Gelling Solutions of Coagulation Equations

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The Smoluchowski equation

Approximation of aggregation models in polymers, planetesimals, aerosols, soots,...

(M. von Smoluchowski Z. Phys. 1916)

$$\frac{\partial f}{\partial t}(t,x) = \frac{1}{2}\int_0^x W(x-y,y) f(t,x-y) f(t,y)\,dy - \\ - f(t,x)\int_0^\infty W(x,y) f(t,y)\,dy$$

$f(t,x)$: density of clusters of size $x > 0$

$W(x,y) =$ homogeneous function of degree λ.

$W(x,y) = W(y,x)$.

Aggregation of "particles" of size $x-y$ and y to give "particles" of size x.

AGGREGATION ?

Different types of aggregating objects: particle-particle (coagulation); particle-cluster; cluster-cluster.

Different types of movement for particles/clusters: Brownian motion (diffusion); ballistic aggregation (linear trajectories motion)...

Different types of aggregation: Diffusion Limited Aggregation (all collisions lead to attachment); Reaction Limited Aggregation (requires repeated collisions before sticking)...

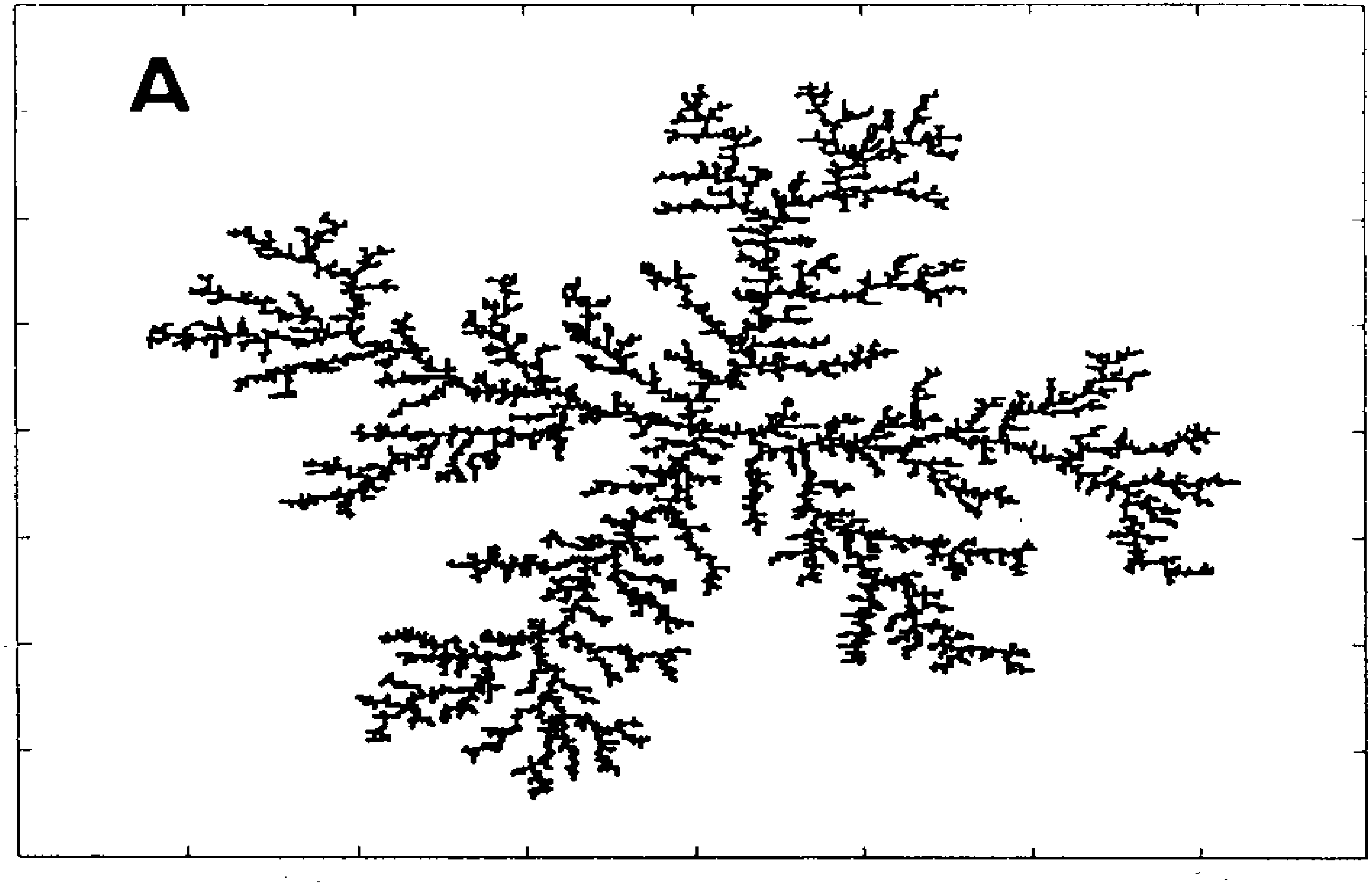


Figure 1: Particle-cluster, Diffusion Limited Aggregation with 10.000 particles

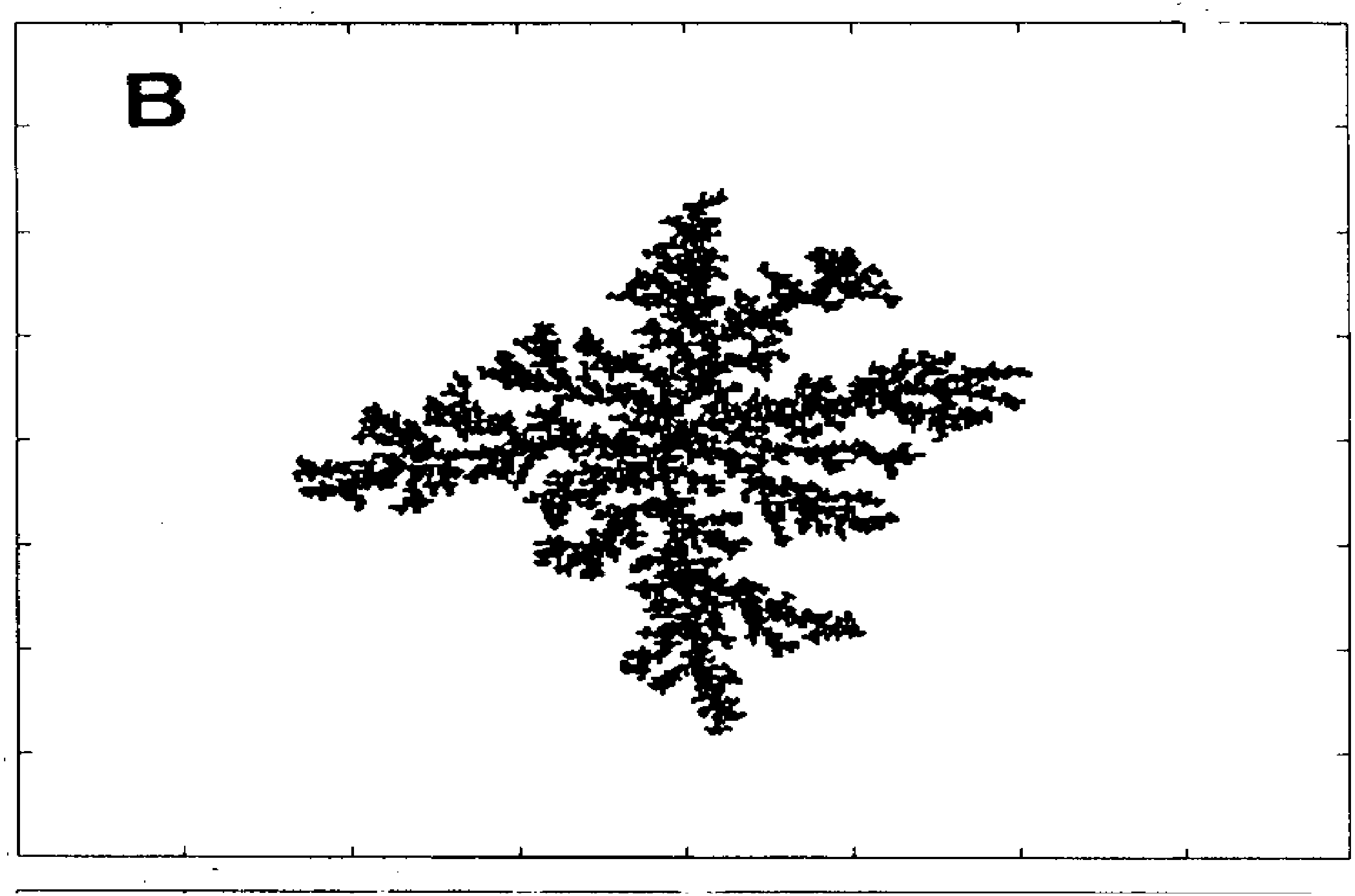


Figure 2: Particle-cluster, Reaction Limited Aggregation with 10.000 particles

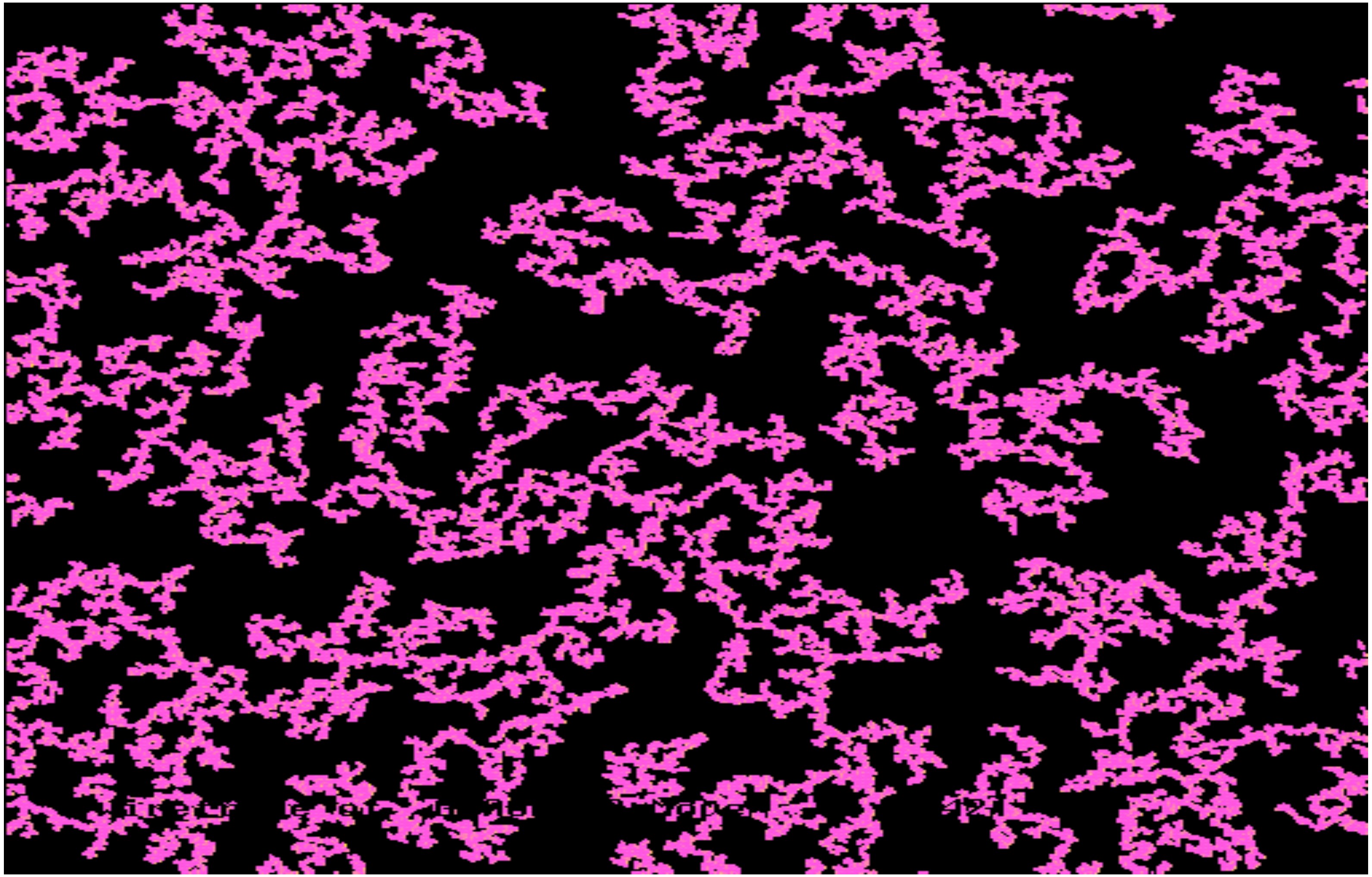


Figure 3: Cluster-cluster, Diffusion Limited Aggregation with 250.000 particles

Smoluchowski equation: Strong simplification.

In particular: No space dependence.

The kernel W has to model all these different situations.

Examples:

1.-Coagulation with size of particles $>$ mean distance between particles:

$$W(x,y) = \left(x^{-\frac{1}{3}} + y^{-\frac{1}{3}}\right)\left(x^{\frac{1}{3}} + y^{\frac{1}{3}}\right) \quad x : \text{volume of particles}$$

2.-If the size of particles $<$ mean distance between particles:

$$W(x,y) = \left(\frac{1}{x} + \frac{1}{y}\right)^{\frac{1}{2}}\left(x^{\frac{1}{3}} + y^{\frac{1}{3}}\right)^2$$

For simplicity: $W(x,y) = x^{\alpha}\,y^{\beta} + x^{\beta}\,y^{\alpha}, \;\; \lambda = \alpha + \beta$

Existence and Uniqueness of Solutions.

- Under general conditions on W (continuity, no growth condition) if

$$f_0 \geq 0, \qquad \int_0^\infty x f_0(x) dx < \infty$$

there exists a solution $f(t,x) \geq 0$ to Smoluchowski equation such that $f(0,x) = f_0(x)$ and, for all $t > 0$:

$$\int_0^\infty x f(t,x) dx \leq \int_0^\infty x f(0,x) dx.$$

Spouge 84; Ball, Carr & Penrose 86; Dubovski 86; Stewart 89; Laurençot 00.

- If $\displaystyle\int_0^\infty (x + x^r) f_0(x) dx < \infty$ for some $r > 2$ the solution is unique if $\lambda \leq 1$.

Uniqueness may fail if $\lambda > 1$ (J. Norris 98).

The model describes collisions between particles:

$\longrightarrow$ Should preserve the total volume:

$$\frac{d}{dt}\int_0^\infty x\, f(t,x)\, dx = 0, \ \forall t > 0. \tag{1}$$

True for $0 < \lambda \leq 1$: the (unique) solution is global in time and satisfies (1).

BUT

If $\lambda > 1 \rightarrow$ Gelation in finite time.

GELATION ?

For some $T^* > 0$ (gelling time):

$$\int_0^\infty x\, f(t,x)\, dx \;=\; \int_0^\infty x\, f(0,x)\, dx, \;\; \forall t \in (0, T^*)$$

$$\int_0^\infty x\, f(t,x)\, dx \;<\; \int_0^\infty x\, f(0,x)\, dx, \;\; \forall t > T^*.$$

Similar to blow up phenomena. Consider for example:

$W(x,y) = x\, y$ and $M_2(t) = \int_0^\infty x^2 f(t,x) dx$ (mean size of particles) :

$$\frac{d}{dt} M_2(t) = M_2^2(t) \quad \text{and} \quad T_{blowup} = T^*.$$

Questions of Mass

- After T^*, the solution f has "lost" part of its mass:

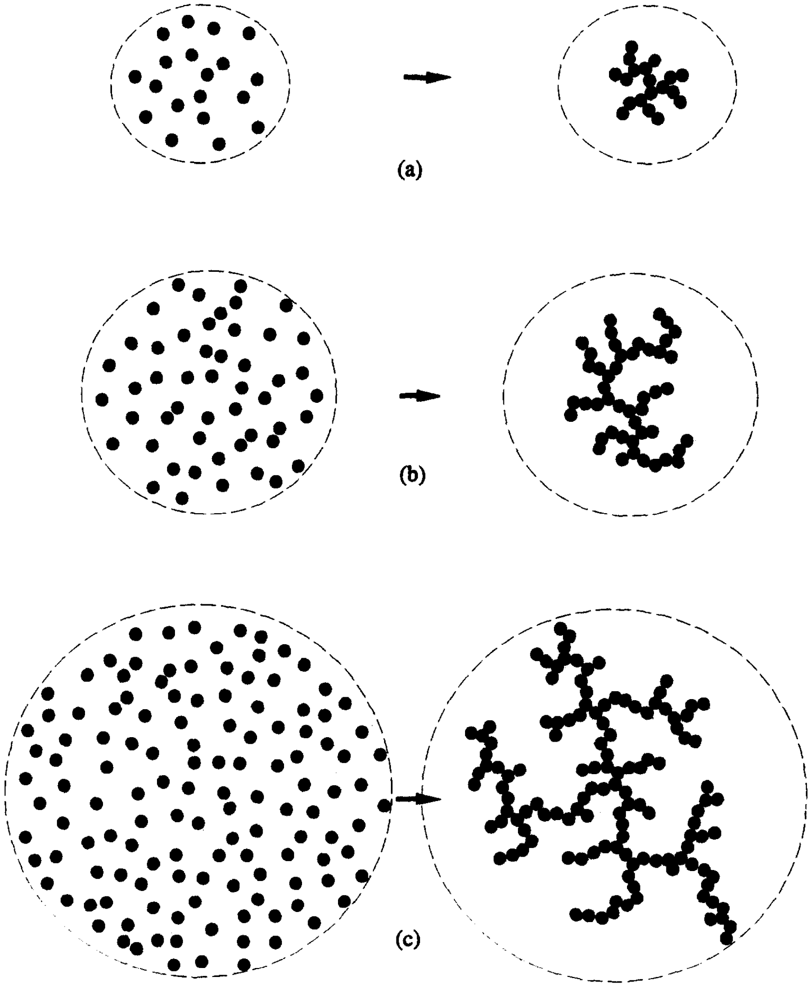
$$\int_0^\infty x\,f(t,x)\,dx < \int_0^\infty x\,f(0,x)\,dx \quad t > T^*.$$

- This mass has gone to particles of "infinite size":

$$\lim_{t\to T^*} \int_0^\infty x^2\,f(t,x)\,dx = +\infty.$$

- The mass of the "particles of infinite size":

$$\int_0^\infty x\,f(0,x)\,dx - \int_0^\infty x\,f(t,x)\,dx.$$



(a): the small cluster $<<$ region where unagregated particles were contained.
(b): larger region gives larger aggregate.
(c): the aggregate is the same size as the original region.

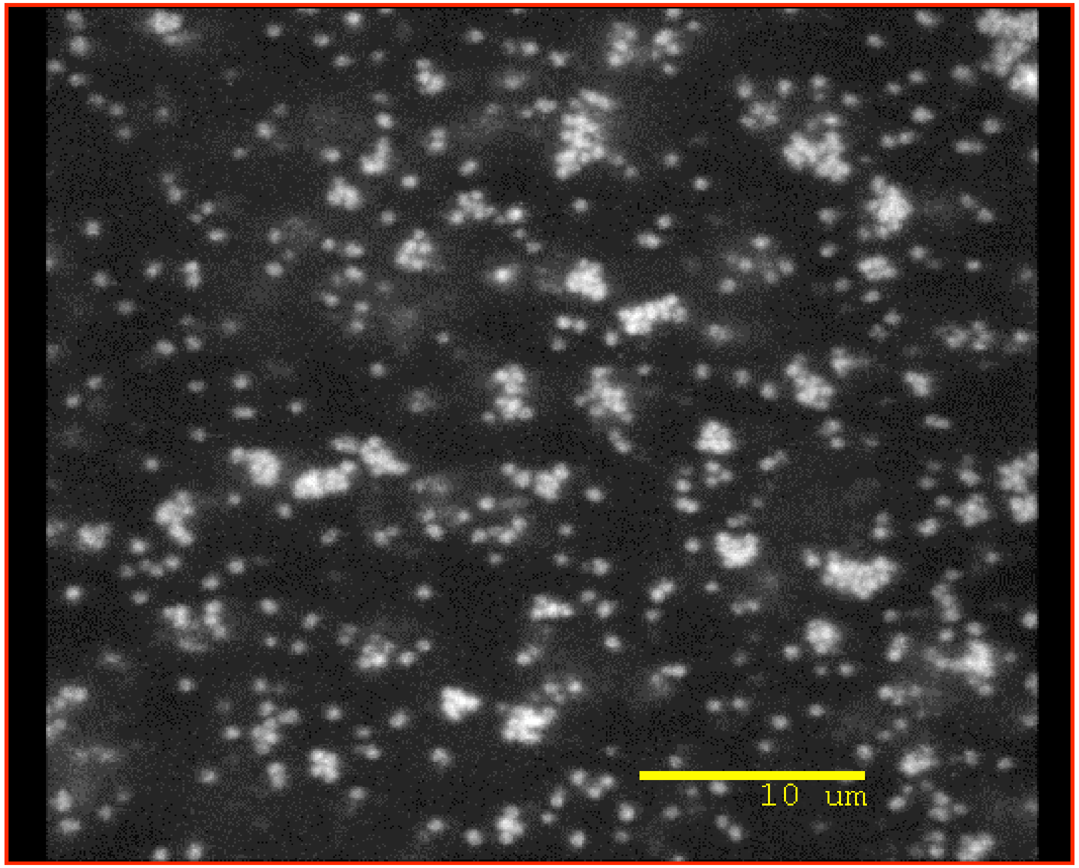


Figure 4: Confocal microscope image of a thin slice through a sample in the fluid-cluster phase. Colloidal suspension of polymethylmethacrylate with polystyrene.

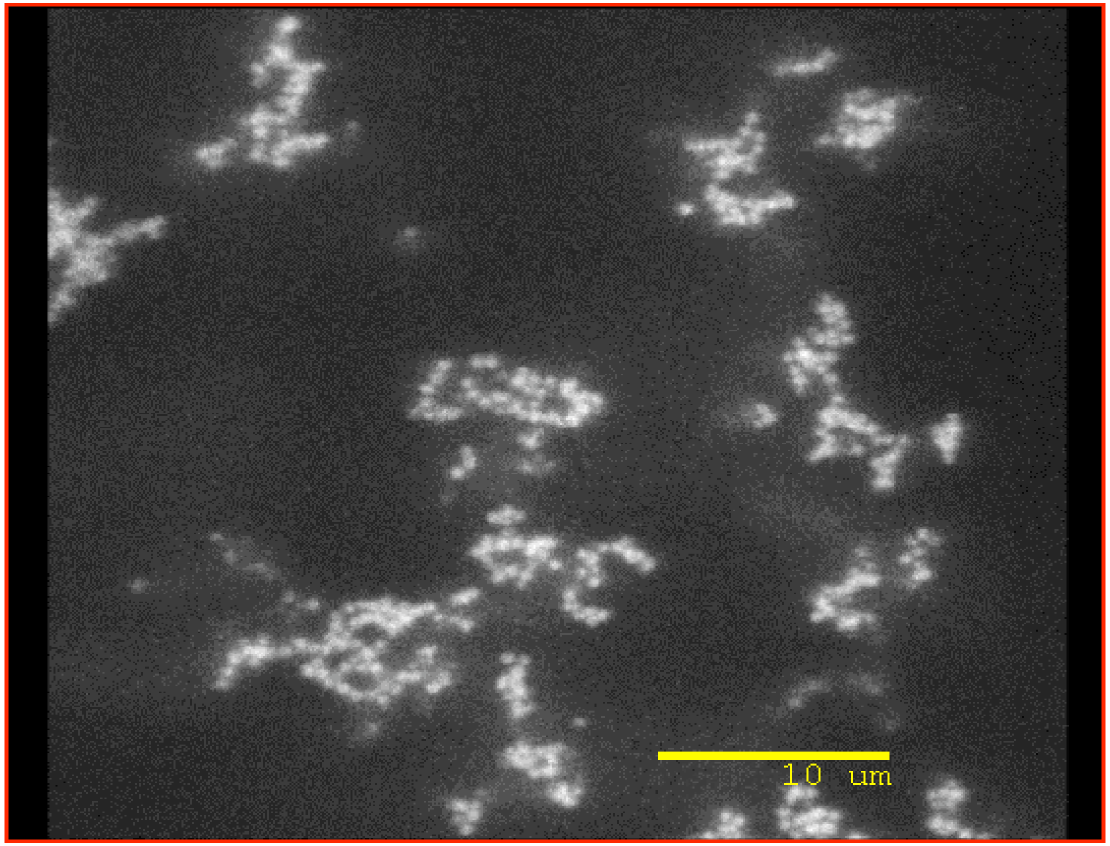


Figure 5: Image of a sample that has formed a gel.
(Dinsmore & Weitz J. Phys.: Condens. Matter '02.)

THE SOL-GEL TRANSITION IN CHEMICAL GELS

J. E. Martin and D. Adolf; Annu. Rev. Phys. Chem. 1991.

• The incipient gel (that fluid formed just at the sol-gel transition) is a viscoelastic intermediate between the liquid and solid state.

• Two conceptual frameworks: the kinetic approach, couched in terms of the Smoluchowski equation, and equilibrium theories, such as the classical Flory-Stockmayer theory and percolation.

• Equilibrium theories have an advantage, as they provide structural information about the sol-gel. • The kinetic approach is useful in distinguishing between the concepts of aggregation and gelation, both of which are relevant to real gels.

• Much of the experimental data agrees with neither theory...

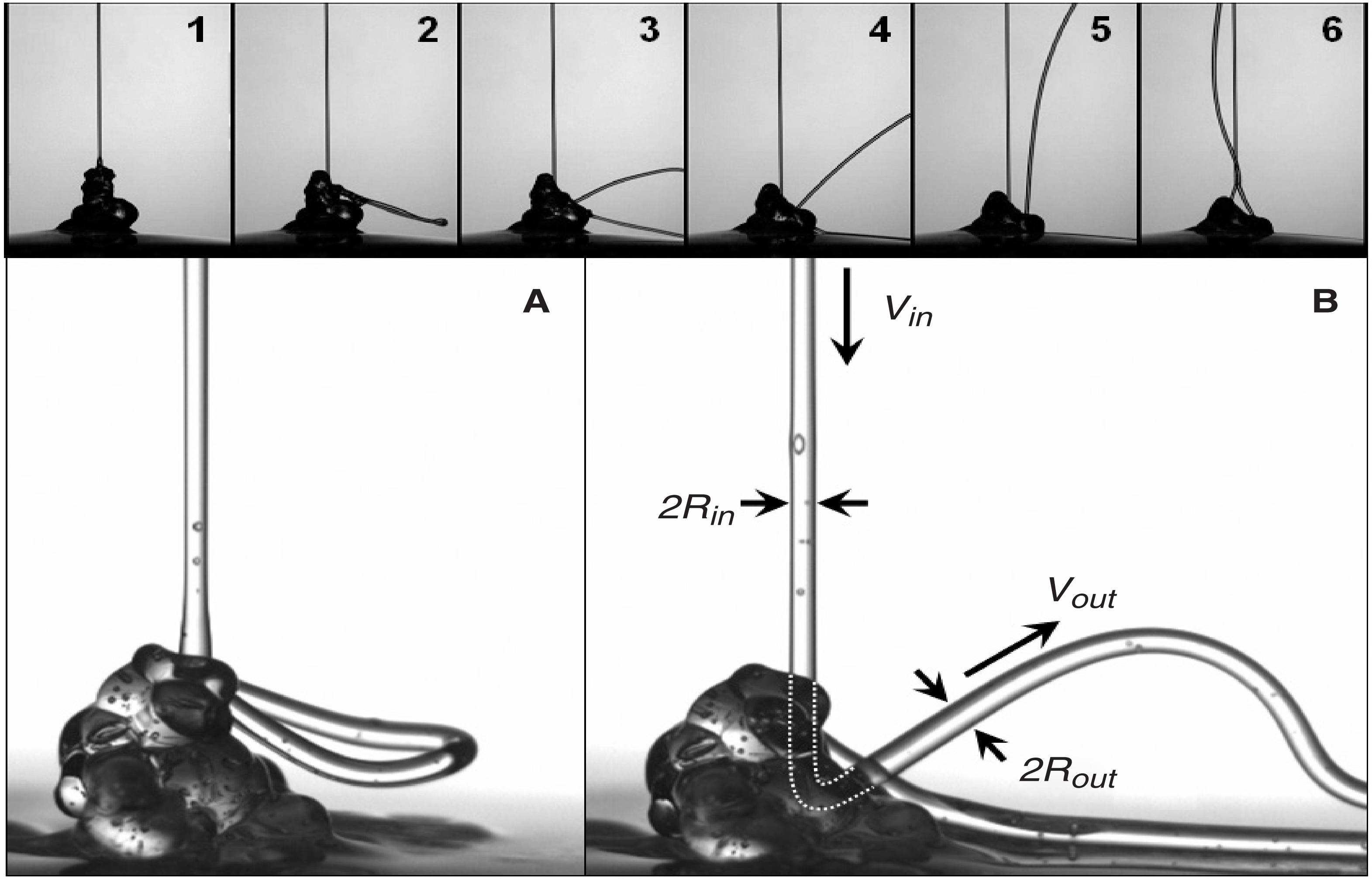


Figure 6: shampoo gel

Solutions after gelation.

What happens after gelation ?

Two different questions:

1.- What happens to the physical system?

2.- What happens to the solutions of the homogeneous Smoluchowski model?

May be related but not always. We consider the second question.

Moreover...

In the reduced Smoluchowski model:

what information may we get?

The rate of "gel" formation ($\equiv$ - the rate of mass loss):

$$\frac{d}{dt}\int_0^\infty x\, f(t,x)\, dx$$

This depends on an essential way on:

How does $f(t,x)$ behaves for $t > T^*$ and x large.

Formal asymptotics.

(P.G.J. van Dongen, J. Phys. A, 1987):

If $T^* < +\infty$:

- $f(t,x) \sim a(t)\, x^{-(3+\lambda)/2}$ as $x \longrightarrow +\infty$

- $\dfrac{d}{dt}\displaystyle\int_0^\infty x\, f(x,t)dx \sim -a^2(t)$

for all $t > T^*$.

Apriori estimates. (E., S. Mischler & B. Perthame C.M.P. 2002).

If $T_0 < T^* < T_1$:

- $\exists C_* > 0,\ C_1 > 0\,;\ \displaystyle\int_0^\infty x\, f(t,x)\, dx \le \frac{C_*}{(1+t)^{1/\lambda}},$

- $\forall R > 0 \quad \displaystyle\sup_{S>R} \int_{T_0}^{T_1} \left(\frac{1}{S^\tau} \int_0^S f(t,y)\, y^{\lambda/2+1/2+\tau}\, dy \right)^2 dt \le C_1,$

- $\forall R > 0 \quad \displaystyle\int_{T_0}^{T_1} \sup_{S>R} \left(\frac{1}{S^\tau} \int_0^S f(t,y)\, y^{\lambda/2+1/2+\tau}\, dy \right)^2 dt \ge C_1^{-1}.$

If one tries power like functions x^θ the only possible:

$$\theta = -\frac{3+\lambda}{2}$$

A well established conjecture is then:

$$f(t,x) \sim a(t)\, x^{-(3+\lambda)/2} \qquad (1)$$

as $x \to +\infty$ and for $t > T^*$.

We do not prove that conjecture.

What do we do ? Answer to the question:

Is there a solution of the S. equation satisfying (1) for x large?

The function:

$$F(x) = x^{-\frac{3+\lambda}{2}}$$

satisfies the Smoluchowski equation in a weak sense:

$$\frac{\partial F}{\partial t} = \int_0^{x/2} \left[(x-y)^{\lambda/2} F(x-y) - x^{\lambda/2} F(x) \right] y^{\lambda/2} F(y) dy$$
$$- F(x) \int_{x/2}^{\infty} (xy)^{\lambda/2} F(y) dy.$$

(Where from now on: $W(x,y) = (xy)^{\lambda/2}$, $1 < \lambda < 2$.)

BUT: $x^{-\frac{3+\lambda}{2}}$ has no finite mass because $x^{1-\frac{3+\lambda}{2}} \notin L^1_{loc}[0,1]$ since:

$$\lambda > 1 \Longrightarrow 1 - \frac{3+\lambda}{2} < -1.$$

Our purpose: To prove the existence of solutions with finite mass which behave like $x^{-\frac{3+\lambda}{2}}$ as $x >> 1$.

To this end:

- We linearise the Smoluchowski equation around $F(x) = x^{-\frac{3+\lambda}{2}}$.

$$f(t,x) = x^{-(3+\lambda)/2} + g(t,x)$$

Study the linear semigroup. Obtain precise estimates.

• Solve the nonlinear Smoluchowski equation with initial data

$$\int_0^{\infty} x\, f_0(x)dx < \infty \quad \text{and} \quad f_0(x) \sim x^{-(3+\lambda)/2} \quad \text{as } x \to +\infty.$$

The solution obtained is local $(t \in (0,T))$, and such that:

$$f(t,x) = a(t)\, x^{-(3+\lambda)/2} + o(x^{-(3+\lambda)/2}) \text{ as } x \to +\infty.$$

In particular: $$x^2 f(x,t) \notin L^1(1,+\infty)$$

and: $$\frac{d}{dt}\int_0^{\infty} x\, f(t,x)\, dx \leq -C\, a(t)^2 \quad \forall t \in (0,T).$$

f is a solution "after the gelling time". $(T^* = 0)$

$$
\begin{aligned}
\frac{d}{dt}\int_0^R x\, f(t,x)\, dx \quad &= \quad \frac{1}{2}\int_0^R x \int_0^x W(x-y,y) f(t,y) f(x-y) dy dx - \\
&- \int_0^R \int_0^\infty x\, f(t,x)\, W(x) W(y) dy dx \\
&< - \int_0^R \int_R^\infty W(x) W(y) f(t,x) f(t,y) dy dx \\
&\sim -a(t) \int_0^R x^{1+\lambda/2} f(t,x) \int_R^\infty y^{-3/2} dy dx \\
&= -a(t) \frac{1}{\sqrt{R}} \int_0^R x^{1+\lambda/2} f(t,x)\, dx \\
&\sim -a^2(t) \ \text{ as } \ R \to +\infty.
\end{aligned}
$$

Fundamental Solutions of the linearized equation.

Theorem.

For all $x_0 > 0$: unique global solution $g(t, \cdot, x_0)$ of linearised equation such that: $x \mapsto x^{3/2}\, g(t, x, x_0)$ is bounded in $\mathbb{R}^+$ and

$$g(0, x, x_0) = \delta(x - x_0).$$

Moreover:

$$g(t, \cdot, x_0) \in \mathcal{C}^\infty(\mathbb{R}^+)$$

and has the self similar form

$$g(t, x, x_0) = \frac{1}{x_0} g\left(t x_0^{\frac{\lambda-1}{2}}, \frac{x}{x_0}, 1\right).$$

Function $g(t, \cdot, 1)$ can be written as follows:

$$
\begin{aligned}
g(t,x,1) &= a_1(t)\, x^{-(3+\lambda)/2} + R_1(t,x) \quad \text{for } x > 2 \\
g(t,x,1) &= a_2(t)\, x^{-3/2} + R_2(t,x) \quad \text{for } 0 < x \le 2
\end{aligned}
$$

where $a_1, a_2 \in \mathbf{C}[0,+\infty)$ have the asymptotics:

$$
a_1(t) = \begin{cases} A_1\, t^{-\frac{1}{\lambda-1}} + \mathcal{O}(t^{-\frac{1}{\lambda-1}-\varepsilon}) & \text{as } t \to +\infty, \\ A_2 t^{\frac{1}{\lambda-1}} + \mathcal{O}\left(t^{\frac{1}{\lambda-1}+\varepsilon'}\right) & \text{as } t \to 0 \end{cases}
$$

$$
a_2(t) = \begin{cases} A_3 t^{n_1} + \mathcal{O}\left(t^{n_1+\varepsilon}\right) & \text{as } t \to +\infty, \\ A_4 t^{n_0} + \mathcal{O}\left(t^{n_0+\varepsilon'}\right) & \text{as } t \to 0, \end{cases}
$$

$$
n_1 = \left[\frac{1}{\lambda-1}\right]; \quad n_0 = \left[\frac{1}{\lambda-1}\right] + 1
$$

$\varepsilon > 0$, $\varepsilon' > 0$ arbitrarily small.

Similar problems (& methods) in different contexts:

1. Gas of Bosons.

The kinetic equation involved is the Uehling Uhlenbeck equation:

$$\text{(U-U)} \begin{cases} \dfrac{\partial f}{\partial t}(t, k_1) = \displaystyle\int\int_{D(k_1)} W(k_1, k_2, k_3, k_4)\, q(f) dk_3 dk_4 \\ q(f) = f_3 f_4 (1 + f_1)(1 + f_2) - f_1 f_2 (1 + f_3)(1 + f_4) \end{cases}$$

$$f_i \equiv f(t, k_i), \quad i = 1, 2, 3, 4.$$

$$D(k_1) \equiv \{(k_3, k_4) : k_3 > 0, k_4 > 0, k_3 + k_4 \geq k_1 > 0\}$$

$$W(k_1, k_2, k_3, k_4) = \frac{\min\left(\sqrt{k_1}, \sqrt{k_2}, \sqrt{k_3}, \sqrt{k_4}\right)}{\sqrt{k_1}}, \quad k_2 = k_3 + k_4 - k_1.$$

L. W. Nordheim (1928), E. A. Uehling & G. E. Uhlenbeck (1933).

Describes a dilute homogeneous isotropic gas of bosons (in polar coordinates).

2.-Oscilating thin plate under external low frequency random forcing.

$$\frac{\partial n}{\partial t}(t,p_1) = \sum_{s_1,s_2,s_3} \int_{D(S,p_1)} |J|^2 q(S,f) dp_2 dp_3 dp_4, \qquad (s_i = \pm 1)$$

$$q(S,f) = n_3 n_4 \left(n_2 + s_1 n_1\right) + n_1 n_2 \left(s_2 n_4 + s_3 n_3\right), \quad S = (s_1, s_2, s_3);$$

$D\left(S,p_1\right):\ \left(p_2,p_3,p_4\right),\ p_i \in \mathbb{R}^3$ such that:

$$p_1 + s_1 p_2 + s_2 p_3 + s_3 p_4 = 0; \quad |p_1|^2 + s_1 |p_2|^2 + s_2 |p_3|^2 + s_3 |p_4|^2 = 0;$$

$J \equiv J(p_1,p_2,p_3,p_4)$ homogeneous function of degree -2.

3. Weakly turbulent Langmuir waves (plasmas).

V. E. Zakharov, Sov. Phys. JETP, 1972.

Almost the same equation: $q(f) = f_3 f_4 (f_1 + f_2) - f_1 f_2 (f_3 + f_4)$.

4. Weakly turbulent Kelvin waves.

Kozik & Svistunov, PRL 2004.

In that case the equation is more complicated:

$$q\,(f) = f_4 f_5 f_6 (f_1 f_2 + f_2 f_3 + f_1 f_3) - f_1 f_2 f_3 (f_4 f_5 + f_5 f_6 + f_4 f_6),$$

$$D\,(k_1) \equiv \{(k_2, \cdots k_6) : k_i > 0, i = 2, \cdots, 6;\ k_1 + k_2 + k_3 - k_4 - k_5 - k_6 = 0;$$
$$\omega_1 + \omega_2 + \omega_3 - \omega_4 - \omega_5 - \omega_6 = 0\},$$

$$\omega_i = C k_i^2 \ln\left(\frac{1}{\xi k_i}\right)$$

and W is a homogeneous function of k_i, $i = 1, \cdots, 6$.

In all cases:

- singularity in finite time is “conjectured”.
- explicit singular particular solution with no finite mass.

Some similar examples are considered by V. E. Zakharov and coauthors in their Weak Turbulence Theory.

They formally study the linear stability of the singular solutions.

A. M. Balk, V. E. Zakharov: *Stability of Weak-Turbulence Kolmogorov Spectra*, A. M. S. Translations Series 2, Vol. 182, 1998, 1-81.

Our contribution (on Smoluchowski equation and the U-U equation):

- Precise estimates of the fundamental solutions of the linear problem.
- Application to the nonlinear problem.

Linearisation of the Coagulation equation.

$$\frac{\partial g}{\partial t} = -2\sqrt{2}x^{(\lambda-1)/2}g(x) + x^{(\lambda-1)/2}\left[\int_0^{1/2}\left((1-y)^{-3/2}-1\right)y^{\lambda/2}g(y)dy + \right.$$

$$\left. + \int_{1/2}^{1}\left(y^{\lambda/2}g(xy)-g(x)\right)(1-y)^{-3/2}dy - \int_{1/2}^{\infty}y^{\lambda/2}g(xy)dy\right].$$

We change variables:

$$G(t,X) = g(t,e^X) \qquad \widehat{G}(t,\xi) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-iX\xi}G(t,X)dX$$

$$\widetilde{G}(z,\xi) = \int_0^{\infty} e^{-tz}\widehat{G}(t,\xi)dt.$$

The Carleman equation.

The equation gives:

$$z\widetilde{G}(z,\xi) = \widetilde{G}\left(z, \xi + \frac{\lambda-1}{2}i\right)\Phi\left(\xi + \frac{\lambda-1}{2}i\right) + \frac{1}{\sqrt{2\pi}}.$$

$$\Phi\left(\xi\right) = \frac{-2\,\Gamma(\frac{1}{2})\,\Gamma\left(i\xi+\lambda+\frac{1}{2}\right)}{\Gamma\left(i\xi+\lambda\right)}; \qquad \Gamma(\cdot):\ \text{Gamma function.}$$

Starting from: $\dfrac{\partial u}{\partial t} = P(D)u,$ for some polynomial P

the same method gives:

$$z\widetilde{G}(z,\xi) = \widetilde{G}(z,\xi)\,P(\xi) + \frac{1}{\sqrt{2\pi}}.$$

Another change of variables in order to apply Wiener-Hopf method:

$$\zeta = T(\xi) \equiv e^{\frac{4\,\pi}{\lambda-1}(\xi-\beta_0 i)} \quad \beta_0 \in \mathbb{R} \;\; to\, be\, fixed$$

$$g(z,\zeta) = \widetilde{G}(z,\xi)$$

$$\widetilde{\varphi}(\zeta) = \Phi(\xi).$$

the pbm. is then: $zg(z,x+i0) = \varphi(x)\, g(z,x-i0) + \frac{1}{\sqrt{2\pi}}$ for all $x \in \mathbb{R}^+$

where, for any $x \in \mathbb{R}^+$:

$$g(z,x+i0) = \lim_{\varepsilon \longrightarrow 0} g(z,xe^{i\varepsilon}), \;\; g(z,x-i0) = \lim_{\varepsilon \longrightarrow 0} g(z,xe^{i(2\pi-\varepsilon)})$$

$$\varphi(x) = \lim_{\varepsilon \longrightarrow 0} \widetilde{\varphi}(xe^{i(2\pi-\varepsilon)}).$$

The explicit solution in ζ variable.

$$g(z,\zeta) = \frac{1}{(2\pi)^{3/2} i} \frac{\zeta^{\frac{1}{8}-\delta}}{z} \int_0^\infty \frac{M(z,\zeta)}{M(z,s+i0)} \frac{s^{-\frac{1}{8}+\delta}\, ds}{(s-\zeta)}$$

$\delta > 0$: arbitrarily small constant and

$$M(z,\zeta) = \exp\left[\frac{1}{2\pi i}\int_0^\infty (\ln(-\varphi(s)) - \ln(-z))\left(\frac{1}{s-\zeta} - \frac{1}{s+1}\right) ds\right]$$

Explicit function in terms of the function φ.

The explicit solution in ξ variable.

$$\widehat{G}(t,\xi) = \frac{\sqrt{2}e^{\frac{4\pi}{\lambda-1}\xi\left(\frac{1}{8}-\delta\right)}}{\sqrt{\pi}i\,(\lambda-1)} \int_{\mathcal{I}m\, y=\beta_0} H(\xi,y)\, t^{-\frac{2i(\xi-y)}{\lambda-1}} e^{-\frac{4\pi}{\lambda-1}y\left(\frac{1}{8}-\delta\right)}\,\Gamma\left(\frac{2i(\xi-y)}{\lambda-1}\right) dy$$

$$H(\xi,y) = \exp\left[\frac{2}{(\lambda-1)i}\int_{\mathcal{I}m\,\eta=\beta_0+\frac{\lambda-1}{2}-\varepsilon} \ln\left(-\Phi(\eta)\right)\Theta(\eta-\xi,\eta-y)d\eta\right].$$

$$\Theta(\sigma,\tau) = \frac{1}{1-e^{-\frac{4\pi}{\lambda-1}\sigma}} - \frac{1}{1-e^{-\frac{4\pi}{\lambda-1}\tau}}.$$

The zeros and poles of Φ play an important role.

Conditions on $\widetilde{G}$

The function G has to be such that

$$\int_0^{1/2} \left((1-y)^{-3/2} - 1\right) y^{\lambda/2} g(y) dy + \int_{1/2}^{\infty} y^{\lambda/2} g(xy) dy < \infty$$

$$\int_{1/2}^{1} \left(y^{\lambda/2} g(xy) - g(x)\right) (1-y)^{-3/2} dy < \infty.$$

Moreover, $\widehat{G}$ and $\widetilde{G}$ must have inverse transforms.

$\longrightarrow$ $\widetilde{G}$ has to be analytic in some strip S of the complex plane.

The strip S is determined by Φ:

- its behaviour as $|\xi| \to \infty$,
- its zeros and poles.

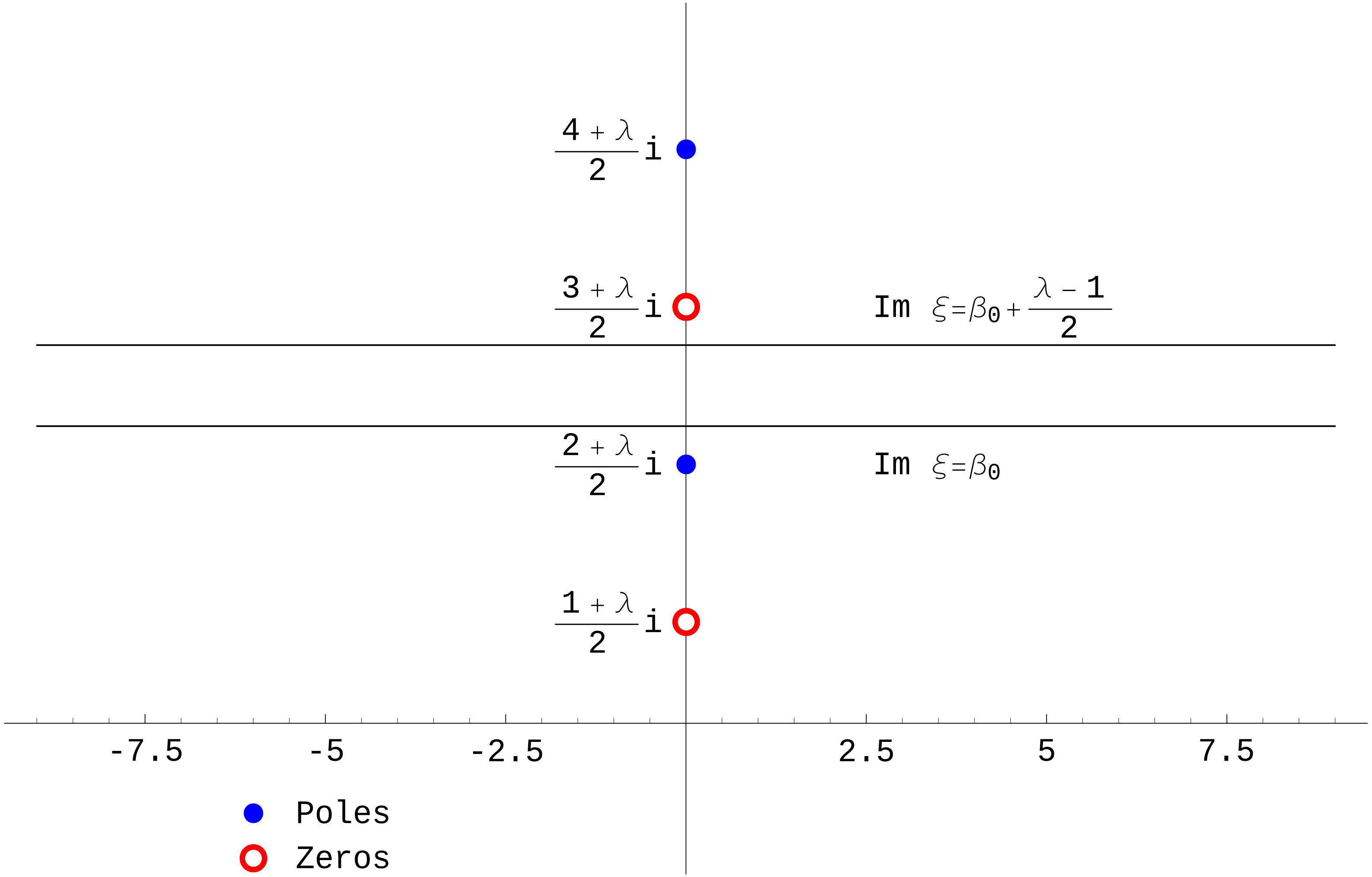


Figure 7: Zeros and poles of the function Φ.

The solution $G(t, X)$ is given by

$$G(t,X) = \frac{1}{\sqrt{2\pi}} \int_{-\infty+bi}^{\infty+bi} e^{iX\xi} \widetilde{G}(t,\xi) d\xi, \qquad \Im mb \in \left(\beta_0, \beta_0 + \frac{\lambda - 1}{2}\right).$$

The function $\widetilde{G}(t, \cdot)$ has singularities at zeros and poles of Φ.

Behaviour of G as $X \to +\infty$. Given by the residue of $\widetilde{G}(t, \cdot)$ at $\xi = (3 + \lambda)i/2$:

$$G(t,X) = A_1\, \sigma_1(t) e^{-\frac{3+\lambda}{2}X} + o(e^{-\frac{3+\lambda}{2}X}) \quad \text{as } X \to +\infty$$

Behaviour of G as $X \to -\infty$. Given by the residue of $\widetilde{G}(t, \cdot)$ at $\xi = 3i/2$:

$$G(t,X) = A_2\, \sigma_2(t) e^{-\frac{3}{2}X} + o(e^{-\frac{3}{2}X}) \quad \text{as } X \to -\infty$$

Regularising effects at $X = 0$:

Due to the behaviour of Φ as $\Re e\xi \to \pm\infty$

$$\Phi(\xi) \quad = \quad -\sqrt{2\pi}(1 \pm i)\sqrt{|\Re e\xi|} + \mathcal{O}\left(\frac{1}{|\xi|^{1/2}}\right), \quad \text{as } \Re e(\xi) \to \pm\infty.$$

The function $\widehat{G}(t,\xi)$ behaves like:

$$\widehat{G}(t,\xi) \sim e^{-(A(\lambda)+iB(\lambda))\sqrt{|\xi|}} \quad \text{as } |\Re e\xi| \to \pm\infty$$

where $A(\lambda) > 0$.

Another example:

For the Uehling Uhlenbeck equation:

$$\lim_{|\Re e \xi| \to +\infty} \Phi(\xi) = -a, \quad a > 0 \text{ constant}$$

$\Longrightarrow$ No regularising effect of the linear equation:

$$g(t, x, 1) = e^{-a\,t} \delta_1(x) + \cdots$$

Open problems

1.- Global existence or not of these solutions ? Does

$$f(t,x) \sim a(t)\, x^{-(3+\lambda)/2}, \quad x >> 1 \quad \text{for all } t > 0\,?$$

2.- Long time behaviour of the solutions ?

3.- If $f_0(x) \sim x^{-\alpha}$ with $\alpha \neq -(3+\lambda)/2$? (work in progress)

4.- Of course:
if f is a solution of the Smoluchowski equation for which $0 < T* < \infty$ does the solution behaves like $x^{-(3+\lambda)/2}$ as $x >> 1$ when $t > T^*$?

Further questions

1.-Validity of the Smoluchowski type model after gelation.

May depend on the behaviour of the kernel $W(x,y)$.
W. Wagner, Electronic Journal of Probability '06.

2.-Non homogeneous equations.

$f(t,x,r)$ density of particles of size x at the point $r \in \mathbb{R}^3$. Brownian motion of particles:

$$\frac{\partial f}{\partial t} - d(k)\,\Delta_r f = Q(f); \qquad d(k):\ \text{diffusion coefficient.}$$