Asymptotic Analysis of an Optimal Boundary Control Problem on Thin Periodic Structures

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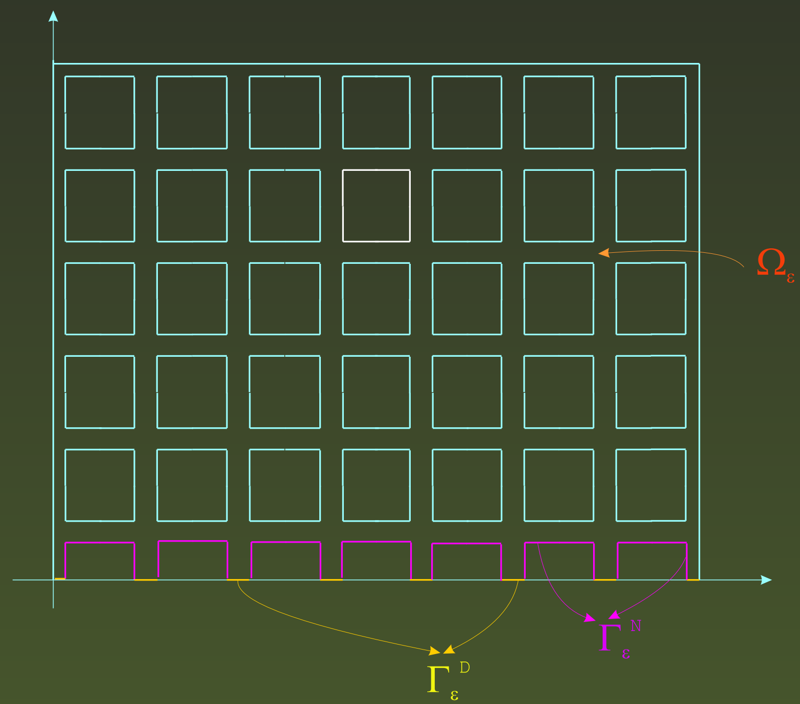
(the joint work with Günter Leugering)

The Definition of the network $\Omega_\varepsilon = \Omega \cap \varepsilon\mathcal{F}^h$



The decomposition of the boundary

$$\partial\Omega_\varepsilon = \Gamma_\varepsilon^D \cup \Gamma_\varepsilon^N \cup S_{int}^\varepsilon \cup S_{ext}.$$

Three Types of the Thin Structures

Having assumed that the parameters h and ε are related by $h = h(\varepsilon) \to 0$ as $\varepsilon \to 0$ we shall divide all thin structures into three classes, namely:

(A_1) sufficiently thick, when $\lim\limits_{\varepsilon \to 0} \frac{\varepsilon}{h(\varepsilon)} = 0$;

(A_2) sufficiently thin, when $\lim\limits_{\varepsilon \to 0} \frac{\varepsilon}{h(\varepsilon)} = +\infty$;

(A_3) structures of critical thickness, when $\lim\limits_{\varepsilon \to 0} \frac{\varepsilon}{h(\varepsilon)} = \theta \in (0, \infty)$.

It is the principal difference between this type of domains and the periodically perforated ones for which the another rule takes a place, namely, $h = h(\varepsilon) \to const \in (0, 1]$ as $\varepsilon \to 0$.

The statement of BVP

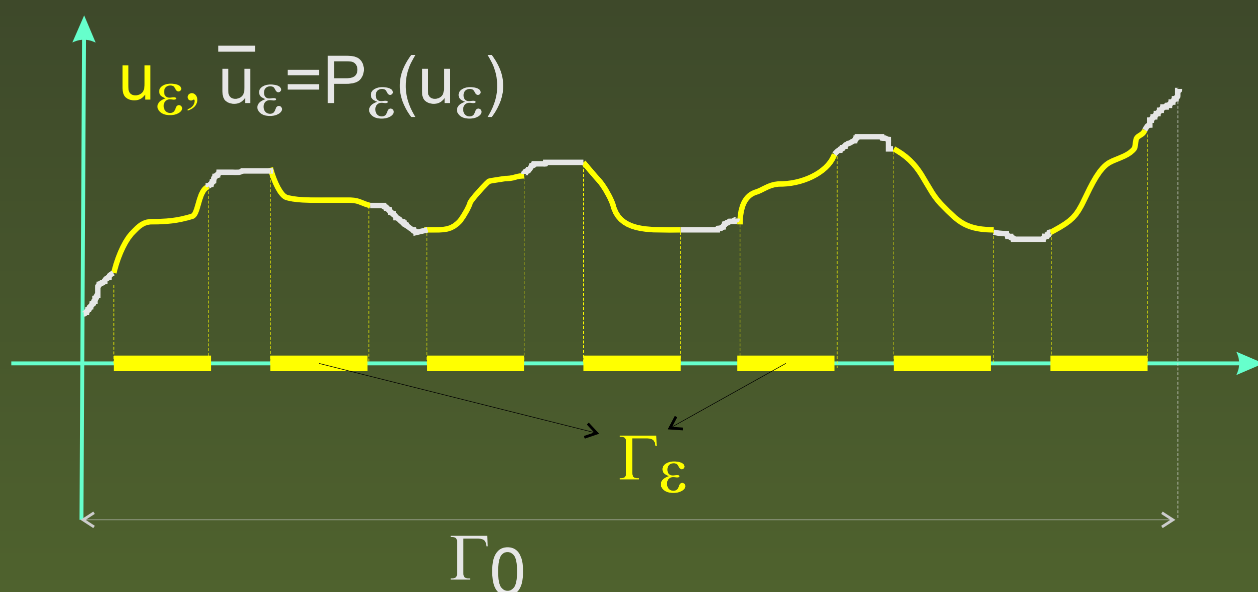
We consider the following initial-boundary value problem in $\Omega_\varepsilon \times (0,T)$

$$
\begin{aligned}
\partial_t y_\varepsilon - \operatorname{div}\left(A(x/\varepsilon)\nabla y_\varepsilon\right) - y_\varepsilon^3 = f_\varepsilon, &\quad \text{in } (0,T)\times\Omega_\varepsilon,\\
\partial_{\nu_A} y_\varepsilon = 0, &\quad \text{on } (0,T)\times S^\varepsilon_{ext},\\
\partial_{\nu_A} y_\varepsilon = \varepsilon^2(-d\, y_\varepsilon + g_\varepsilon), &\quad \text{on } (0,T)\times S^\varepsilon_{int},\\
y_\varepsilon = u_\varepsilon, &\quad \text{on } (0,T)\times \Gamma^D_\varepsilon,\\
\partial_{\nu_A} y_\varepsilon = \varepsilon p_\varepsilon, &\quad \text{on } (0,T)\times \Gamma^N_\varepsilon,\\
y_\varepsilon(0,x) = y_{\varepsilon,0}\,, &\quad \text{for a.a. } x\in\Omega_\varepsilon.
\end{aligned}
$$

Admissible controls

Neumann control: $p_\varepsilon \in L^2(0,T;L^2(\Gamma_\varepsilon^N))$;
Dirichlet control: $u_\varepsilon \in U_\varepsilon$, where

$$U_\varepsilon = \left\{ u|_{\Gamma_\varepsilon^D} \ : \ u \in L^2(0,T;\ H^1(\Gamma_0)),\ \|u\|_{L^2(0,T;\,H^1(\Gamma_0))} \leqslant C$$



$$P_\varepsilon : L^2(0,T;H^1(\Gamma_\varepsilon)) \longrightarrow L^2(0,T;H^1(\Gamma_0)).$$

The Cost Functional

$$I_\varepsilon(u_\varepsilon, p_\varepsilon, y_\varepsilon) = \frac{1}{h(\varepsilon)} \int_{\Omega_\varepsilon} (y_\varepsilon(T,\cdot) - y_\varepsilon^T)^2 \, dx$$

$$+ \frac{1}{h(\varepsilon)} \int_0^T \int_{\Omega_\varepsilon} (y_\varepsilon - y_\varepsilon^*)^6 \, dx dt + \kappa(\varepsilon) \int_0^T \int_{\Gamma_\varepsilon^D} u^2 \, dx_1 dt$$

$$+ \int_0^T \int_{\Gamma_\varepsilon^N} p_\varepsilon^2 \, d\mathcal{H}^1 dt \longrightarrow \inf$$

where $y_\varepsilon^* \in L^6((0,T) \times \Omega_\varepsilon)$ and $y_\varepsilon^T \in L^2(\Omega)$ are given functions, $\kappa(\varepsilon)$ is a given value.

The Optimal Control Problem (OCP)

Find a triplet $(u_\varepsilon^0, p_\varepsilon^0, y_\varepsilon^0)$ such that

$$(\mathbb{P}_\varepsilon) : \; I_\varepsilon(u_\varepsilon^0, p_\varepsilon^0, y_\varepsilon^0) = \inf_{(u_\varepsilon, p_\varepsilon, y_\varepsilon) \in \Xi_\varepsilon} I_\varepsilon(u_\varepsilon, p_\varepsilon, y_\varepsilon),$$

where

$$\Xi_\varepsilon \subset L^2(0,T; H^1(\Gamma_\varepsilon^D)) \times L^2(0,T; L^2(\Gamma_\varepsilon^N)) \times L^2(0,T; H^1(\Omega_\varepsilon))$$

is the set of admissible triplets.

The distinctive features of the OCP

1. We have the optimal control problem for blowing up systems with mixed boundary controls;
2. The control zones Γ_ε^D and Γ_ε^N each of controls are imposed on the different parts of the boundary ($\Gamma_\varepsilon^D \cap \Gamma_\varepsilon^N = \varnothing$), have an highly oscillating geometrical form, are non-connected, and they coincide in the limit, i.e. $\lim_{\varepsilon\to 0}\Gamma_\varepsilon^D = \Gamma_0 = \lim_{\varepsilon\to 0}\Gamma_\varepsilon^N$;
3. The thin domain Ω_ε are such that its thickness $h = h(\varepsilon)$ is related with the parameter of periodicity ε by the supposition: $h(\varepsilon) \to 0$ as $\varepsilon \to 0$.

The main goal

1. Provide the asymptotic analysis of the $\mathbb{P}_\varepsilon$ as $\varepsilon \longrightarrow 0$ and obtain the full identification of the structure of a limit problem

$$\left(\mathbb{P}^{hom}\right) \ : \quad \left\{ \left(\mathrm{CF}^{hom}\right), \ \left(\mathrm{CC}^{hom}\right), \ \left(\mathrm{SE}^{hom}\right) \right\}.$$

2. Study the dependence of the $\mathbb{P}^{hom}$ on a "volume" of the boundary $\Gamma_\varepsilon^D \subset \partial\Omega_\varepsilon$ occupied by the Dirichlet control zone, and how h tends to zero as $\varepsilon \longrightarrow 0$ (so-called "scaling effect").

The main supposition

For given $f \in L^2((0,T) \times \Omega)$, $g_\varepsilon \in L^2(0,T; L^2(S^\varepsilon_{int}))$, $u \in L^2(0,T; H^1(\Gamma^D_\varepsilon))$, and $p_\varepsilon \in L^2(0,T; L^2(\Gamma^N_\varepsilon))$ the BVP, in general, may not have a global solution $y_\varepsilon \in L^2(0,T; H^1(\Omega_\varepsilon))$ in the time.

Nevertheless we shall always suppose that for every $\varepsilon > 0$ there exists a triplet

$$(u^*_\varepsilon, p^*_\varepsilon, y^*_\varepsilon) \in U_\varepsilon \times L^2(0,T; L^2(\Gamma^N_\varepsilon)) \\ \times \left[L^6(0,T; L^6(\Omega_\varepsilon)) \cap L^2(0,T; H^1(\Omega_\varepsilon)) \right]$$

such that $(u^*_\varepsilon, p^*_\varepsilon, y^*_\varepsilon) \in \Xi_\varepsilon$.

Existence of optimal solutions

Theorem 1. *For every value* $\varepsilon > 0$ *there exists a solution* $(u^0_\varepsilon, p^0_\varepsilon, y^0_\varepsilon) \in \Xi_\varepsilon$ *for the optimal control problem* $\mathbb{P}_\varepsilon$ *if and only if* $\Xi_\varepsilon \neq \emptyset$.

The Diagram of Homogenization

$$
\begin{array}{ccc}
(\mathbb{P}_\varepsilon) \Longrightarrow (\widehat{\mathbb{P}}_\varepsilon) & & (\mathbb{P}^{hom}) \Longleftarrow (\widehat{\mathbb{P}}^{hom}) \\
\Big\downarrow & & \Big\uparrow \\
(\mathbb{CP}_\varepsilon) & & (\mathbb{CP}^{hom}) \\
\Big\downarrow & & \Big\uparrow \\
\left\langle \inf_{(a_\varepsilon, v_\varepsilon, w_\varepsilon, z_\varepsilon) \in \Sigma_\varepsilon} J_\varepsilon(a_\varepsilon, v_\varepsilon, w_\varepsilon, z_\varepsilon) \right\rangle & \xrightarrow{\varepsilon \to 0} & \left\langle \inf_{(a,v,w,z) \in \Sigma_0} J_0(a, v, w, z) \right\rangle
\end{array}
$$

The main variational property:

If $(u_\varepsilon^0, p_\varepsilon^0, y_\varepsilon^0) \in \Xi_\varepsilon$ is an optimal triplet of $\mathbb{P}_\varepsilon$-problem, and if $(u_\varepsilon^0, p_\varepsilon^0, y_\varepsilon^0)$ tends (in some sense) to (u^0, p^0, y^0) then (u^0, p^0, y^0) is an optimal solution for the limit problem.

Virtual Extension $\widehat{\mathbb{P}}_\varepsilon$ of the $\mathbb{P}_\varepsilon$-problem

$$\partial_t y_\varepsilon - \operatorname{div}\left(A(x/\varepsilon)\nabla y_\varepsilon\right) - y_\varepsilon^3 = g_\varepsilon, \text{ in } (0,T)\times\Omega_\varepsilon,$$

$$\widehat{I}_\varepsilon(q_\varepsilon, u_\varepsilon, p_\varepsilon\,, y_\varepsilon) = \frac{1}{h(\varepsilon)}\int_{\Omega_\varepsilon}(y_\varepsilon(T,x) - y_\varepsilon^T)^2\,dx$$
$$+\frac{1}{h(\varepsilon)}\int_0^T\int_{\Omega_\varepsilon}(y_\varepsilon - y_\varepsilon^*)^6\,dxdt + \kappa(\varepsilon)\int_0^T\int_{\Gamma_\varepsilon^D} u^2\,dx_1dt$$
$$+\int_0^T\int_{\Gamma_\varepsilon^N} p_\varepsilon^2\,d\mathcal{H}^1 + \frac{\gamma(\varepsilon)}{h(\varepsilon)}\int_0^T\int_{\Omega_\varepsilon}(q_\varepsilon - f_\varepsilon)^2\,dxdt \longrightarrow \inf,$$

where $\gamma(\varepsilon) > 0$ is a penalizing coefficient, and
$q_\varepsilon \in L^2(0,T;L^2(\Omega_\varepsilon))$ is a virtual control.

Motivation to the problem $\widehat{\mathbb{P}}_\varepsilon$

The main reasons to consider the $\widehat{\mathbb{P}}_\varepsilon$-problem instead of the original one are

1. In contrast to the original problem $\mathbb{P}_\varepsilon$ the sets of admissible solution $\widehat{\Xi}_\varepsilon$ for $\widehat{\mathbb{P}}_\varepsilon$ are always non-empty for every $\varepsilon > 0$;
2. The problem $\widehat{\mathbb{P}}_\varepsilon$ is always solvable;
3. Due to the structure of the cost functional $\widehat{I}_\varepsilon$ one can take the penalizing coefficient $\gamma(\varepsilon) > 0$ such that any optimal solution of $\widehat{\mathbb{P}}_\varepsilon$-problem will take the form $(f_\varepsilon, u^0_\varepsilon, p^0_\varepsilon, y^0_\varepsilon)$, where $(u^0_\varepsilon, p^0_\varepsilon, y^0_\varepsilon)$ be an optimal triplet for the original problem $\mathbb{P}_\varepsilon$.

Definition of C-extended optimal control problems

Definition 1. *We say that some optimal control problem $\mathbb{CP}_\varepsilon$ is a C-extension of $\widehat{\mathbb{P}}_\varepsilon$ one if their sets of admissible states coincide.*

Remark. We accept the notion of "C-extension" or "extension by control" since not every admissible control for $\mathbb{CP}_\varepsilon$-problem is that for $\widehat{\mathbb{P}}_\varepsilon$. It means that these problems may have different sets of admissible solutions in a general.

The structure of the C-extended problem $\mathbb{CP}_\varepsilon$

$$\partial_t z_\varepsilon - \operatorname{div}\left(A(x/\varepsilon)\nabla z_\varepsilon\right) = a_\varepsilon, \text{ in } (0,T)\times\Omega_\varepsilon,$$

$$J_\varepsilon(a_\varepsilon, v_\varepsilon, w_\varepsilon, z_\varepsilon) = \frac{1}{h(\varepsilon)}\int_{\Omega_\varepsilon}\left(z_\varepsilon(T,x) - y_\varepsilon^T\right)^2 dx$$
$$+ \frac{1}{h(\varepsilon)}\int_0^T\int_{\Omega_\varepsilon}(z_\varepsilon - y_\varepsilon^*)^6\, dxdt + \kappa(\varepsilon)\int_0^T\int_{\Gamma_\varepsilon^D} v_\varepsilon^2\, dx_1 dt$$
$$+ \int_0^T\int_{\Gamma_\varepsilon^N} w_\varepsilon^2\, d\mathcal{H}^1 + \frac{\gamma(\varepsilon)}{h(\varepsilon)}\int_0^T\int_{\Omega_\varepsilon}(a_\varepsilon - z_\varepsilon^3 - f_\varepsilon)^2\, dxdt \to \inf,$$

where $a_\varepsilon \in L^2((0,T)\times\Omega_\varepsilon)$ is a distributed control.

Lemma 2. *$\mathbb{CP}_\varepsilon$-problem is a C-extension of $\widehat{\mathbb{P}}_\varepsilon$-one.*

Properties of $\mathbb{CP}_\varepsilon$-problem

Theorem 2. *Assume that the original problem $\mathbb{P}_\varepsilon$ is uniformly regular, that is:*

(H1) $\Xi_\varepsilon \neq \emptyset$ *for every* $\varepsilon > 0$*;*

(H2) there is a sequence of admissible solutions $\{(u^*_\varepsilon, p^*_\varepsilon, y^*_\varepsilon) \in \Xi_\varepsilon\}_{\varepsilon>0}$ *such that*

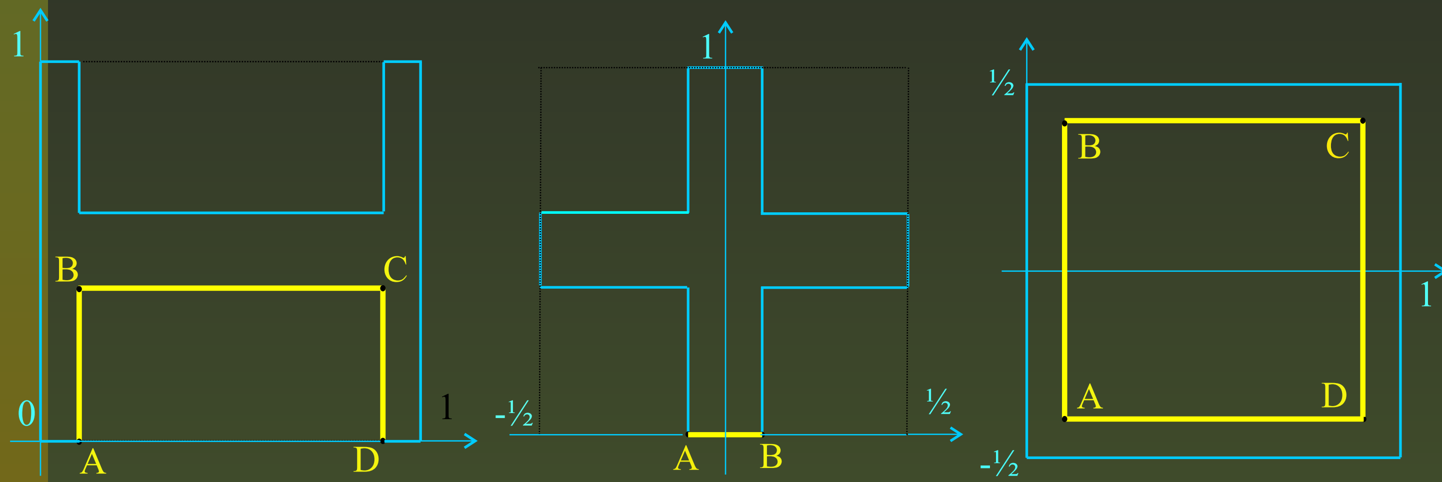
$$\limsup_{\varepsilon\to 0} \int_0^T \left[\|u^*_\varepsilon\|^2_{H^1(\Gamma^D_\varepsilon)} + \|p^*_\varepsilon\|^2_{L^2(\Gamma^N_\varepsilon)} + \|y^*_\varepsilon\|^6_{L^6(\Omega_\varepsilon)}\right] dt < +\infty.$$

Then there is a constant $\gamma^* > 0$ *such that having assumed* $\gamma = \gamma^*$ *one has for every* $\varepsilon > 0$*: "*$(u^0_\varepsilon, p^0_\varepsilon, y^0_\varepsilon) \in \Xi_\varepsilon$ *is an optimal solution to the original problem* $\mathbb{P}_\varepsilon$ *if and only if the quaternary* $(f_\varepsilon + (y^0_\varepsilon)^3, u^0_\varepsilon, p^0_\varepsilon, y^0_\varepsilon)$ *is an optimal one to* $\mathbb{CP}_\varepsilon$*-problem" .*

The Diagram of Homogenization

$$
\begin{array}{ccc}
(\mathbb{P}_\varepsilon) \Longrightarrow (\widehat{\mathbb{P}}_\varepsilon) & & (\mathbb{P}^{hom}) \Longleftarrow (\widehat{\mathbb{P}}^{hom}) \\
\big\downarrow & & \big\uparrow \\
(\mathbb{CP}_\varepsilon) & & (\mathbb{CP}^{hom}) \\
\big\downarrow & & \big\uparrow \\
\left\langle \inf\limits_{(a_\varepsilon, v_\varepsilon, w_\varepsilon, z_\varepsilon) \in \Sigma_\varepsilon} J_\varepsilon(a_\varepsilon, v_\varepsilon, w_\varepsilon, z_\varepsilon) \right\rangle & \xrightarrow{\varepsilon \longrightarrow 0} & \left\langle \inf\limits_{(a, v, w, z) \in \Sigma_0} J_0(a, v, w, z) \right\rangle
\end{array}
$$

The Periodicity Cells for Ω_ε



(a) The cell Y_1 (b) The cell Y_2 (c) The cell Y_3

Each of these figures indicates zones Λ_N^h, Λ_D^h, and Λ_R^h, where the corresponding boundary conditions for $\mathbb{CP}_\varepsilon$-problem are located. Here the index h indicates the dependence of these sets on the thickness of the thin grid $\mathcal{F}^h$.

Singular Measures

Let ν^h, μ^h, λ^h be the probability measures, concentrated and uniformly distributed on the sets Λ_N^h, Λ_D^h, and Λ_R^h, respectively.

$$\int_{Y_1} d\nu^h = \int_{Y_2} d\mu^h = \int_{Y_3} d\lambda^h = 1.$$

Then for any smooth function $\varphi \in C^\infty(\mathbb{R}^2)$ we have

$$\int_{\Lambda_N^h} \varphi \, d\mathcal{H}^1 = 2(1-h) \int_{Y_1} \varphi \, d\nu^h, \quad \int_{\Lambda_D^h} \varphi \, dx_1 = h \int_{Y_2} \varphi \, d\mu^h,$$

$$\int_{\Lambda_R^h} \varphi \, d\mathcal{H}^1 = 4(1-h) \int_{Y_3} \varphi \, d\lambda^h.$$

Scaling Measures

Since the homothetic contraction of the plane at ε^{-1} times takes the grid $\mathcal{F}^h$ to $\mathcal{F}^h_\varepsilon = \varepsilon\mathcal{F}^h$, we introduce so-called "scaling" measures ν^h_ε, μ^h_ε, and λ^h_ε by the following rules

$$\lambda^h_\varepsilon(B) = \varepsilon^2\lambda^h(\varepsilon^{-1}B), \quad \nu^h_\varepsilon(B) = \begin{cases} \varepsilon\nu^h(\varepsilon^{-1}(B\cap\mathcal{O}_\varepsilon)), & \text{if } B\cap\mathcal{O}_\varepsilon \neq \varnothing; \\ 0, & \text{otherwise;} \end{cases}$$

$$\mu^h_\varepsilon(B) = \begin{cases} \varepsilon\mu^h(\varepsilon^{-1}(B\cap\mathcal{O}_\varepsilon)), & \text{if } B\cap\mathcal{O}_\varepsilon \neq \varnothing; \\ 0, & \text{otherwise,} \end{cases}$$

where B is any planar Borel set, and

$$\mathcal{O}_\varepsilon = \{(x_1, x_2) \in \mathbb{R}^2 \ : \ 0 \leqslant x_2 < \varepsilon\}.$$

λ^h_ε is an ε-periodic measure on $\mathbb{R}^2$, whereas ν^h_ε and μ^h_ε are ε-periodic measure along x_1-axis.

Weak Convergence of the Singular Measures

We relate the parameters h and ε assuming that $h = h(\varepsilon) \longrightarrow 0$ as $\varepsilon \longrightarrow 0$.

Proposition 1.

1. The measure $\lambda_\varepsilon^{h(\varepsilon)}$ weakly converges to the planar Lebesgue measure as $\varepsilon \longrightarrow 0$ (in symbols $d\lambda_\varepsilon^{h(\varepsilon)} \rightharpoonup dx$), that is

$$\lim_{\varepsilon \longrightarrow 0} \int_{\mathbb{R}^2} \varphi \, d\lambda_\varepsilon^{h(\varepsilon)} = \int_{\mathbb{R}^2} \varphi \, dx \; \forall \, \varphi \in C_0^\infty(\mathbb{R}^2);$$

2. $$d\nu_\varepsilon^{h(\varepsilon)} \overset{\varepsilon \longrightarrow 0}{\rightharpoonup} \delta_{\{x_2=0\}} dx, \quad d\mu_\varepsilon^{h(\varepsilon)} \overset{\varepsilon \longrightarrow 0}{\rightharpoonup} \delta_{\{x_2=0\}} dx,$$

   where by $\delta_{\{x_2=0\}} dx$ we denote the product of the linear Lebesgue measure dx_1 and the Dirac measure $\delta_{\{x_2=0\}} dx_2$.

Reformulation of C-extended optimal control problem $\mathbb{CP}_\varepsilon$

Find a quaternary $(\widehat{a}_\varepsilon^{\,0}, \widehat{v}_\varepsilon^{\,0}, \widehat{w}_\varepsilon^{\,0}, \widehat{z}_\varepsilon^{\,0})$ such that

$$(\widehat{a}_\varepsilon^{\,0}, \widehat{v}_\varepsilon^{\,0}, \widehat{w}_\varepsilon^{\,0}, \widehat{z}_\varepsilon^{\,0}) \in \mathbb{Z}_\varepsilon \equiv$$

$$\equiv L^2(0,T; L^2(\Omega, d\eta_\varepsilon^h)) \times L^2(0,T; \left[H^1(\Gamma_0) \cap L^2(\Omega, d\mu_\varepsilon^h)\right])$$

$$\times L^2(0,T; L^2(\Omega, d\nu_\varepsilon^h)) \times \left[\, L^2(0,T; H^1(\Omega, d\eta_\varepsilon^h)) \cap L^2(0,T; L^2(\Omega, d\lambda_\varepsilon^h\right.$$

$$\widehat{J}_\varepsilon(\widehat{a}_\varepsilon^{\,0}, \widehat{v}_\varepsilon^{\,0}, \widehat{w}_\varepsilon^{\,0}, \widehat{z}_\varepsilon^{\,0}) = \inf_{(\widehat{a}_\varepsilon, \widehat{v}_\varepsilon, \widehat{w}_\varepsilon, \widehat{z}_\varepsilon) \in \widehat{\Sigma}_\varepsilon} \widehat{J}_\varepsilon(\widehat{a}_\varepsilon, \widehat{v}_\varepsilon, \widehat{w}_\varepsilon, \widehat{z}_\varepsilon), \qquad (1)$$

where $|h(\varepsilon)(2-h(\varepsilon))|d\eta_\varepsilon^h = \chi_\varepsilon^h\,dx$, and the cost functional $\widehat{J}_\varepsilon$ and the set of admissible solutions $\widehat{\Sigma}_\varepsilon$ have the following analytical representation

The Cost Functional $\widehat{J}_\varepsilon$

$$\widehat{J}_\varepsilon(\widehat{a}_\varepsilon, \widehat{v}_\varepsilon, \widehat{w}_\varepsilon, \widehat{z}_\varepsilon) = (2 - h(\varepsilon)) \int_\Omega (\widehat{z}_\varepsilon(T,x) - y_\varepsilon^T)^2 \, d\eta_\varepsilon^h$$

$$+ (2 - h(\varepsilon)) \int_0^T \int_\Omega (\widehat{z}_\varepsilon - y_\varepsilon^*)^6 \, d\eta_\varepsilon^h dt + \kappa(\varepsilon) h(\varepsilon) \int_0^T \int_\Omega \widehat{v}_\varepsilon^2 \, d\mu_\varepsilon^h \, dt$$

$$+ 2 \, (1 - h(\varepsilon)) \int_0^T \int_\Omega \widehat{w}_\varepsilon^2 \, d\nu_\varepsilon^h dt$$

$$+ \gamma^* \, (2 - h(\varepsilon)) \int_0^T \int_\Omega (\widehat{a}_\varepsilon - \widehat{z}_\varepsilon^3 - f_\varepsilon)^2 \, d\eta_\varepsilon^h dt.$$

The Set of Admissible Solutions $\widehat{\Sigma}_\varepsilon$

$$(\widehat{a}_\varepsilon, \widehat{v}_\varepsilon, \widehat{w}_\varepsilon, \widehat{z}_\varepsilon) \in \widehat{\Sigma}_{\ \varepsilon} \quad \text{if}$$

$$\left\{ \begin{array}{c} \widehat{z}_\varepsilon = \widehat{v}_\varepsilon \quad \mu_\varepsilon^h - \text{ a.e. on } (0,T)\times\Omega, \\ \widehat{z}_\varepsilon(0,\cdot) = y_{\varepsilon,0} \quad \eta_\varepsilon^h - \text{a.e. in } \Omega; \\ \|\widehat{v}_\varepsilon\|_{L^2(0,T;H^1(\Gamma_0))} \leqslant \mathbf{C}_0, \\ -\int_0^T\int_\Omega \widehat{z}_\varepsilon \varphi\, \psi'\, d\eta_\varepsilon^h dt + \int_0^T\int_\Omega (A(x/\varepsilon)\nabla\widehat{z}_\varepsilon\cdot\nabla\varphi)\psi\, d\eta_\varepsilon^h dt \\ +4d\,\beta(\varepsilon)\ \int_0^T\int_\Omega \widehat{z}_\varepsilon\varphi\psi\, d\lambda_\varepsilon^h dt = \ \int_0^T\int_\Omega \widehat{a}_\varepsilon\,\varphi\,\psi\, d\eta_\varepsilon^h dt \\ +4\,\beta(\varepsilon)\ \int_0^T\int_\Omega \widehat{g}_\varepsilon\varphi\psi\, d\lambda_\varepsilon^h dt + 2\,\beta(\varepsilon)\ \int_0^T\int_\Omega \widehat{w}_\varepsilon\varphi\psi\, d\nu_\varepsilon^h dt, \\ \forall\,\varphi\in C_0^\infty(\mathbb{R}^2;\Gamma_\varepsilon^D),\ \forall\,\psi\in C_0^\infty(0,T). \end{array} \right\}$$

Here $\beta(\varepsilon) = \frac{1-h(\varepsilon)}{2-h(\varepsilon)}\cdot\left[\frac{\varepsilon}{h(\varepsilon)}\right]$.

Convergence in the variable spaces

Let $\left\{ z_\varepsilon^h \in L^2(0,T;L^2(\Omega,d\lambda_\varepsilon^h)) \cap L^2(0,T;H^1(\Omega,d\eta_\varepsilon^h)) \right\}$ be a bounded sequence, i.e. $\limsup_{\varepsilon\to 0} \int_0^T \int_\Omega (z_\varepsilon^h)^2 \, d\lambda_\varepsilon^h dt < +\infty$.

1. The weak convergence $z_\varepsilon^h \rightharpoonup z$ in $L^2(0,T;L^2(\Omega,d\lambda_\varepsilon^h))$ means that

$$z \in L^2((0,T)\times\Omega) \text{ and } \lim_{\varepsilon\to 0} \int_0^T \int_\Omega z_\varepsilon^h \varphi\psi \, d\lambda_\varepsilon^h dt = \int_0^T \int_\Omega z\varphi\psi \, dxdt$$

   for any $\varphi \in C_0^\infty(\Omega)$ and $\psi \in C_0^\infty(0,T)$;

2. The strong convergence $z_\varepsilon^h \to z$ in $L^2(0,T;L^2(\Omega,d\lambda_\varepsilon^h))$ means that

$$z \in L^2((0,T)\times\Omega) \text{ and } \lim_{\varepsilon\to 0} \int_0^T \int_\Omega z_\varepsilon^h y_\varepsilon^h \, d\lambda_\varepsilon^h dt = \int_0^T \int_\Omega zy \, dxdt$$

   if $y_\varepsilon^h \rightharpoonup y$ in $L^2(0,T;L^2(\Omega,d\lambda_\varepsilon^h))$.

Convergence of the State Functions

Definition 2. Let $\left\{z_\varepsilon^h\right\}_{\varepsilon>0}$ be a bounded sequence in

$$\mathbb{M}_\varepsilon = L^2(0,T;H^1(\Omega,d\eta_\varepsilon^h)) \cap L^2(0,T;L^2(\Omega,d\lambda_\varepsilon^h)),$$

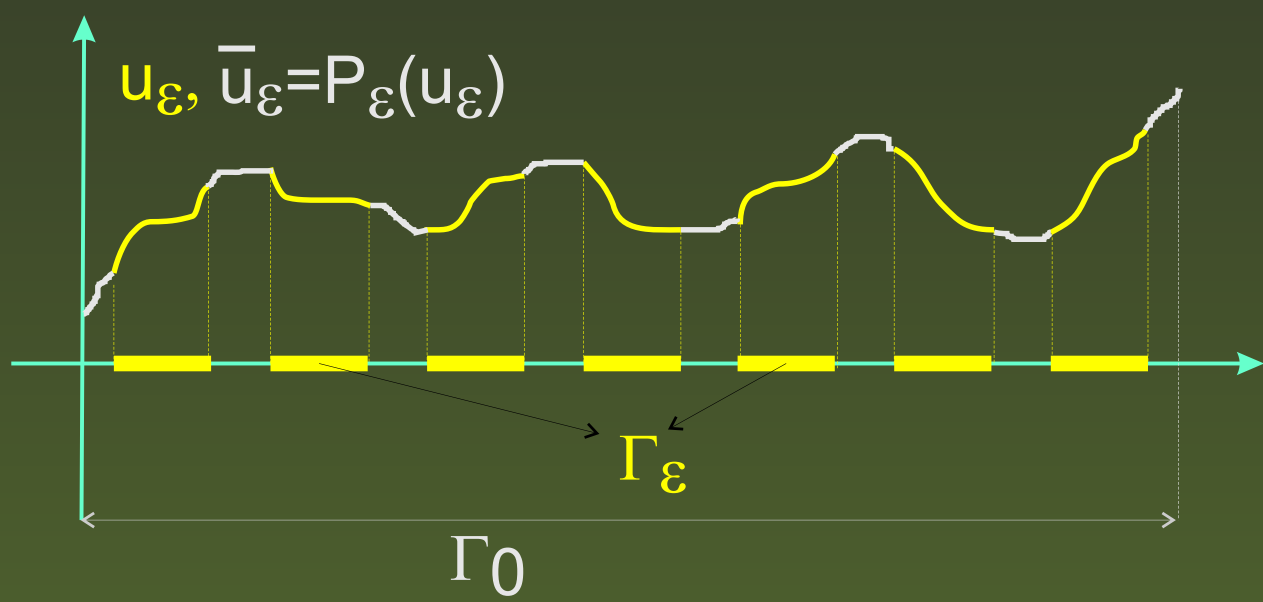
i.e. $\limsup_{\varepsilon\to 0} \int_0^T \left[\|z_\varepsilon^h(\varepsilon)\|^2_{H^1(\Omega,d\eta_\varepsilon^{h(\varepsilon)})} + \|z_\varepsilon^h(\varepsilon)\|^2_{L^2(\Omega,d\lambda_\varepsilon^{h(\varepsilon)})} \right] dt < +\infty.$
We say that this sequence is weakly convergent in the variable space $\mathbb{M}_\varepsilon$ if there is a function $z \in L^2(0,T;H^1(\Omega))$ such that

$$\begin{cases} z_\varepsilon^h \to z & \text{in } L^2(0,T;L^2(\Omega,d\eta_\varepsilon^h)); \\ z_\varepsilon^h \rightharpoonup z & \text{in } L^2(0,T;L^2(\Omega,d\lambda_\varepsilon^h)); \\ \int_0^T \int_\Omega |z_\varepsilon^h - z|^2 \, d\eta_\varepsilon^h dt \to 0 & \text{as } \varepsilon \to 0. \end{cases}$$

Convergence Formalism of Dirichlet Boundary Controls. 1.

Dirichlet control

$$\widehat{v}_\varepsilon^0 \in L^2(0,T;H^1(\Gamma_0)) \cap L^2(0,T;L^2(\Omega, d\mu_\varepsilon^h)) \quad \forall\, \varepsilon > 0.$$



Convergence Formalism of Dirichlet Boundary Controls. 2.

Definition 3. We say that a function u^* is s_a-limit for the sequence of Dirichlet controls $\left\{\widehat{v}_\varepsilon \in L^2(0,T;H^1(\Gamma_\varepsilon))\right\}_{\varepsilon>0}$ if some sequence of its images $\{P_\varepsilon(\widehat{v}_\varepsilon)\}_{\varepsilon>0}$ converges to u^* weakly in $L^2(0,T;H^1(\Gamma_0))$. And we say that the sequence $\left\{\widehat{v}_\varepsilon \in L^2(0,T;L^2(\Omega,d\mu_\varepsilon^h))\right\}_{\varepsilon>0}$ is s_b-convergent to an element $u^{**} \in L^2(0,T;L^2(\Gamma_0))$ if this sequence is uniformly bounded and $\widehat{v}_\varepsilon \rightharpoonup u^{**}$ in the variable space $L^2(0,T;L^2(\Omega,d\mu_\varepsilon^h))$.

Theorem 4. Let $\left\{\widehat{v}_\varepsilon \in \widehat{\mathcal{U}}_\varepsilon\right\}_{\varepsilon>0}$ be a sequence of admissible Dirichlet controls. Then one can extract an s_b-convergent subsequence of $\{\widehat{v}_\varepsilon\}_{\varepsilon>0}$ for which its s_a- and s_b-limits coincide.

w-Convergence of Admissible Solutions to the $\widehat{\mathbb{CP}}_\varepsilon$-problem

Definition 4. We say that a bounded sequence $\{(a_\varepsilon, v_\varepsilon, w_\varepsilon, z_\varepsilon) \in \mathbb{Z}_\varepsilon\}_{\varepsilon>0}$ is w-convergent to a quaternary (a, v, w, z) in the variable space $\mathbb{Z}_\varepsilon$ as ε tends to zero (in symbols, $(a_\varepsilon, v_\varepsilon, w_\varepsilon, z_\varepsilon) \xrightarrow{w} (a, v, w, z)$) , if :

$$(a, v, w, z) \in L^2((0,T)\times\Omega) \times L^2(0,T;H^1(\Gamma_0)) \times L^2((0,T)\times\Gamma_0) \times L^2(0,T;H^1$$

$$a_\varepsilon \rightharpoonup a \quad \text{in } \ L^2(0,T;L^2(\Omega, d\eta_\varepsilon^h));$$

$$v_\varepsilon \to v \quad \text{in } \ L^2(0,T;L^2(\Omega, d\mu_\varepsilon^h)), \quad v \in L^2(0,T;H^1(\Gamma_0));$$

$$w_\varepsilon \rightharpoonup w \quad \text{in} \quad L^2(0,T;L^2(\Omega, d\nu_\varepsilon^h));$$

$$z_\varepsilon^h \to z \text{ in } \ L^2(0,T;L^2(\Omega, d\eta_\varepsilon^h)); \quad z_\varepsilon^h \rightharpoonup z \text{ in } \ L^2(0,T;L^2(\Omega, d\lambda_\varepsilon^h));$$

$$\int_0^T \int_\Omega |z_\varepsilon^h - z|^2 \, d\eta_\varepsilon^h dt \to 0 \quad \text{as } \ \varepsilon \to 0.$$

Compactness property

Theorem 3. Let $\{(a_\varepsilon, v_\varepsilon, w_\varepsilon, z_\varepsilon) \in \mathbb{Z}_\varepsilon\}_{\varepsilon>0}$ be a bounded sequence of admissible solutions to C-extended problems $\widehat{\mathbb{CP}}_\varepsilon$. Then this sequence is relatively compact with respect to the w-convergence in the variable space $\mathbb{X}_\varepsilon$ and moreover if $(a_\varepsilon, v_\varepsilon, w_\varepsilon, z_\varepsilon) \xrightarrow{w} (a, v, w, z)$ then

$$z_\varepsilon \to z \ \text{ in } \ L^6(0,T; L^6(\Omega, d\eta_\varepsilon^h)),$$

$$(z_\varepsilon)^3 \to z^3 \ \text{ in } \ L^2(0,T; L^2(\Omega, d\eta_\varepsilon^h)).$$

Result of Recovery of the Limit Singular OCP $\mathbb{P}^{hom}$

The singular OCP $\mathbb{P}^{hom}$ takes the following form

$$I_0(u,y) = 2\int_\Omega (y(T,x) - y^T)^2\,dx + 2\int_0^T\int_\Omega (y-y^*)^6\,dxdt$$

$$+\kappa^*\int_0^T\int_{\Gamma_0} u^2\,dx_1dt \longrightarrow \inf,$$

$$\left.\begin{aligned} \partial_t y - \operatorname{div}\left(A^{hom}\nabla y\right) + 2d\xi^* y - y^3 = f + 2\xi^* g, &\quad \text{in } (0,T)\times\Omega,\\ \partial_{\nu_{A^{hom}}} y = 0, &\quad \text{on } (0,T)\times\partial\Omega\setminus\Gamma_0,\\ y = u, &\quad \text{on } (0,T)\times\Gamma_0,\\ y(0,x) = y_0 &\quad \text{for a.a. } x\in\Omega, \end{aligned}\right\}$$

$$u \in L^2(0,T;H^1(\Gamma_0)), \quad \|u\|_{L^2(0,T;H^1(\Gamma_0))} \leqslant \mathbf{C}_0.$$

where $\xi^* = \lim_{\varepsilon\to 0}\frac{\varepsilon}{h(\varepsilon)}$, $\kappa^* = \lim_{\varepsilon\to 0}\kappa(\varepsilon)h(\varepsilon)$.

Variational Properties of the Homogenized Problem $\mathbb{P}^{hom}$

Theorem 5. Let $(u_\varepsilon^0, p_\varepsilon^0, y_\varepsilon^0) \in \Xi_\varepsilon$ be an optimal solution to the original problem for every $\varepsilon > 0$. Then there can be found a sequence of indices $\{\varepsilon_k\}$ converging to 0 as $k \to \infty$, and functions $y^0 \in L^2(0,T;H^1(\Omega))$, and $u^0 \in L^2(0,T;H^1(\Gamma_0))$ such that

$$\widehat{u}^0_{\varepsilon_k} \longrightarrow u^0 \quad \text{strongly in } L^2(0,T;L^2(\Omega, d\mu^h_{\varepsilon_k}));$$

$$\widehat{p}^0_{\varepsilon_k} \longrightarrow 0 \quad \text{weakly in } L^2(0,T;L^2(\Omega, d\nu^h_{\varepsilon_k}));$$

$$\widehat{y}^0_{\varepsilon_k} \longrightarrow y^0 \quad \text{strongly in } L^2(0,T;L^2(\Omega, d\eta^h_{\varepsilon_k}));$$

$$\left(\widehat{y}^0_{\varepsilon_k}\right)^3 \longrightarrow \left(y^0\right)^3 \quad \text{strongly in } L^2(0,T;L^2(\Omega, d\eta^h_{\varepsilon_k}));$$

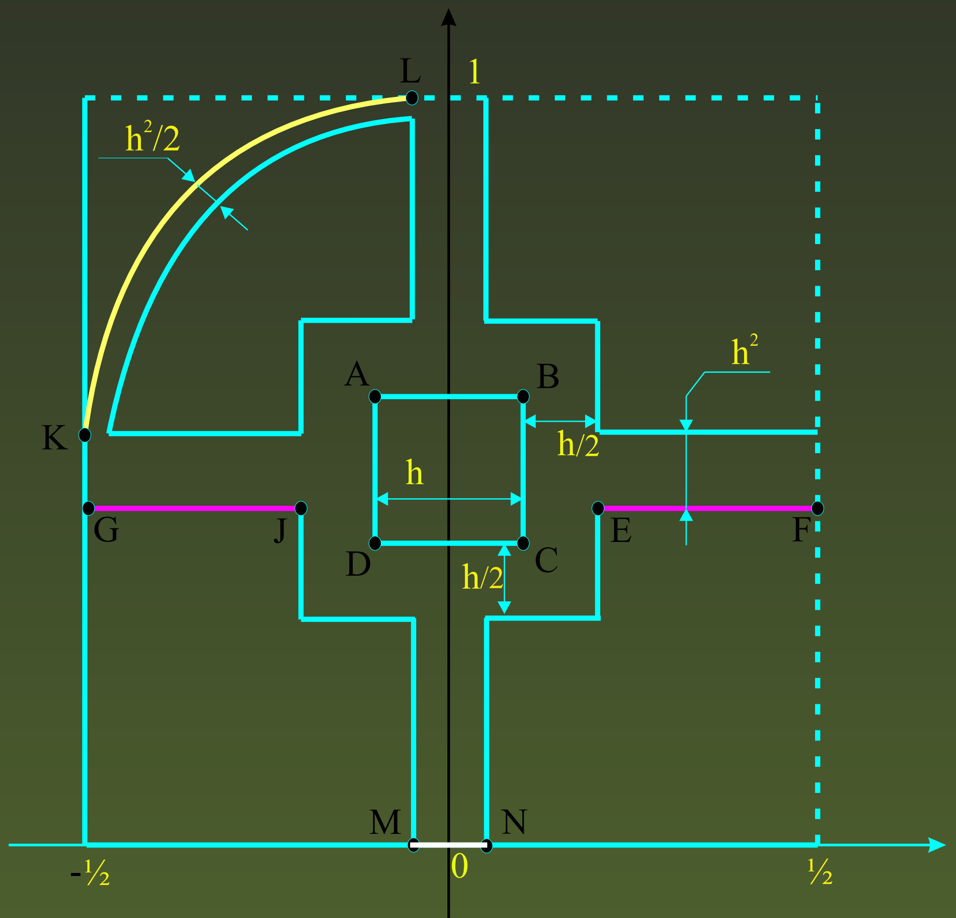
$$\left(\nabla \widehat{y}^0_{\varepsilon_k} - \nabla y^0\right) \stackrel{2}{\rightharpoonup} r \in L^2(0,T;L^2(\Omega, V_{pot}));$$

$$\lim_{\varepsilon \to 0} I_\varepsilon(u_\varepsilon^0, p_\varepsilon^0, y_\varepsilon^0) = I_0(u^0, y^0),$$

where the pair (u^0, y^0) is an optimal solution to $\mathbb{P}^{hom}$.

Example of the Fattened Graph which is not a Thin Grid.

(d) The structure of the $\mathcal{F}^h$.



(e) The periodicity cell of $\mathcal{F}^h$.

Example of Optimal Control Problem ($h(\varepsilon) = \sqrt{\varepsilon/(40+\pi)}$)

$$
\left.\begin{array}{rcll}
\partial_t y_\varepsilon - \operatorname{div}\left(A(x/\varepsilon)\nabla y_\varepsilon\right) - y_\varepsilon^3 & = & f_\varepsilon, & \text{in } (0,T)\times\Omega_\varepsilon, \\
\partial_{\nu_A} y_\varepsilon & = & 0, & \text{on } (0,T)\times S^\varepsilon_{ext}, \\
\partial_{\nu_A} y_\varepsilon & = & \varepsilon^2(-y_\varepsilon + g_\varepsilon), & \text{on } (0,T)\times S^\varepsilon_{int}, \\
y_\varepsilon & = & 0, & \text{on } (0,T)\times \Gamma^D_\varepsilon, \\
\partial_{\nu_A} y_\varepsilon & = & \varepsilon p_\varepsilon, & \text{on } (0,T)\times \Gamma^N_\varepsilon, \\
y_\varepsilon(0,x) & = & 0 & \text{for a.a. } x\in\Omega_\varepsilon,
\end{array}\right\}
$$

$$
p_\varepsilon \in L^2(0,T;L^2(\Gamma^N_\varepsilon)),
$$

$$
I_\varepsilon(u_\varepsilon,p_\varepsilon,y_\varepsilon) = h^{-1}\int_{\Omega_\varepsilon} y_\varepsilon^2(T,x)\,dx + h^{-1}\int_0^T\int_{\Omega_\varepsilon} y_\varepsilon^6\,dxdt
$$

$$
+\int_0^T\int_{\Gamma^N_\varepsilon} p_\varepsilon^2\,d\mathcal{H}^1 dt \longrightarrow \inf .
$$

Result of Homogenization for $h(\varepsilon) = \sqrt{\frac{\varepsilon}{(40+\pi)}}$

$$\left.\begin{array}{rl} \partial_t y - \operatorname{div}\left(A^{hom}\nabla y\right) + 2\pi y - y^3 = f + 2\pi\, g, & \text{in } (0,T)\times\Omega, \\ \partial_{\nu_{A^{hom}}} y = 0, & \text{on } (0,T)\times(\partial\Omega\setminus\Gamma_0), \\ y = 0, & \text{on } (0,T)\times\Gamma_0, \\ y(0,x) = 0 & \text{a.e. } x\in\Omega, \end{array}\right\}.$$

Variational properties: if $\left\{(p_\varepsilon^0, y_\varepsilon^0)\right\}$ is a sequence of optimal solutions to the above stated problem on Ω_ε, then there can be found a sequence of indices $\{\varepsilon_k\}$ converging to 0 as $k\to\infty$ such that $(0, p^0_{\varepsilon_k}, y^0_{\varepsilon_k}) \longrightarrow (0,0,y^0)$, where the function $y^0 \in L^2(0,T;H^1(\Omega))$ is a solution to the $\mathbb{P}^{hom}$.

Thank you for your uncommon patience!