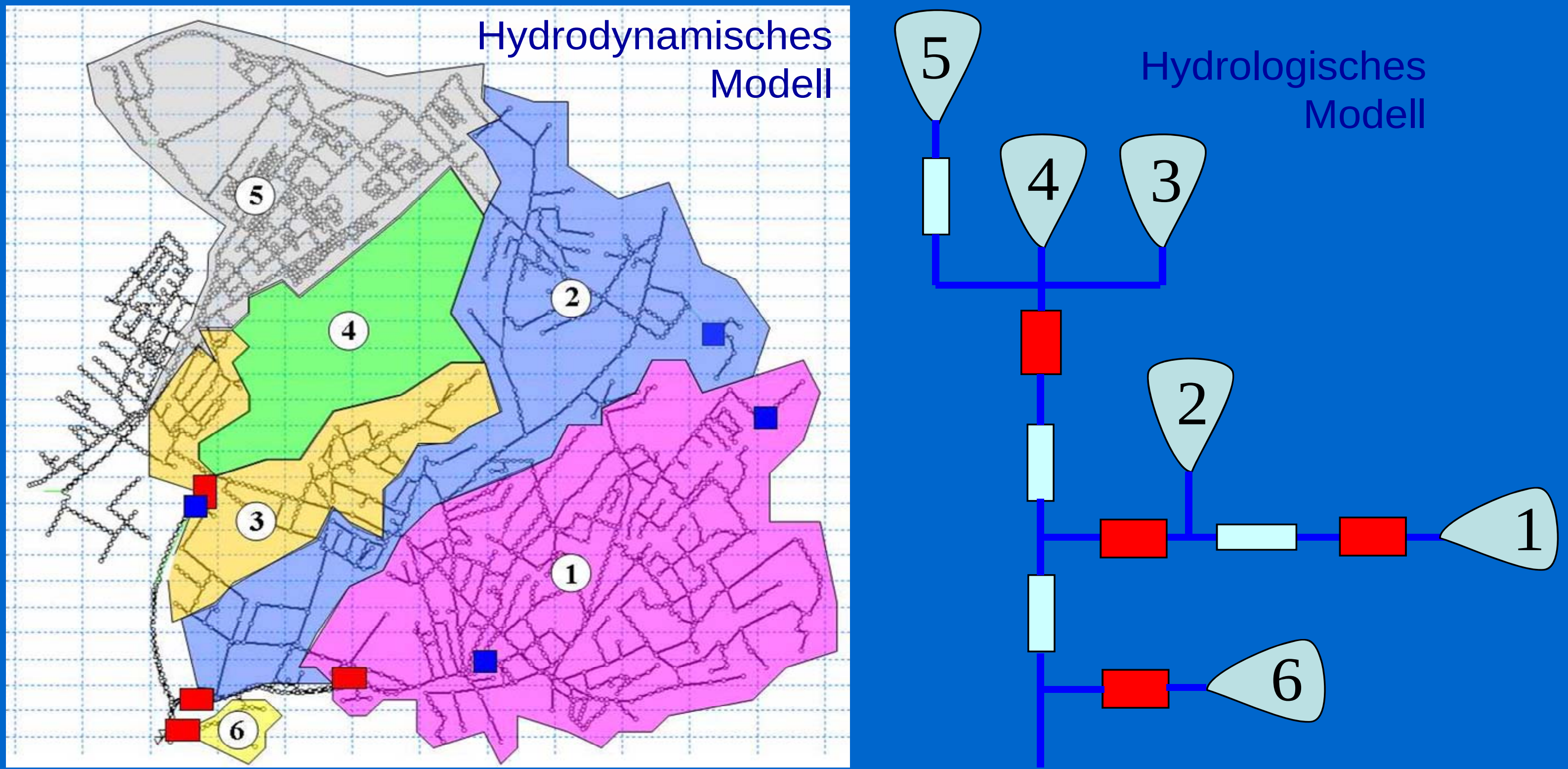
- Controllability/reachability/stabilizability
- Optimal control:
  - Optimal tracking at end-nodes
  - Maximal throughput
  - Minimal operating cost
  - Robust control
  - Real-time control
  - Open-loop/closed loop
  - Suboptimal control: instantaneous control

‚The problem‘

‚Large‘ network problems

- Gas- or water networks: ~ 5000-10000 channels or major pipes
- ‚Microstructures‘: possibly millions of edges -no local analysis wanted (and, in fact, possible)!
- Hence: model-hierarchies needed. Two-level optimization based on discrete-versus continuous graph problem
- Model reduction by the way of domain decomposition (G.L. and Lagnese 04 and very recently Halpern and Szeftel 07) and homogenization (G.L. and Kogut 06/07)

Hydrodynamic versus hydrological modelling



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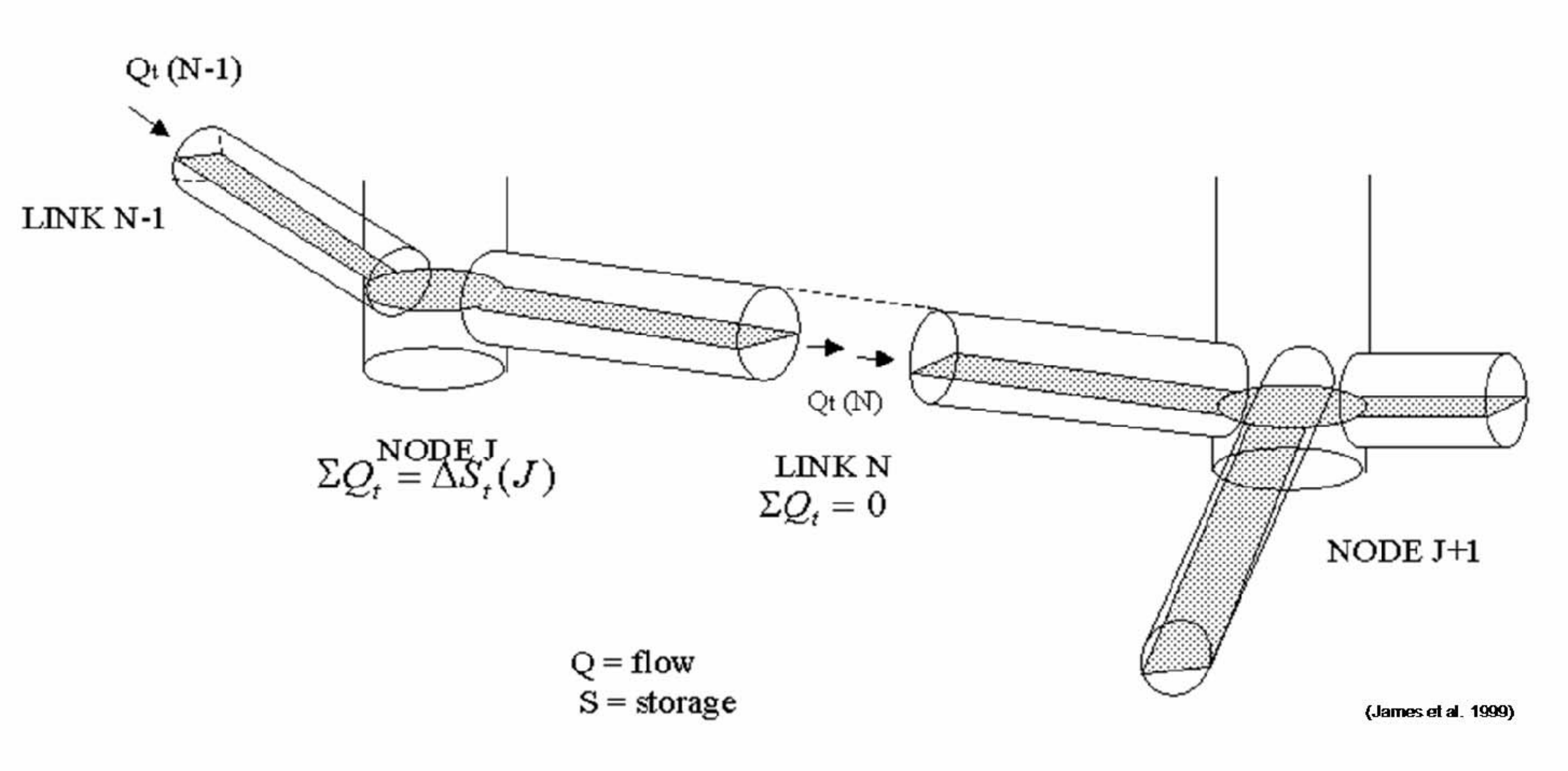
Conceptual hydrodynamical modelling



Shallow water equations

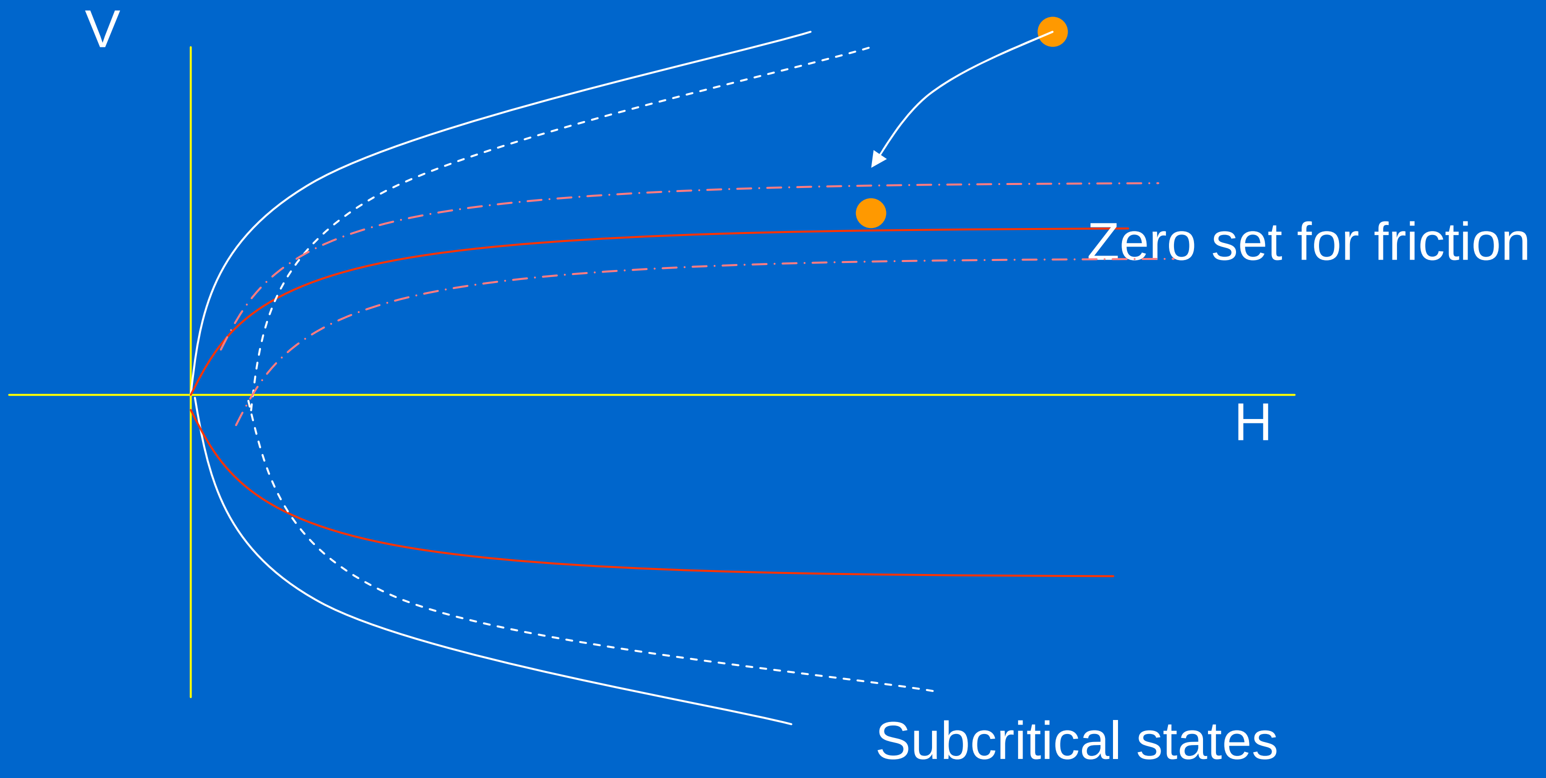
Equations of motion (de St. Venant 1871)

$$\frac{\partial}{\partial t}\begin{pmatrix} A \\ V \end{pmatrix} + \frac{\partial}{\partial x}\begin{pmatrix} AV \\ \frac{1}{2}V^2 + gh(x,A) + gY_b(x) \end{pmatrix} = \begin{pmatrix} 0 \\ gI_f \end{pmatrix}$$

Notation: $U = \begin{pmatrix} A \\ V \end{pmatrix}$, $F(U) = \begin{pmatrix} AV \\ \frac{1}{2}V^2 + gh(x,A) + gY_b(x) \end{pmatrix}$,

$$\boxed{\partial_t U + \partial_x F(U) = S(U)}$$

Control of subcritical points to equlibrium:
see more in a special talk by Martin Gugat tomorrow…

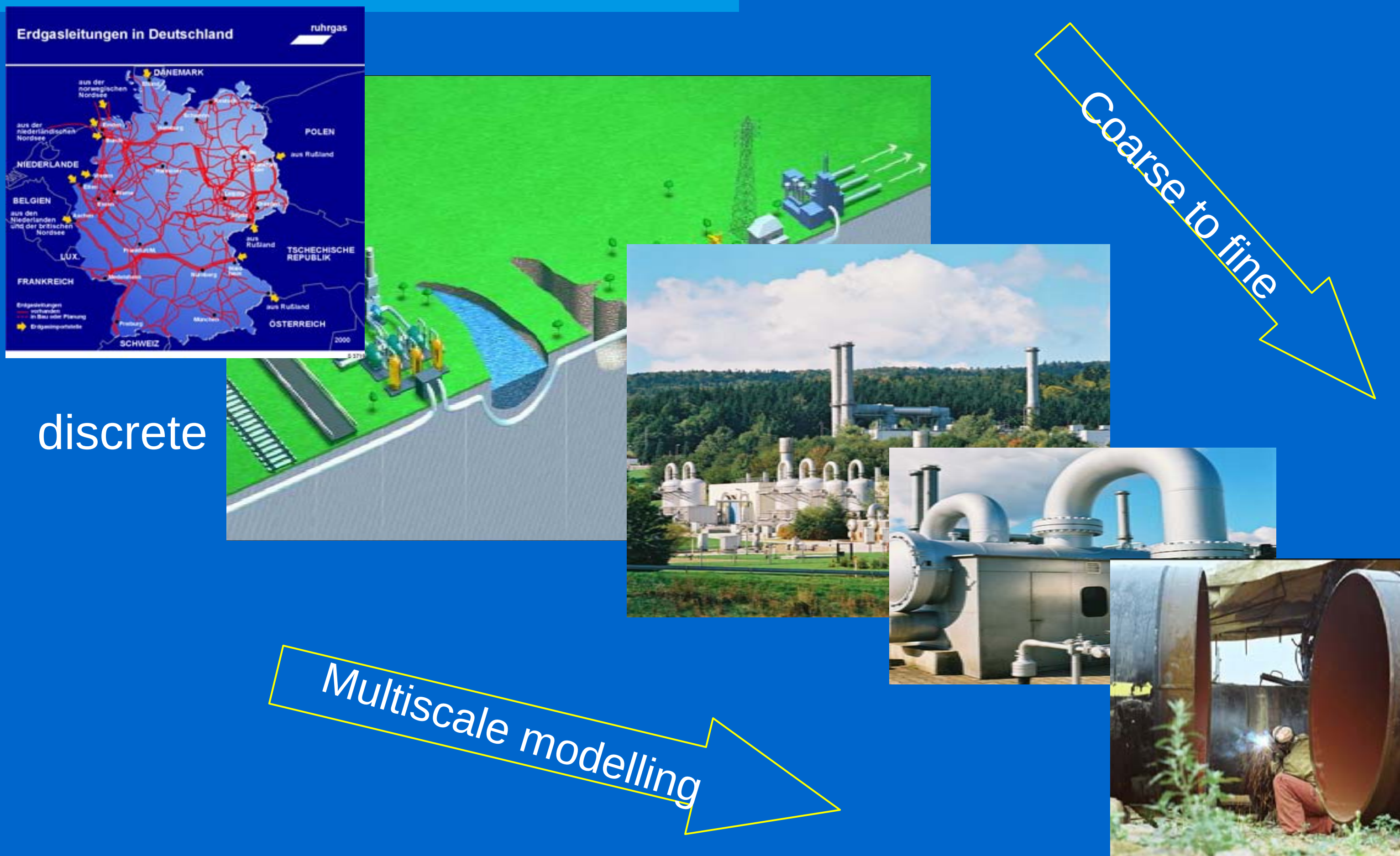


Controllability and optimal control (mostly for the de Saint-Venant system)

- Exact controllability only for classical solutions so far
- Semi-global exact controllability results for level networks without friction term by G.L. and Schmidt 02, G.L. and M. Gugat 03 and G.L. and M. Gugat and G. Schmidt 04
- Semi-global exact controllability with fricition for level networks: Ta tsien Li and Wang 07,Ta tsien Li 05 , Ta tsien Li and Bopeng Rao 02, 03
- Semi-global exact controllability of constant subcritical states to a neighbourhood of equlibrium points for non-level systems with friction: G.L. and M;. Gugat 2007
- Stabilization: J.-M. Coron et.al. 00-05
- Exact controllability of weak solutions for a single edge:

  O. Glass 05 (Networks open), F. Ancona, Piccoli, Colombo…

Industrial application: Optimization and control of gas-networks (joint with: M.Gugat, J.Lang, A. Martin)



discrete

continuous

Modelling of gas-networks (similar for the shallow water equations)

$$\frac{\partial p}{\partial t} + \frac{c^2}{S}\frac{\partial Q}{\partial x} = 0$$

$$\frac{\partial Q}{\partial t} + S\frac{\partial p}{\partial x} + \frac{c^2}{S}\frac{\partial}{\partial x}\frac{Q^2}{p} = -\left(\frac{\lambda c^2}{2DS}\frac{Q|Q|}{p} + \frac{g\sin(\alpha)}{c^2}p\right)$$

Vector form: system of nonlinear hyperbolic conservation laws with source terms

$$\frac{\partial}{\partial t}\begin{pmatrix} p \\ Q \end{pmatrix} + \frac{\partial}{\partial x}\begin{pmatrix} \frac{c^2}{S}Q \\ Sp + \frac{c^2}{S}\frac{Q^2}{p} \end{pmatrix} = -\begin{pmatrix} 0 \\ \frac{\lambda c^2}{2DS}\frac{Q|Q|}{p} + g\sin\alpha p \end{pmatrix}$$

$S \approx$ cross sectional area, $D \approx$ diameter,
$c \approx$ isothermal speed of sound,
$p \approx$ pressure, $Q \approx$ flux $\lambda \approx$ friction factor

The fully nonlinear network model

$$\min_{u,\delta}\left\{\sum_{J,i\in\mathcal{C}\cap\mathcal{I}_J}\int_0^T\int_0^{\ell_i} j(p_i,Q_i,u_j,w_J)\,dxdt\right\} \text{ s.t.}$$

$$\frac{\partial}{\partial t}\begin{pmatrix}p_i\\ Q_i\end{pmatrix}+\frac{\partial}{\partial x}\begin{pmatrix}\frac{c_i^2}{S_i}Q_i\\ S_ip_i+\frac{c^2}{S_i}\frac{Q_i^2}{p_i}\end{pmatrix} = -\begin{pmatrix}0\\ \frac{\lambda c_i^2}{2D_iS_i}\frac{Q_i|Q_i|}{p_i}+g\sin\alpha_i p_i\end{pmatrix}$$

$$Q_i(t,v_J)=s_J(t),\ i\in\mathcal{I}_J,\ J\in\mathcal{N}_S,\ p_i(v_J,t)=w_J(t),\ i\in\mathcal{I}_J,\ J\in\mathcal{N}_C$$

$$\sum_{i\in\mathcal{I}_J} d_{iJ}Q_i(v_j)=u_J,\ i\in\mathcal{I}_J,\ J\in\mathcal{N}_{mult}$$

$$p_i(v_J)=p_j(v_J),\ i,j\in\mathcal{I}_J,\ J\in\mathcal{N}_{mult}$$

$$p_i(0,x)=p_{i0}(x),\ Q_i(0,x)=Q_{i0}(x),\ x\in[0,\ell_i]$$

A nonlinear network flow model

$$\min_{p,Q,s,D} \left\{ \sum_{i\in\mathcal{N}_S} c_i s_i + \sum_{i\in\mathcal{N}_C,\, j\in\mathcal{I}_i} f_i(p_i, Q_i) \right\}$$

$$\max_{s,p,Q,D} \sum_{j\in\mathcal{I}_i,\, i\in\mathcal{N}_{out}} e_i Q_{ji} \qquad \text{subject to}$$

$$K_{ij}(D_{ij})(p_i^2 - p_j^2) = \text{sign } (Q_{ij}) Q_{ij}^2, \quad j \in \mathcal{I}_i,\ i \in \mathcal{N}_{mult}$$

$$\underline{p}_i \leq p_i \leq \bar{p}_i, \quad i \in \mathcal{N}$$

$$\underline{s}_i \leq s_i \leq \bar{s}_i, \quad i \in \mathcal{N}_S$$

Osciadasz 96, de Wolfe et.al 98……..piecewise linear approximations discussed by A. Martin and his group, linear model G.L. and Dymkou 07

Optimal branching in transportation networks: minimal resistance graphs

Optimal branching in transportation networks: minimal resistance graphs

We consider an equality constraint w.r.t. to the conservation of total volume or surface:

$$V_\beta(s,\ell) = \sum_{i\in I} \int_0^{\ell_i} s_i^\beta \, dx = \sum_{i\in I} s_i^\beta \ell_i$$

where $\beta > 0$ ($\beta = 1$ volume, $\beta = \frac{1}{2}$ surface).
The problem to consider:

$$\min_s F(Q; s, \ell) \text{ s.t.}$$
$$V_\beta(s,\ell) = c, \; s > 0$$
$$\sum_{i\in \mathcal{I}_J} d_{iJ} Q_i = 0, \; \forall J \in \mathcal{J}_M$$

Optimal branching in transportation networks: optimality conditions
(see Durand 06 for a similar analysis)

We ignore the positivity of s_i (clear for proper cost) and obtain the KKT-condiitons:

$$\nabla_s F(Q; s, \ell) + \lambda \nabla_s V_\beta(s, \ell) = 0 \text{ where}$$

$$\frac{\partial F_i}{\partial s_i} = -\alpha \frac{Q_i^2}{K_i^2 s_i^{\alpha+1}},$$

$$\frac{\partial V_\beta}{\partial s_i} = \beta s_i^{\beta-1},\ i \in I$$

This gives

$$Q_i = K_i \sqrt{\frac{\lambda\beta}{\alpha}} s_i^{\frac{\alpha+\beta}{2}},\ i \in \mathcal{I}, J\mathcal{J}_M$$

Optimal branching in transportation networks: Murray's law (1926)

We insert the result into the Kirchhoff condition (balance law at multiple nodes)

$$0 = \sum_{i \in \mathcal{I}_J} d_{iJ} Q_i = \frac{\lambda \beta}{\alpha} \sum_{i \in \mathcal{I}_J} d_{iJ} K_i s_i^{\frac{\alpha+\beta}{2}}, \ J \in \mathcal{J}_M$$

This is a an instance (generalization) of Murray's law obtained for blood flow in 1926. Notice that the Lagrange multiplier λ satisfies

$$\lambda = \frac{F\alpha}{c\beta}$$

Sensitivity of minimal resistance graphs with respect to topology changes

We may consider three types of changes:

- move a multiple node
- introduce an edge and reduce edge degree
- introduce a cycle (create a hole) reducing edge degree

Sensitivity of minimal resistance graphs with respect to topology changes

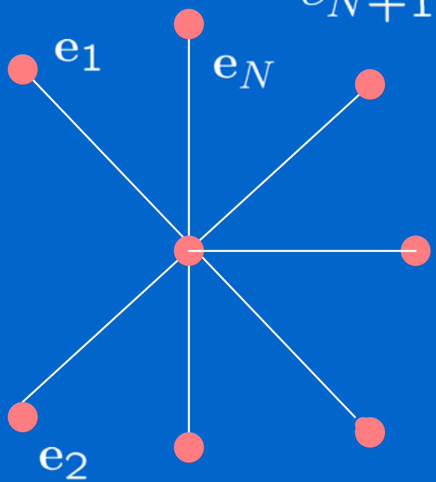
Sensitivity of minimal resistance graphs with respect to topology changes

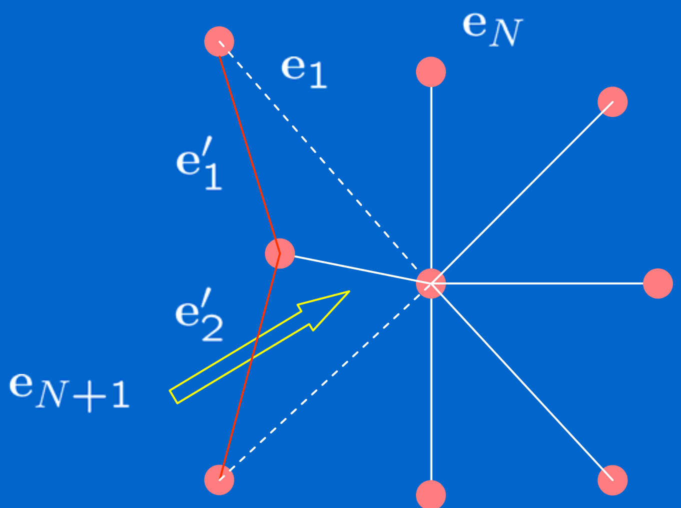
- reduce N-fold joint to a 3-fold joint plus an N-1-fold joint by introducing a new edge

$$\mathrm{e}_1' := \frac{\ell_1 \mathrm{e}_1 - \rho \mathrm{e}_{N+1}}{\|\ell_1 \mathrm{e}_1 - \rho \mathrm{e}_{N+1}\|}$$

$$\mathrm{e}_2' := \frac{\ell_2 \mathrm{e}_2 - \rho \mathrm{e}_{N+1}}{\|\ell_2 \mathrm{e}_2 - \rho \mathrm{e}_{N+1}\|}$$

$$e_{N+1} := \rho \mathrm{e}_{N+1}$$





Topological derivative for minimal resistance graphs

We evaluate the (minimal resistance) for the perturbed graph:

$$F^\rho = \sum_{i=1}^{N+1} \frac{Q_i^2}{K_i^2 s_i \alpha} \ell_i^\rho$$

$$= \sum_{i=1}^{N} \frac{Q_i^2}{K_i^2 s_i \alpha} \ell_i - \rho \left\{ \frac{Q_1^2 \cos\theta_1}{K_1^2 s_1^\alpha} + \frac{Q_2^2 \cos\theta_2}{K_2^2 s_2^\alpha} - \frac{Q_{N+1}^2}{s_{N+1}^\alpha} \right\}$$

$$=: F^0 - \rho \mathcal{T}(Q, s, \theta_1, \theta_2)$$

We call $\mathcal{T}$ the topological derivative with respect to the inserted edge e_{N+1}.

Topological derivative for minimal flow resistance graphs

In order to stay feasible w.r.t. the total volume (or surface), we have:

$$V_\beta^\rho(s,\ell^\rho) = \sum_{i=1}^{N+1} s_i^\beta \ell_i^\rho = c$$

$$\Leftrightarrow s_{N+1}^\beta = s_1^\beta \cos\theta_1 + s_2^\beta \cos\theta_2 (+O(\rho))$$

We obtain for $\mathcal{T}$:

$$\mathcal{T} = \frac{\lambda\beta}{\alpha s_{N+1}^\alpha}\left\{ (s_{N+1}^{\frac{\alpha+\beta}{2}})^2 - (d_{1I} s_1^{\frac{\alpha+\beta}{2}} + d_{2I} s_2^{\frac{\alpha+\beta}{2}})^2 \right\}$$

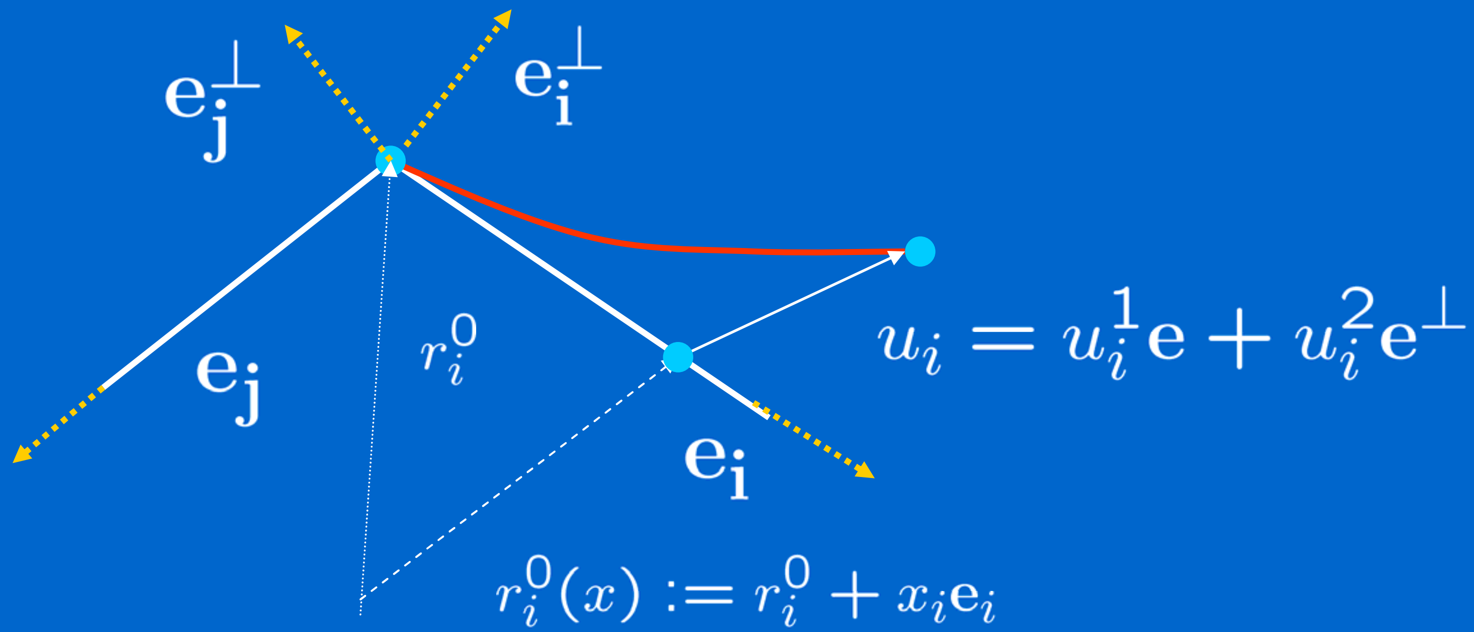
Some special cases

- If $\alpha = 2$, $\beta = 1$ then $\mathcal{T} > 0$ iff
$\theta_1 + \theta_2 =: \gamma < acos(2^{\frac{3\beta-\alpha}{\alpha+\beta}} - 1) = 74.9^o$ in the first case and $\gamma < 90^o$ in the second case. However, there is at least on angle $\leq 2\pi/N$. Thus all multiple nodes with edge degree > 4 are released to 3-node plus N-1-node until the edge degree is 4. In a flow network at least 2 adjacent pipes should carry flow in opposite directions. Thus only 4-nodes with such pattern survive.
- If $\alpha = 2, \beta = \frac{1}{2}$ the topological gradient is negative until the edge degree is is 3

Mechanical networks

- In progress:
  - Trusses (similar analysis as above, not yet cpmpleted)
  - 2-D elasticity for grid structures
- Open:
  - 3-D elasticity for grid structures
  - Cosserat networks
  - Same for beams
  - Same for plates, shells and combinations…

In-plane analysis (3-d case similar)



Elasticity on graphs

$$-(K_i \partial_{xx} u_i) = f_i \quad x \in (0, \ell_i)$$

$$\partial_x u_i(v_N) = g_i \quad \forall i \in \mathcal{I}_N,$$

$$u_i(v_J) = u_j(v_J) \quad i,j \in \mathcal{I}_J,$$

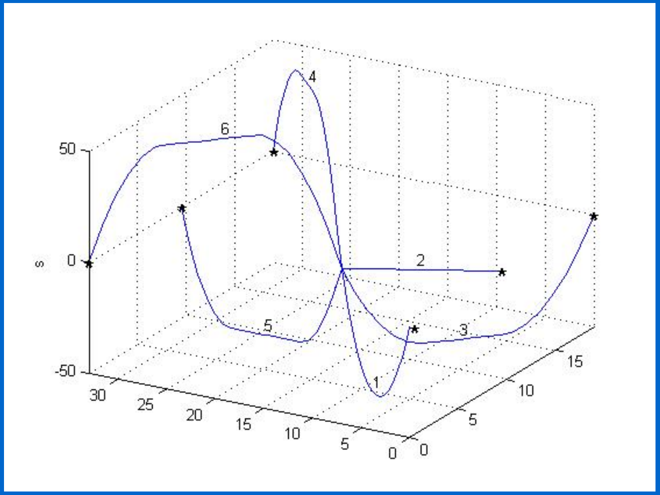
$$\sum_{i \in \mathcal{I}_J} d_{iJ} \rho_i K_i \partial_x u_i(v_J) = 0 \quad J \in \mathcal{J}_M,$$

$$K_i := h_i[(1 - \frac{1}{s_i})I + \frac{1}{s_i}\mathrm{e}_i \mathrm{e}_i^T], \; h_i = \frac{E_i}{\ell_i^2}$$

Notice: $s_i \geq 1$, if $s_i = 1$ no vertical stiffness

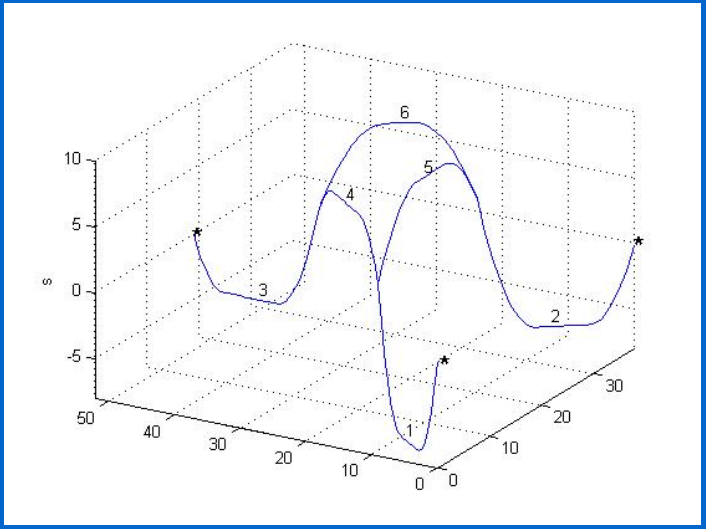
Can also consider the wave equation!

An example of static networks under contact conditions



Graph-like networks of strings

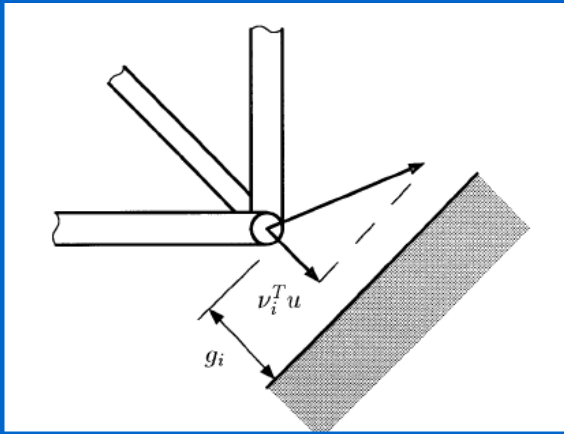
We ask for optimal topologies under various constraints



truss modelling (including contact)

Edgewise linear elasticity on a graph together with $s_i = 1$ leads to trusses:

$$K(t)u = F, \quad K(t) = \sum_{i=1}^{m} t_i K_i, \quad K_i = \frac{E_i}{\ell_i^2}\mathbf{e}_i\mathbf{e}_i^T$$



truss-design
(see also the presentation of G. Bouchitte)

$$\min_t f^T u \text{ s.t.}$$

$$K(t)u = f + C^T p$$

$$Cu \leq g,\ p \leq 0,\ p^T(C^T u - g) = 0,$$

$$t \in \{t = (t_i)_i | \sum_{i=1}^{m} t_i = V,\ t_i \geq 0\};$$

$$g \in \{g = (g_i)_i | \sum_{i=1}^{r} g_i = 0\}$$

Klarbring et. al. 95, Ben-Tal, Zowe, Outrata, Kocvara, Achtziger....05.

Problems studied: • sizing • optimal location of nodes in reference configuation • robust design (Ben-Tal ...07)

truss-design: simplified problem

The minimal compliance problem can be expressed as follows:

$$\min_{f,s} J(f,s) := \frac{1}{2}\sum_{k=1}^{N} \frac{\ell_k}{s_k E_k} f_k^2 \text{ s.t.}$$

$$\sum_{k=1}^{N} f_k \mathbf{e}_k = -F$$

$$\sum_{k=1}^{N} s_k \ell_k = V$$

when we consider the associated optimal displacements and if we perform a similar analysis as in the scalar flow case.

Free material optimization
(with M. Stingl and M. Kovcvara, EU-Project 07)

$$\min_{E\in\mathcal{E}} \max_{k\in\mathcal{K}} \int_\Gamma f\cdot u_k dx \qquad \text{s.t.}$$

$$u_k \in \left\{H^1(\Omega) | u = 0 \text{ on } \Gamma_0\right\} := \mathcal{V} \text{ satisfies}$$

$$a_E(u,v) := \int_\Omega \langle Ee(u), e(u)\rangle dx = \int_\Gamma f_k \cdot v dx, \qquad \forall v \in \mathcal{V}$$

$$E \in \mathcal{E} := \left\{K_i \in L^\infty(0,\ell_i)^{d\times d},\ K_i = K_i^T,\ K_i \succcurlyeq \rho_l,\ \text{tr }(K_i) \leq \rho_u\right\}$$

$$v(E) := \int_\Omega \text{tr }(E) dx \leq V$$

- computational complexity cubically w.r.t. number of loads
- dual approach difficult to apply for further constraints....
- hence develop direct approach analoguous to SCP, MMA

FMO: discretization

- local stiffness: $A_i(E) = \sum_{j=1}^{ni} B_{i,j}^T E_i B_{i,j}$
- global stiffness: $A(E) = \sum_{i=1}^{m} A_i(E)$
- compliance: $f_k^T u_k, \quad A(E)u_k = f_k, \; k \in \mathcal{K},$

$c_k(E) = f_k^T A(E)^{-1} f_k, \; k \in \mathcal{K}$

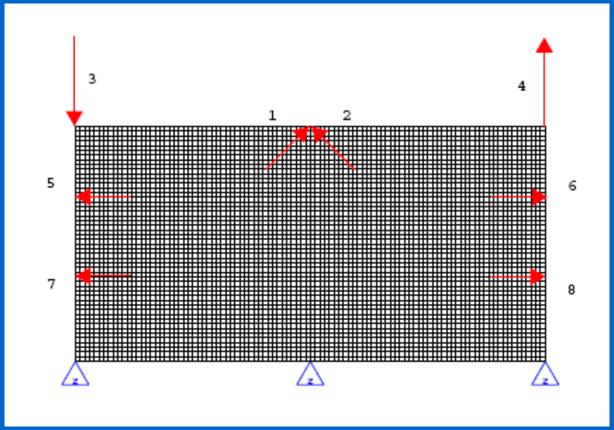
The discrete multiple load worst case design problem reads as:

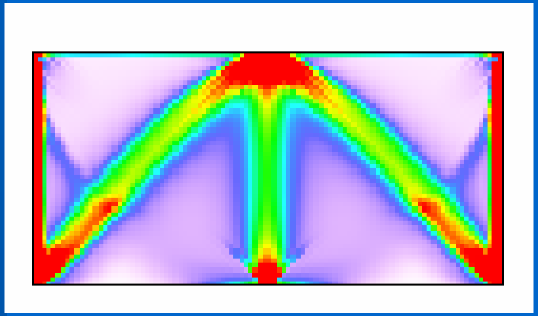
$$\min_{E \in \mathcal{E}} \max_{k \in \mathcal{K}} c_k(E) \quad \text{s.t.}$$

$$\sum_{i=1}^{m} \mathrm{tr}(E_i) \leq V$$

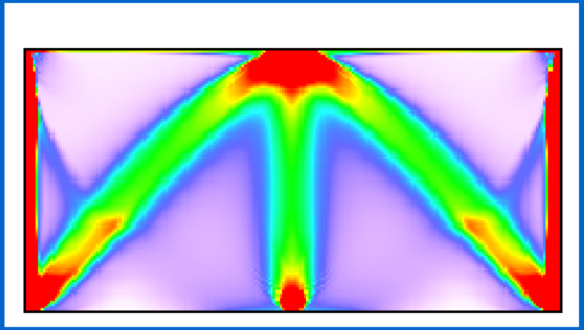
FMO: properties

Free material optimization (computations: M. Stingl 07)

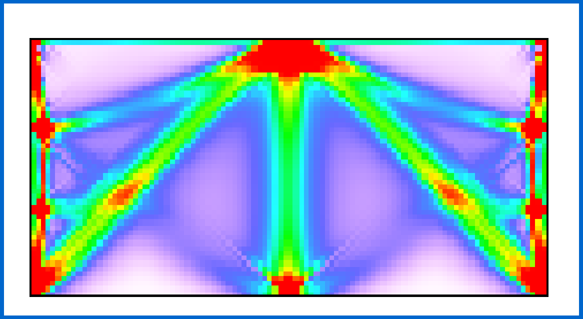




4 loads 5000 elements

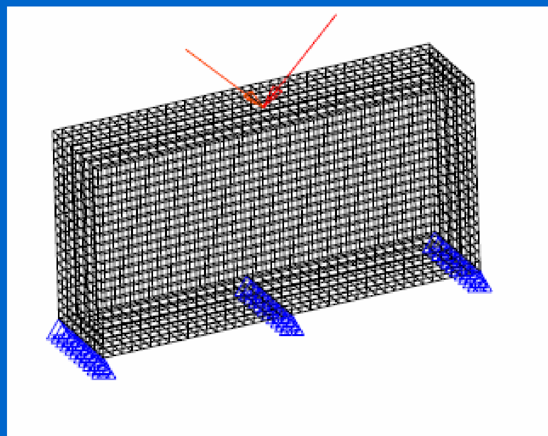


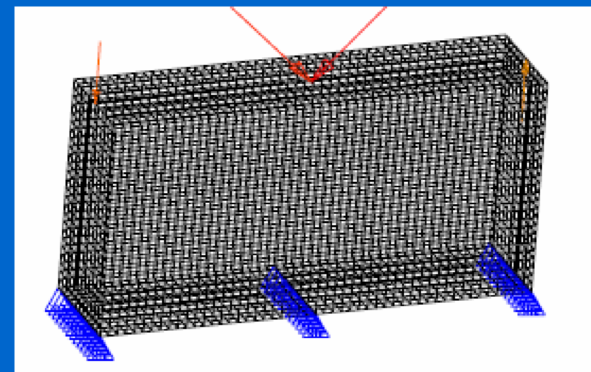
20000 elements

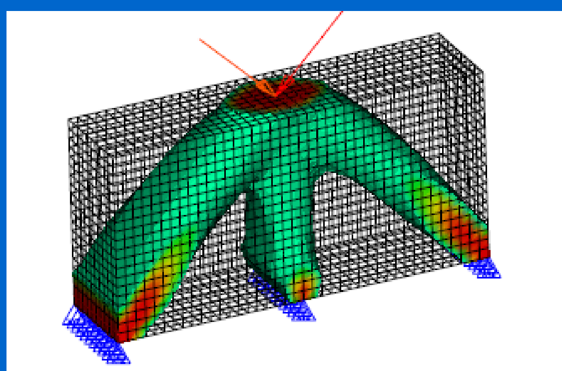


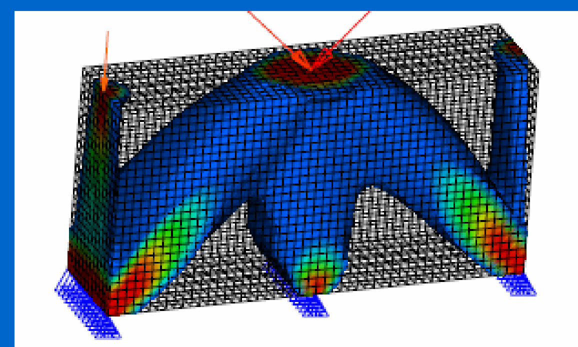
8 loads 5000 elements

3-D examples: multilpe load cases





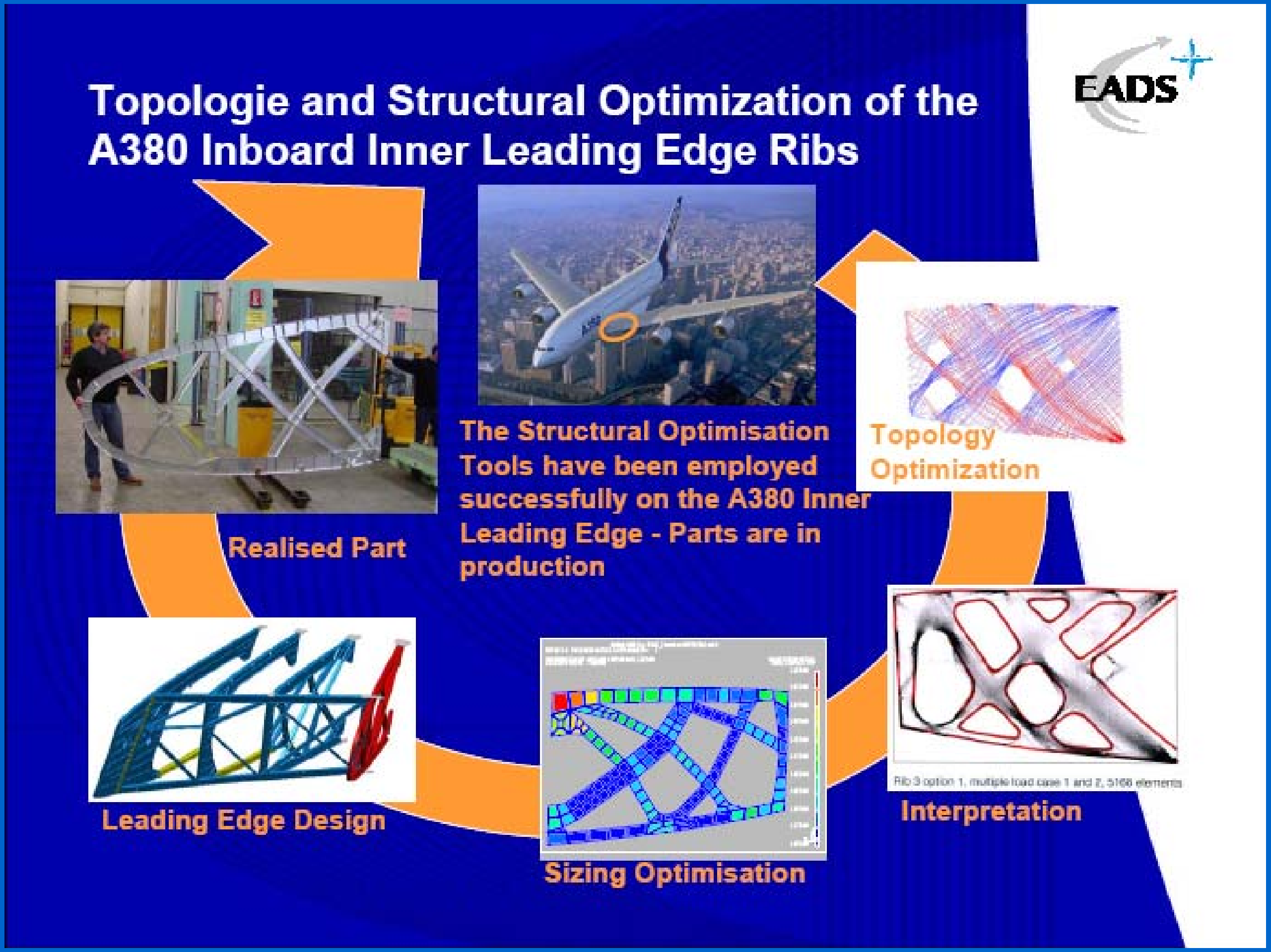




10000 elements, 1.5 h

20000 elements 4 h

FMO: industrial application

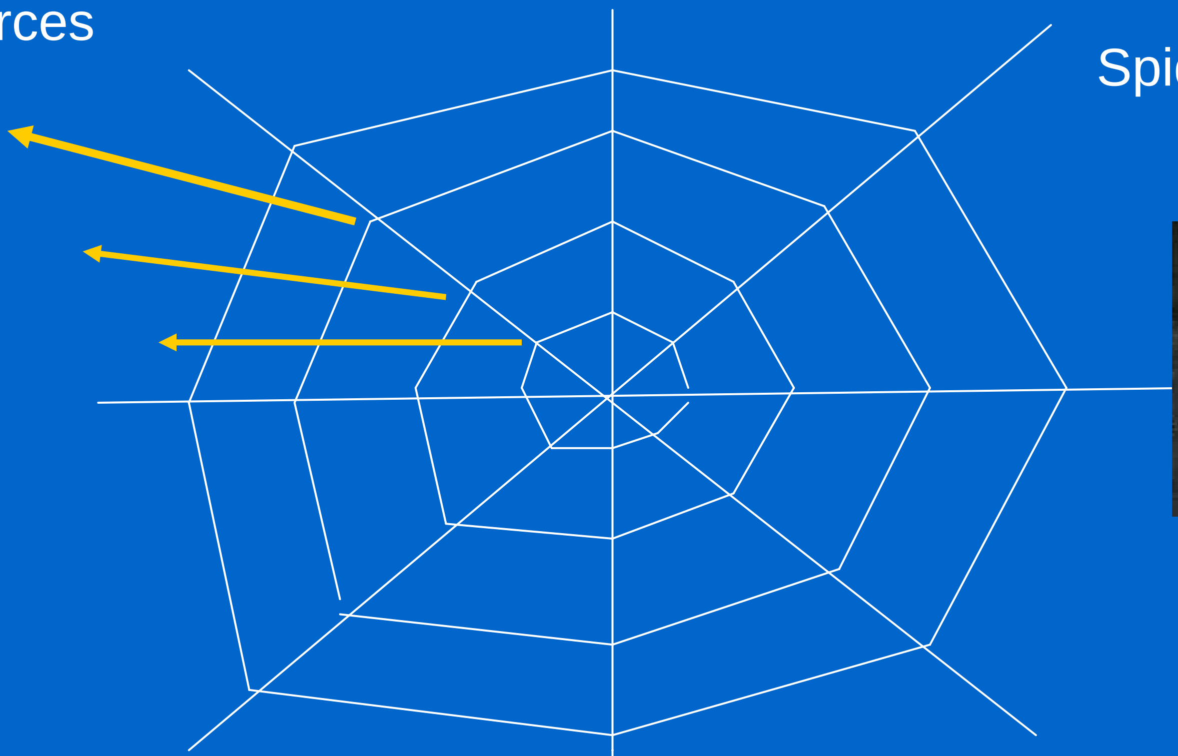


Extension to continuous prolems: sizing, material and topology optimization (open)

‚digging holes‘ in graphs:

G.L. and J. Sokolowski 2006/2007

Forces

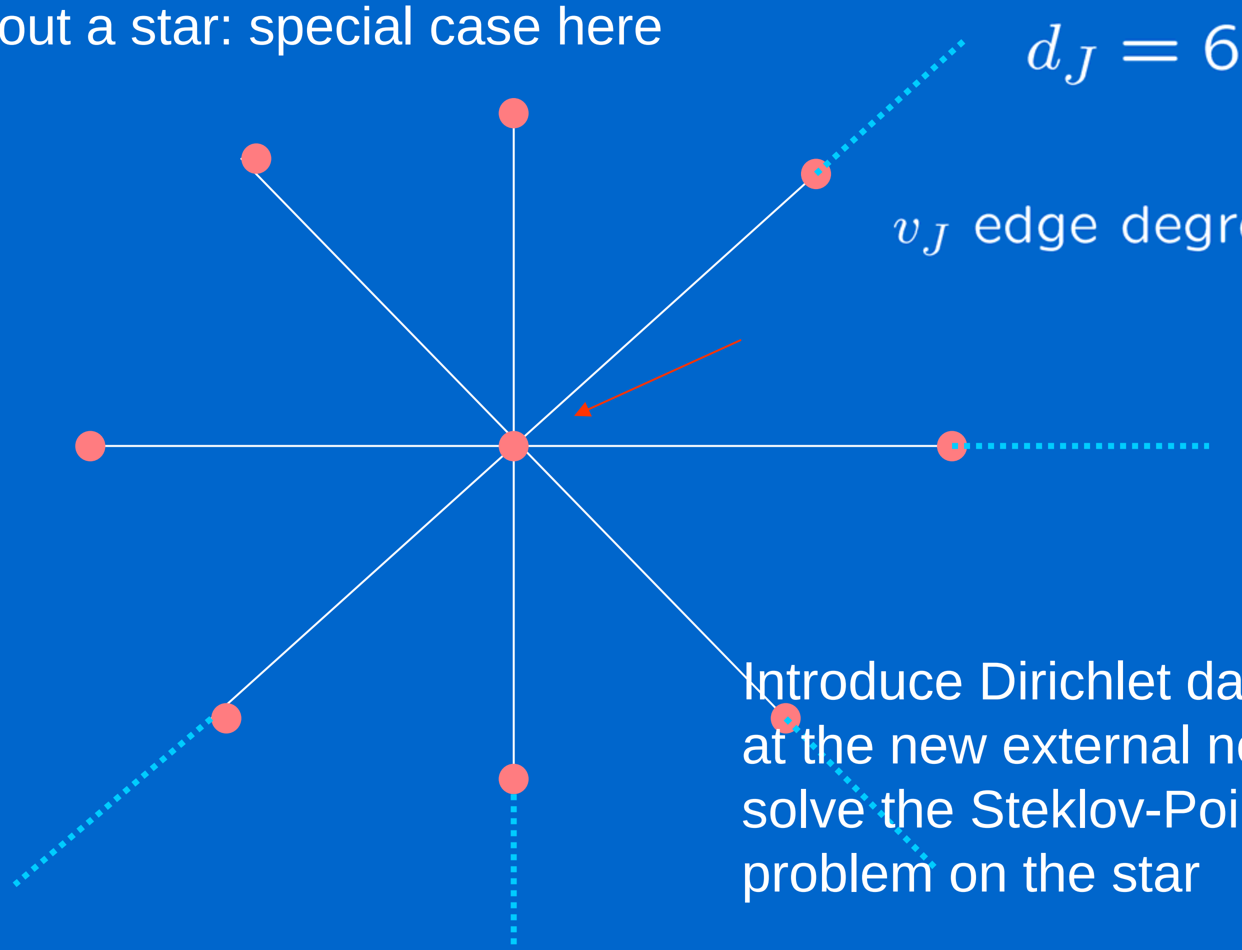


Spider‘s web

Is it energetically reasonable to cut a hole at the center ?

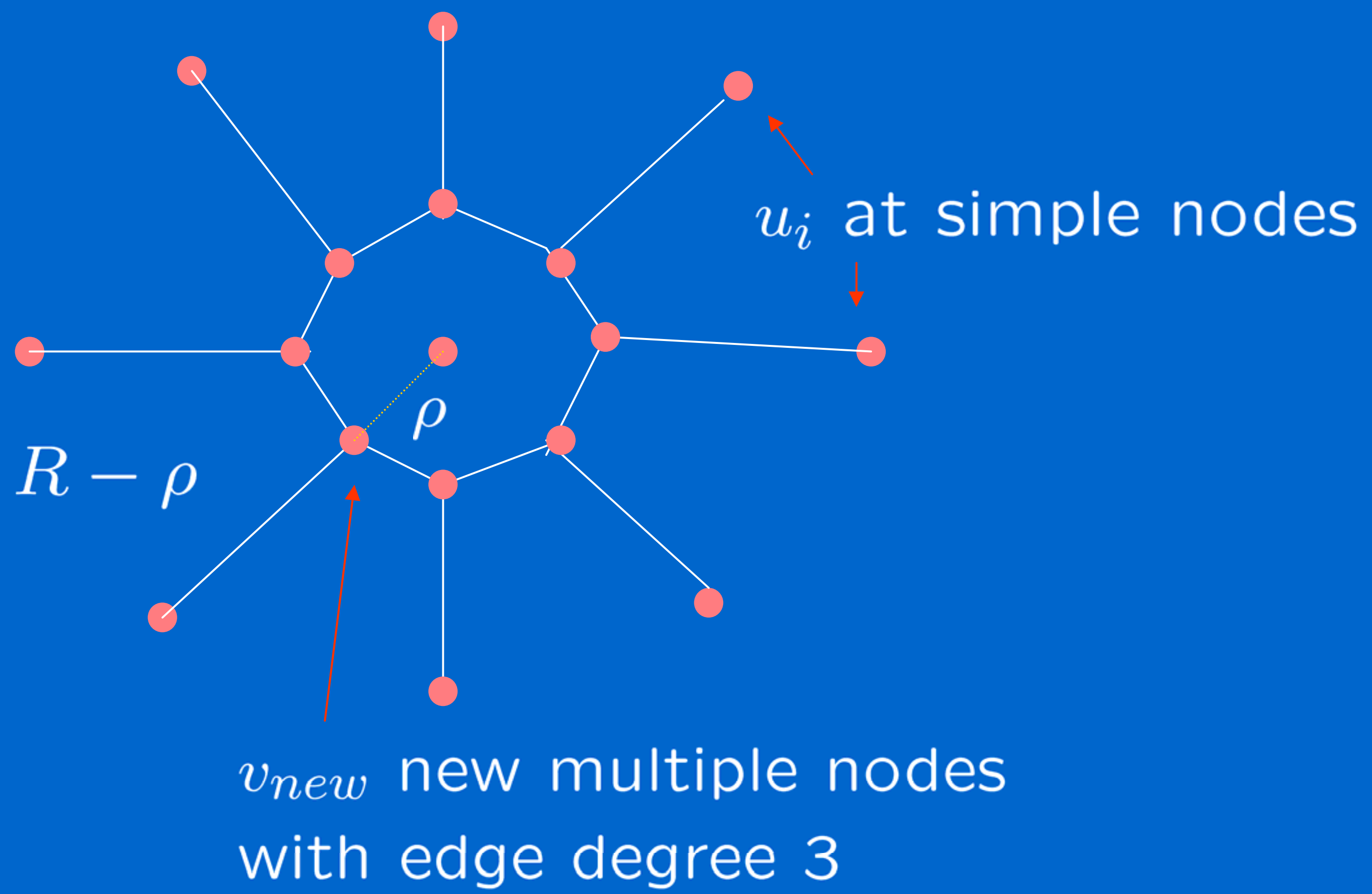
A local subproblem: Steklov-Poincaré

- Cut out a star: special case here



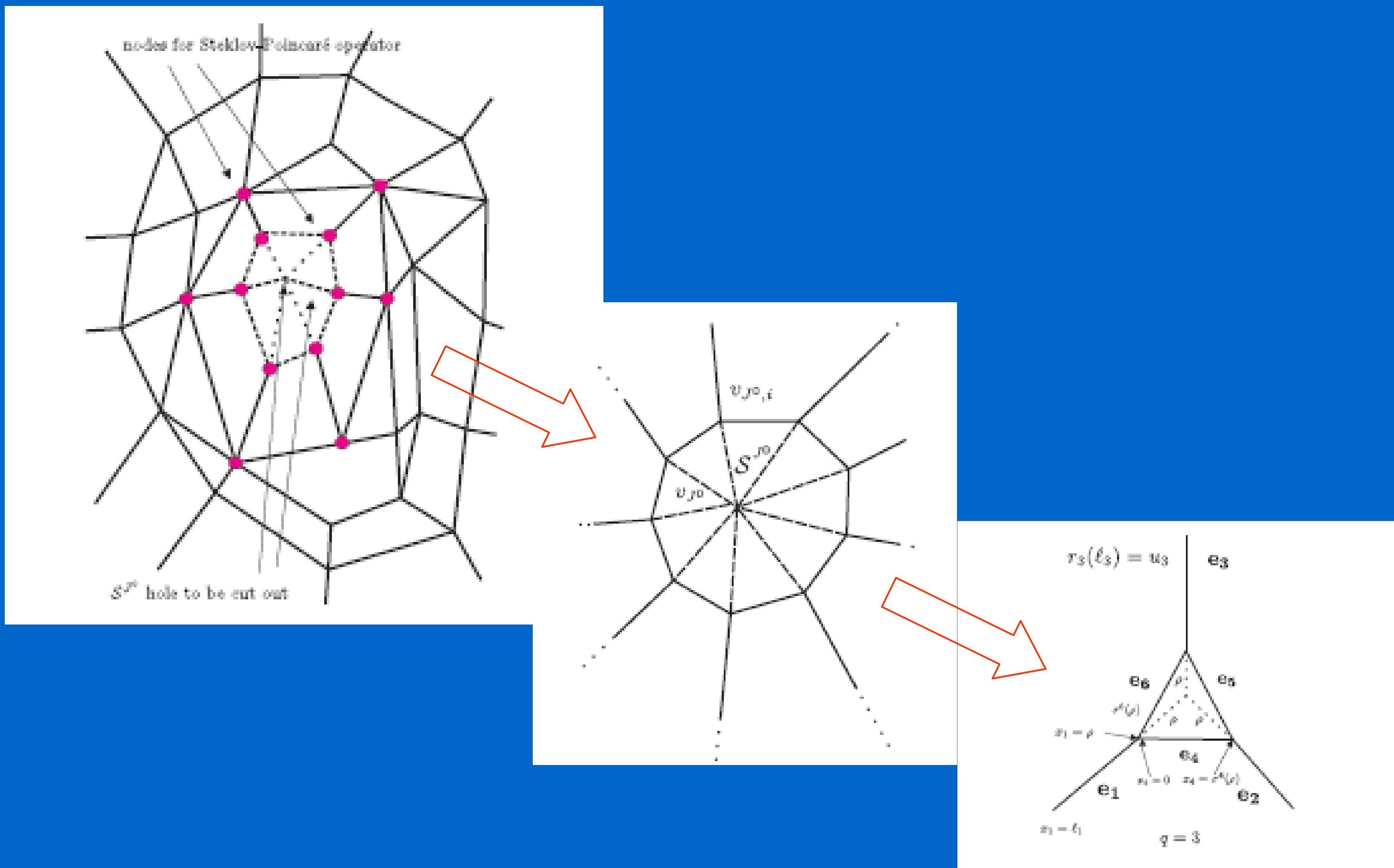
Introduce Dirichlet data at the new external nodes, solve the Steklov-Poincare problem on the star

Resolve the center



Elasticity on graphs: weak formulation

Cut out a subgraph including the hole



Asymptotic solution for the homogeneous star-graph with symmetric hole

The Steklov- Poincaré-map is then obtained using

$$
\begin{aligned}
(r_i')^\rho(\ell) = &\coth(\ell)(u_i - \tfrac{1}{3}\sum_{j=1}^{3} u_j) + \tanh(\ell)\tfrac{1}{3}\sum_{j=1}^{3} u_j \\
&+\rho\Big\{(1-\tanh^2(\ell))[(1-\tfrac{1}{3}\sigma)\coth^2(\ell)(u_i - \tfrac{1}{3}\sum_{j=1}^{3} u_j)) \\
&\qquad + (\sigma-1)\tfrac{1}{3}\sum_{j=1}^{3} u_j]\Big\},\ i = 1,\dots q.
\end{aligned}
$$

Energy criterion

$$\mathcal{E}_0(v_J) := \frac{1}{2}\sum_{i=1}^{n}\int_0^{\ell_i} K_i r_i' \cdot r_i' + c_i |r_i|^2 dx$$

$$= \langle \mathcal{S}_0^J(u), u \rangle$$

$$\mathcal{E}_\rho(v_J) \;\; := \;\; \frac{1}{2}\sum_{i=1}^{n+d_J}\int_0^{\ell_i^\rho} K_i (r_i^\rho)' \cdot (r_i^\rho)' + |r_i^\rho|^2 dx$$

$$= \frac{1}{2}\sum_{J_+}\sum_{i\in\mathcal{I}_{J_+}} d_{iJ_+} (r_i^\rho)'(v_{J_+}) r_i^\rho(v_{J_+})$$

$$= \mathcal{E}_0(v_J) + \rho\{\mathcal{T}_{\mathcal{E}}(u)\} + O(\rho^2)$$

Topological gradient: potential energy as criterion (also compliance) G.L. and Sokolowski 2006, 2007

$$\frac{d}{d\rho}\mathcal{E}_\rho(v_J)|_{\rho=0} := \lim_{\rho\to 0_+} \frac{\mathcal{E}_\rho(v_J)-\mathcal{E}_0(v_J)}{\rho}$$
$$= \mathcal{T}(v_J, u)$$

If now $\mathcal{T}(v_J, u)$ is negative, we may introduce a hole!

Then we may perform a descent step analogu-ous to Sokolowski and Zochowski(1999-2005), Hintermueller(2004), Allaire, Jouve ...(2004)....

The topological derivative for the homogeneous star-graph with symmetric hole

$$\langle \mathcal{S}_i^\rho(u), u\rangle = \langle \mathcal{S}_i^0(u), u\rangle + \rho\left\{(1-\tfrac{1}{3}\sigma)\sum_{i=1}^{3}\|r_i'(0)\|^2 + (\sigma-1)\sum_{i=1}^{3}\|r_i(0)\|^2\right\}$$

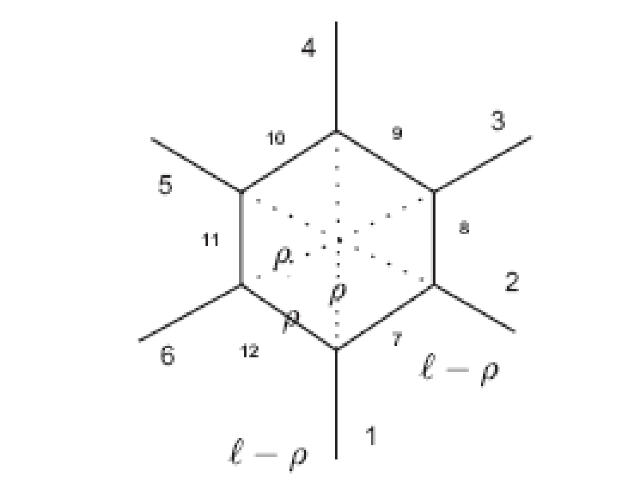
This says that the energy function in the homogeneous case, when cutting out a symmetric hole e.g. $\sigma^i = \sigma = \sqrt{3}$, $i = 1, 2, 3$, we have

$$\mathcal{T}_E(r, v_{J^0}) = \left\{(1-\frac{1}{3}\sigma)\sum_{i=1}^{3}\|r_i'(0)\|^2 + (\sigma-1)\sum_{i=1}^{3}\|r_i(0)\|^2\right\}$$

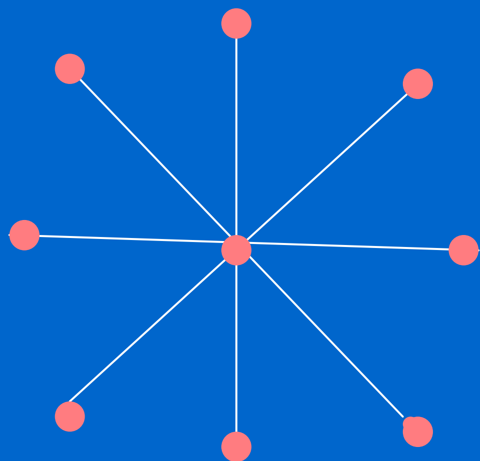
We can handle topological gradients for other cost functionals.
Note, however, the difficulty resulting from the fact, that G_ρ is not a subset of G, unlike in the higher-dimensional case.

Still to do……..

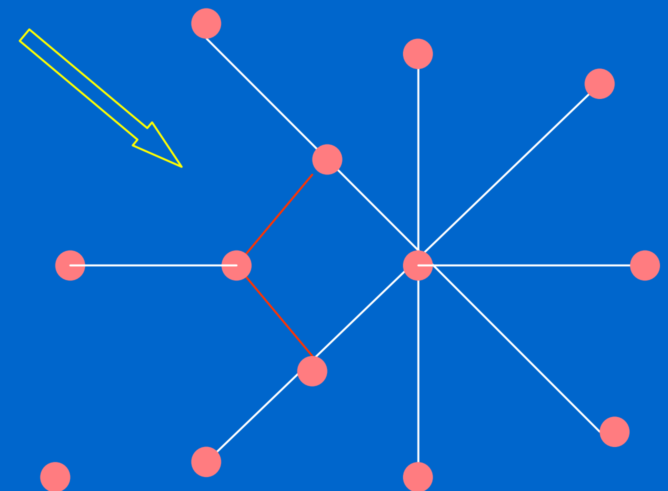
- higher edge degrees (s.t. mass is actually removed)
- multiple loads (asymmetric loading, worst-case designs)
- nonsymmetric holes (holes may degenerate!)
- 3-d networks (also for curved edges)
- non-homogenous networks……….



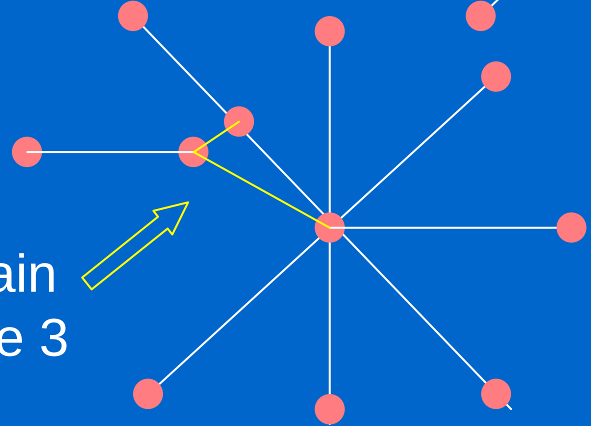
Asymmetric holes and more…..



Asymmetric
hole



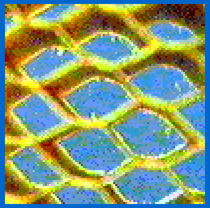
We may let the two edges coincide and obtain a node with edge degree 3 and one with N-3

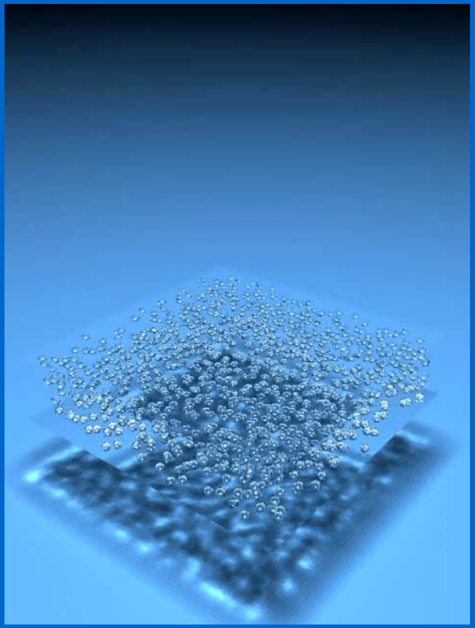


2-D networks in 3-space: Scope

Modeling includes:

- Scalar problems (diffusion) on networks
- Membrane networks
- Networks of Reissner-Mindlin plates
- Network of shells (fairly open)
- Can be extended to time-dependent problems





Courtesy U.Rüde

Thank you for your attention!!