Optimal Perfectly Matched Layers for time-harmonic scattering problems

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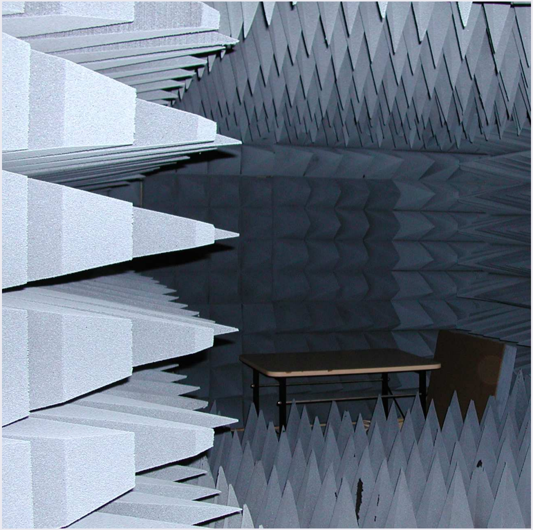
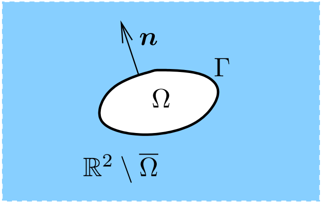
Outline

An anechoic room

Perfectly Matched Layer = Dissipative media without reflection



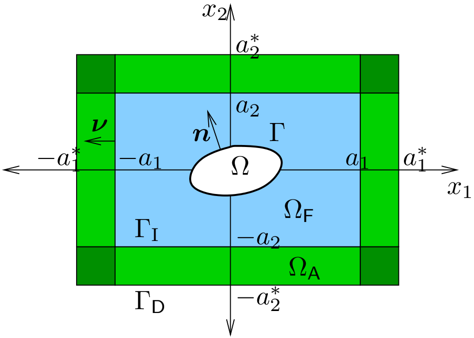


Acoustic scattering problem

$$\begin{cases} \Delta p + k^2 p = 0 & \text{in } \mathbb{R}^2 \setminus \overline{\Omega}, \\ \dfrac{\partial p}{\partial \boldsymbol{n}} = g & \text{on } \Gamma, \\ \displaystyle\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial p}{\partial r} - ikp\right) = 0. \end{cases}$$

Computational solution

- **Difficulty:** solving the problem in an unbounded domain.
- **Solution:** surround the physical domain of interest with PML.



Associated to each coordinate, we define

$$\gamma_j(x_j) := 1 + \frac{i}{\omega}\sigma_j(x_j), \ \ j = 1, 2,$$

where

$$\sigma_j : \ (-a_j^*, a_j^*) \longrightarrow \mathbb{R}, \ \ j = 1, 2,$$

are the **absorbing functions**.

2D PML model in cartesian coordinates (Berenger, 1994)

$$\frac{1}{\gamma_1(x_1)}\frac{\partial}{\partial x_1}\left[\frac{1}{\gamma_1(x_1)}\frac{\partial p}{\partial x_1}\right] + \frac{1}{\gamma_2(x_2)}\frac{\partial}{\partial x_2}\left[\frac{1}{\gamma_2(x_2)}\frac{\partial p}{\partial x_2}\right] + k^2 p = 0.$$

- $\Omega_{\rm F} = (-a_1, a_1) \times (-a_2, a_2) \setminus \overline{\Omega}$ and $p_{\rm F} = p|_{\Omega_{\rm F}}$,
- $\Omega_{\rm A} = (-a_1^*, a_1^*) \times (-a_2^*, a_2^*) \setminus \overline{\Omega}_{\rm F}$ and $p_{\rm A} = p|_{\Omega_{\rm A}}$.

Statement of the coupled fluid/PML problem

$$-k^2 p_{\rm F} - \operatorname{div}(\mathbf{grad}\, p_{\rm F}) = 0 \text{ in } \Omega_{\rm F},$$

$$-k^2 \gamma_1\gamma_2 p_{\rm A} - \frac{\partial}{\partial x_1}\left[\frac{\gamma_2}{\gamma_1}\frac{\partial p_{\rm A}}{\partial x_1}\right] - \frac{\partial}{\partial x_2}\left[\frac{\gamma_1}{\gamma_2}\frac{\partial p_{\rm A}}{\partial x_2}\right] = 0 \text{ in } \Omega_{\rm A},$$

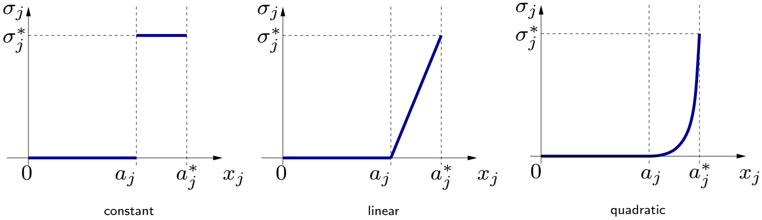
$$\frac{\partial p_{\rm F}}{\partial \boldsymbol{n}} = g \text{ on } \Gamma,$$

$$p_{\rm F} = p_{\rm A} \text{ on } \Gamma_{\rm I},$$

$$\frac{\nu_1}{\gamma_1}\frac{\partial p_{\rm A}}{\partial x_1} + \frac{\nu_2}{\gamma_2}\frac{\partial p_{\rm A}}{\partial x_2} = \frac{\partial p_{\rm F}}{\partial \boldsymbol{\nu}} \text{ on } \Gamma_{\rm I},$$

$$p_{\rm A} = 0 \text{ on } \Gamma_{\rm D}.$$

- Classical choices of absorbing function σ_j:



Properties of absorbing function σ_j

$$\begin{cases} \sigma_j : (0, a_j^*) \longrightarrow \mathbb{R}, \quad \text{not decreasing}, \\ \sigma_j(x_j) = 0, \quad 0 < x_j < a_j, \\ \sigma_j \in \mathcal{C}\big((a_j, a_j^*), \mathbb{R}\big), \\ \sigma_j(x_j) > 0, \quad a_j < x_j < a_j^*. \end{cases}$$

Behavior of the theoretical error (Collino, Monk 1996)

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- $\displaystyle\int_{a_j}^{a_j^*} \sigma_j(s)ds$ is **small**: we recover the solution of a Helmholtz problem in a bounded domain.

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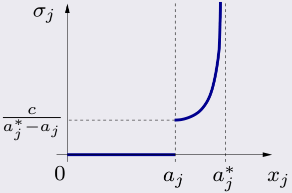
Consequence

The optimal σ_j^* depends on the data and on the **mesh**.

Our choice for σ

We propose a non-integrable function σ

$$\sigma_j(x_j) = \frac{c}{a_j^* - x_j},\ a_j < x_j < a_j^* \ \Rightarrow \int_{a_j}^{a_j^*} \sigma(s)ds = +\infty.$$



Numerical advantages

- We recover the exact solution.
- Robustness of the choice of σ_j (mesh independent).
- Better numerical performance than the classical ones.

- Triangular mesh in the fluid domain Ω_{F},
- Rectangular mesh in the PML domain Ω_{A}.

We define the discrete spaces:

$$\mathrm{V}_h(\Omega_{\mathsf{F}}) = \{q \in \mathcal{C}(\Omega_{\mathsf{F}}) : q|_K \in P_1(K), \forall K \in \mathcal{T}_h\},$$
$$\mathrm{W}_h(\Omega_{\mathsf{A}}) = \{q \in \mathcal{C}(\Omega_{\mathsf{A}}) : q|_K \in Q_1(K), \forall K \in \mathcal{T}_h,\ q = 0 \text{ on } \Gamma_{\mathsf{D}}\}.$$

Discrete fluid/PML problem

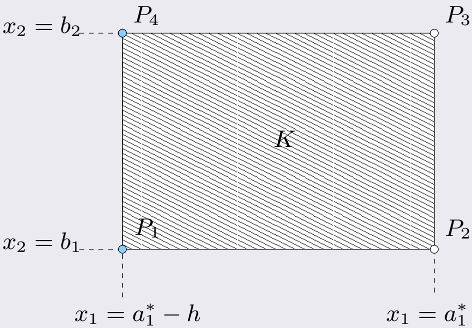
Find $(p_{\mathrm{F}}^h, p_{\mathrm{A}}^h) \in \mathrm{V}_h(\Omega_{\mathrm{F}}) \times \mathrm{W}_h(\Omega_{\mathrm{A}})$ such that

$$-\int_{\Omega_{\mathrm{F}}} k^2 p_{\mathrm{F}}^h \bar{q}_{\mathrm{F}}^h \, dV + \int_{\Omega_{\mathrm{F}}} \mathbf{grad}\, p_{\mathrm{F}}^h \cdot \mathbf{grad}\, \bar{q}_{\mathrm{F}}^h \, dV - \int_{\Omega_{\mathrm{A}}} k^2 \gamma_1 \gamma_2 p_{\mathrm{A}}^h \bar{q}_{\mathrm{A}}^h \, dV$$
$$+ \int_{\Omega_{\mathrm{A}}} \frac{\gamma_2}{\gamma_1} \frac{\partial p_{\mathrm{A}}^h}{\partial x_1} \frac{\partial \bar{q}_{\mathrm{A}}^h}{\partial x_1} \, dV + \int_{\Omega_{\mathrm{A}}} \frac{\gamma_1}{\gamma_2} \frac{\partial p_{\mathrm{A}}^h}{\partial x_2} \frac{\partial \bar{q}_{\mathrm{A}}^h}{\partial x_2} \, dV = \int_{\Gamma_{\mathrm{I}}} g \frac{\partial \bar{q}_{\mathrm{F}}^h}{\partial \boldsymbol{n}} \, dS,$$

for all $(q_{\mathrm{F}}, q_{\mathrm{A}}) \in \mathrm{V}_h(\Omega_{\mathrm{F}}) \times \mathrm{W}_h(\Omega_{\mathrm{A}})$.

Element matrices

Every element matrix is bounded with $\sigma_j(x_j) = \dfrac{c}{a_j^* - x_j}$.



- $\phi_j \in Q_1(K)$: canonical basis function associated to P_j,
- ϕ_2 and ϕ_3 do not belong to $\mathrm{W}_h(\Omega_\mathrm{A})$.

Element matrices

If $|b_2| < a_2^*$, $|b_1| < a_2^*$,

$$\begin{cases} \phi_j = O(|x_1 - a_1^*|), \\ \dfrac{\partial \phi_j}{\partial x_1} = O(1), \ \dfrac{\partial \phi_j}{\partial x_2} = O(|x_1 - a_1^*|), \\ \gamma_1 = O\left(\dfrac{1}{|x_1 - a_1^*|}\right), \ \gamma_2 = O(1), \end{cases}$$

for $j = 1, 4$ and when x_1 tends to a_1^*.

The three integrands are bounded for $i, j = 1, 4$:

$$\int_K k^2 \gamma_1 \gamma_2 \phi_i \bar{\phi}_j \, dV, \quad \int_K \frac{\gamma_2}{\gamma_1} \frac{\partial \phi_i}{\partial x_1} \frac{\partial \bar{\phi}_j}{\partial x_1} \, dV, \quad \int_K \frac{\gamma_1}{\gamma_2} \frac{\partial \phi_i}{\partial x_2} \frac{\partial \bar{\phi}_j}{\partial x_2} \, dV.$$

Spherical waves generated by a monopole

Neumann data on the unit circumference,

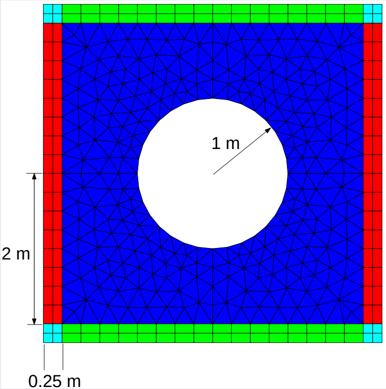
$$\frac{\partial p}{\partial \boldsymbol{n}}(\boldsymbol{x}) = \frac{i}{4}\mathrm{H}_1^{(1)}\left(\frac{\omega}{c}\left|\boldsymbol{x}-\boldsymbol{a}\right|\right),$$

being $\boldsymbol{a} = (0.5, 0)$, $\omega = 750\,\mathrm{rad/s}$ and $c = 340\,\mathrm{m/s}$.

Absorbing functions:

$$\sigma_1(x_1) := \frac{c}{a_1^* - |x_1|},$$
$$\sigma_2(x_2) := \frac{c}{a_2^* - |x_2|}.$$

Mesh



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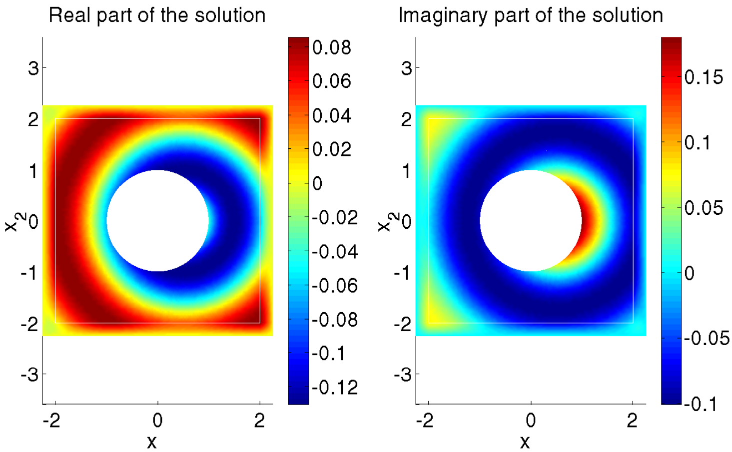
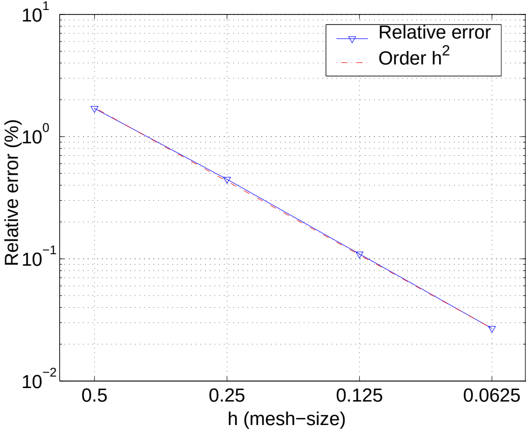


Error curve

Order 2 of convergence for L^2-relative error in the fluid domain Ω_F.



Comparison between a quadratic σ and our proposal of unbounded function.

		Unbounded		Quadratic		
ω(rad/s)	d.o.f.	Error(%)	κ	σ^*	Error(%)	κ
250	464	0.763	6.7e+02	$22.28\,c$	11.644	4.7e+02
	1720	0.131	5.1e+03	$29.57\,c$	3.675	5.0e+03
	6768	0.029	4.1e+04	$38.37\,c$	1.134	4.6e+04
750	464	1.700	1.1e+02	$27.67\,c$	7.602	1.1e+02
	1720	0.447	7.0e+02	$35.52\,c$	2.291	9.4e+02
	6768	0.109	5.6e+03	$43.49\,c$	0.698	8.2e+03
1250	464	6.958	2.7e+02	$27.89\,c$	11.620	2.9e+02
	1720	1.946	1.1e+03	$36.94\,c$	3.336	1.7e+03
	6768	0.430	9.7e+03	$45.70\,c$	0.919	1.5e+03

- **Remark:** the optimal σ^* depends on the mesh and the data of the problem.

Comparison using exact and numerical integration with an unbounded function σ.

ω(rad/s)	d.o.f.	Gauss-Legendre 4 nodes	Gauss-Legendre 9 nodes	Exact Integration
250	464	0.763689	0.770405	0.763485
	1720	0.130572	0.130413	0.130580
	6768	0.028858	0.028755	0.028860
750	464	1.699869	1.699611	1.699889
	1720	0.446922	0.446910	0.446922
	6768	0.109444	0.109467	0.109443
1250	464	6.957597	6.958152	6.958012
	1720	1.946417	1.946320	1.946313
	6768	0.429920	0.429913	0.429912

- **Conclusion:** the quadrature rules do not perturb the accuracy of PML.

Waves generated by a curved tuning fork

Geometry:

- Fork thickness 0.25 m
- Interior radius 1 m

Neumann data:

$$\frac{\partial p}{\partial \boldsymbol{n}}(\boldsymbol{x}) = \frac{\partial}{\partial \boldsymbol{n}}\left(e^{i\frac{\omega}{c}x_1}\right)$$

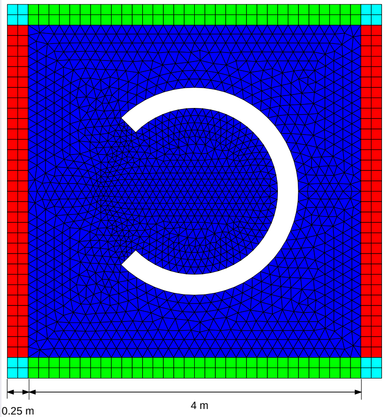
with $\omega = 2\pi c\,\mathrm{rad/s}$, $c = 340\,\mathrm{m/s}$.

Absorbing functions:

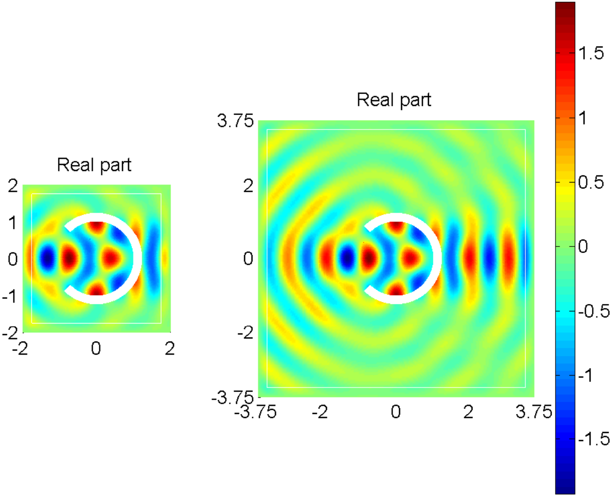
$$\sigma_1(x_1) := \frac{c}{a_1^* - |x_1|},$$

$$\sigma_2(x_2) := \frac{c}{a_2^* - |x_2|}.$$

Mesh



Relative difference in L^2-norm: 0.233%



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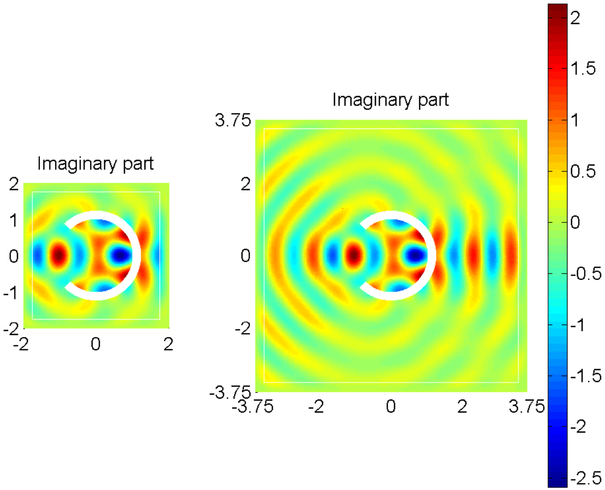
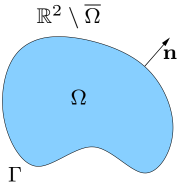


Conclusions

- We propose a new construction of the PML using a singular absorbing function σ.
- The method can be implemented in practice since the integrals of the discrete variational problem are bounded for a standard choice of FEM spaces.
- We have implemented this method in a computer code obtaining a better performance than the classical PML techniques.

- A. Bermúdez, L. Hervella-Nieto, A. Prieto, R. Rodríguez. *Optimal perfectly matched layers with unbounded decay functions for time-harmonic acoustic scattering problems.* Journal of Computational Physics 223 (2007) 469-488.

Statement of the scattering problem

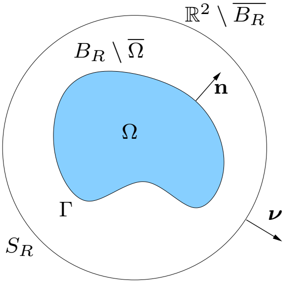


Given $f \in \mathrm{H}^{\frac{1}{2}}(\Gamma)$, find $p \in \mathrm{H}^1_{\mathrm{loc}}(\mathbb{R}^2 \setminus \overline{\Omega})$ such that

$$\begin{cases} \Delta p + k^2 p = 0 & \text{in } \mathbb{R}^2 \setminus \overline{\Omega}, \\ p = f & \text{on } \Gamma, \\ \lim_{r\to\infty} \sqrt{r}\left(\frac{\partial p}{\partial r} - ikp\right) = 0. \end{cases}$$

- **Difficulty:** solving the problem in an unbounded domain.
- **Solution:** use the Dirichlet-to-Neumann (DtN) operator.

Scattering problem with the DtN operator



Given $f \in \mathrm{H}^{\frac{1}{2}}(\Gamma)$, find $p \in \mathrm{H}^1(B_R \setminus \overline{\Omega})$ such that

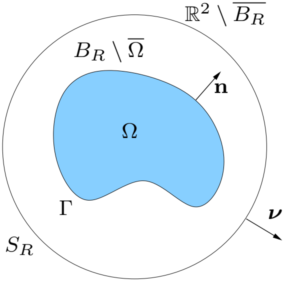
$$\begin{cases} \Delta p + k^2 p = 0 & \text{in } B_R \setminus \overline{\Omega}, \\ p = f & \text{on } \Gamma, \\ \dfrac{\partial p}{\partial \boldsymbol{\nu}} = Gp & \text{on } S_R. \end{cases}$$

where

G is the **DtN operator** given by a problem stated in $\mathbb{R}^2 \setminus \overline{B_R}$. If $g \in \mathrm{H}^{\frac{1}{2}}(S_R)$

$$Gg = \sum_{n=-\infty}^{\infty} g_n k \frac{[H_n^{(1)}]'(kR)}{H_n^{(1)}(kR)} \mathrm{e}^{in\theta}, \; g_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(\theta)\, \mathrm{e}^{-in\theta}\, d\theta.$$

DtN operator

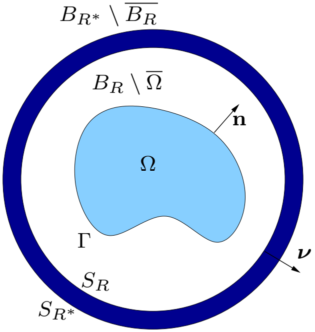


$$G : \quad \mathrm{H}^{\frac{1}{2}}(S_R) \longrightarrow \mathrm{H}^{-\frac{1}{2}}(S_R)$$
$$g \longmapsto \left.\frac{\partial \tilde{p}}{\partial \boldsymbol{\nu}}\right|_{S_R}$$

where $\tilde{p} \in \mathrm{H}^1_{\mathrm{loc}}(\mathbb{R}^2 \setminus \overline{B_R})$ is the unique solution of

$$\begin{cases} \Delta \tilde{p} + k^2 \tilde{p} = 0 & \text{in } \mathbb{R}^2 \setminus \overline{B_R}, \\ \tilde{p} = g & \text{on } S_R, \\ \lim_{r\to\infty} \sqrt{r}\left(\frac{\partial \tilde{p}}{\partial r} - ik\tilde{p}\right) = 0. \end{cases}$$

- **Drawback:** this problem is stated in an unbounded domain.
- **Solution:** use the Perfectly Matched Layer (PML) technique.



$$\begin{array}{rccc} \hat{G}: & \mathrm{H}^{\frac{1}{2}}(S_R) & \longrightarrow & \mathrm{H}^{-\frac{1}{2}}(S_R) \\ & g & \longmapsto & \dfrac{1}{\gamma(R)} \dfrac{\partial \hat{p}}{\partial \boldsymbol{\nu}}\Big|_{S_R} \end{array}$$

where $\hat{p}$ is the solution of

$$\begin{cases} \dfrac{\partial}{\partial r}\left[\dfrac{\hat{\gamma}}{\gamma}\dfrac{\partial \hat{p}}{\partial r}\right] + \dfrac{\gamma}{\hat{\gamma}}\dfrac{\partial^2 \hat{p}}{\partial \theta^2} + k^2 \gamma \hat{\gamma} \hat{p} = 0, \\ \hat{p} = g \quad \text{on } S_R, \end{cases}$$

where $\hat{r}(r) = r\hat{\gamma}(r) = r + \dfrac{i}{\omega}\displaystyle\int_R^r \sigma(s)\,ds$ and $\gamma(r) = \dfrac{d\hat{r}}{dr}(r)$.

Questions

- Choice of absorbing function σ.
- Relation between G and $\hat{G}$.

Let us define the weighted Sobolev space

$$\mathrm{V} := \left\{ v \in \mathrm{H}^1_{loc}(\mathrm{D}) : \|q\|^2_{\mathrm{V}} := \int_R^{R^*} \int_{-\pi}^{\pi} \left| \frac{\hat{\gamma}(r) r}{\gamma(r)} \right| \left| \frac{\partial q}{\partial r} \right|^2 \, d\theta \, dr \right.$$
$$\left. + \int_R^{R^*} \int_{-\pi}^{\pi} \left| \frac{\gamma(r)}{\hat{\gamma}(r) r} \right| \left| \frac{\partial q}{\partial \theta} \right|^2 \, d\theta \, dr + \int_R^{R^*} \int_{-\pi}^{\pi} |\hat{\gamma}(r)\gamma(r) r| \, |q|^2 \, d\theta \, dr < +\infty \right\}.$$

Existence and uniqueness of smooth solutions

If $g \in \mathrm{H}^s(S_R)$ with $s > 3/2$, then there exists an unique solution $\hat{p} \in \mathrm{V} \cap \mathcal{C}^1(B_{R^\star} \setminus B_R) \cap \mathcal{C}^2(B_{R^\star} \setminus \overline{B_R})$ satisfying

$$\begin{cases} \dfrac{\partial}{\partial r}\left[\dfrac{\hat{\gamma}}{\gamma} \dfrac{\partial \hat{p}}{\partial r} \right] + \dfrac{\gamma}{\hat{\gamma}} \dfrac{\partial^2 \hat{p}}{\partial \theta^2} + k^2 \gamma \hat{\gamma} \hat{p} = 0 & \text{in } B_{R^\star} \setminus \overline{B_R}, \\ \hat{p} = g & \text{on } S_R. \end{cases}$$

Series representation of PML solution

If $g \in \mathrm{H}^s(S_R)$ with $s > 3/2$, then the unique solution of PML problem $\hat{p} \in \mathrm{V} \cap \mathcal{C}^1(B_{R^\star} \setminus B_R) \cap \mathcal{C}^2(B_{R^\star} \setminus \overline{B_R})$ satisfying

$$\begin{cases} \dfrac{\partial}{\partial r}\left[\dfrac{\hat{\gamma}(r)}{\gamma(r)}\dfrac{\partial \hat{p}}{\partial r}\right] + \dfrac{\gamma(r)}{\hat{\gamma}(r)}\dfrac{\partial^2 \hat{p}}{\partial \theta^2} + k^2\gamma(r)\hat{\gamma}(r)\hat{p} = 0 & \text{in } B_{R^\star} \setminus \overline{B_R}, \\ \hat{p} = g \quad \text{on } S_R, \end{cases}$$

is represented by the series

$$\hat{p}(\boldsymbol{x}) = \sum_{n=-\infty}^{\infty} \frac{g_n}{H_n^{(1)}(kR)} H_n^{(1)}(k\hat{r}_{\boldsymbol{x}})\, \mathrm{e}^{in\theta_{\boldsymbol{x}}},$$

where g_n is the n-th Fourier coefficient of g.

Proof

Sketch of the proof:

(i) The outgoing fundamental solution of the PML problem is

$$\Phi(\boldsymbol{x},\boldsymbol{y}) = \frac{i}{4} H_0^{(1)}(k\mathrm{d}(\boldsymbol{x},\boldsymbol{y})),$$

where $\mathrm{d}(\boldsymbol{x},\boldsymbol{y}) = \sqrt{\hat{r}_{\boldsymbol{x}}^2 + \hat{r}_{\boldsymbol{y}}^2 - 2\hat{r}_{\boldsymbol{x}}\hat{r}_{\boldsymbol{y}}\cos(\theta_{\boldsymbol{y}} - \theta_{\boldsymbol{x}})}$ is a complex 'distance' which plays the role of $|\boldsymbol{x}-\boldsymbol{y}|$.

(ii) Two *"trace results"*: if $\hat{p} \in \mathrm{V} \cap \mathcal{C}^1(B_{R^\star} \setminus B_R) \cap \mathcal{C}^2(B_{R^\star} \setminus \overline{B_R})$ is a solution of PML equation and $q \in \mathrm{V}$, then

$$\lim_{\tilde{R}\to R^*} \int_{S_{\tilde{R}}} |\hat{\gamma}|\, |q|^2 \, dS = 0, \qquad \lim_{\tilde{R}\to R^*} \int_{S_{\tilde{R}}} \frac{\hat{\gamma}}{\gamma}\frac{\partial \hat{p}}{\partial r} q \, dS = 0.$$

Proof

Sketch of the proof:

(iii) Using (i) and (ii), we obtain the Green's formula for the PML problem

$$\hat{p}(\boldsymbol{x}) = \frac{1}{\gamma(R)} \int_{S_R} \left\{ \frac{\partial \Phi(\boldsymbol{x}, \boldsymbol{y})}{\partial r_{\boldsymbol{y}}} \hat{p}(\boldsymbol{y}) - \frac{\partial \hat{p}}{\partial r_{\boldsymbol{y}}}(\boldsymbol{y}) \Phi(\boldsymbol{x}, \boldsymbol{y}) \right\} dS_{\boldsymbol{y}}.$$

(iv) Using (iii), the addition theorem shows a series representation of Φ:

$$\Phi(\boldsymbol{x}, \boldsymbol{y}) = \frac{i}{4} \sum_{n=-\infty}^{\infty} H_n^{(1)}\left(k \hat{r}_{\boldsymbol{x}}\right) J_n\left(k \hat{r}_{\boldsymbol{y}}\right) \mathrm{e}^{in(\theta_{\boldsymbol{x}} - \theta_{\boldsymbol{y}})}.$$

(v) We conclude the result using (i), (iii) and (iv).

Coupled fluid/PML problem

For $f \in \mathrm{H}^{\frac{1}{2}}(\Gamma)$, there exists a unique solution $(p, \hat{p}) \in \mathrm{H}^1(B_R \setminus \overline{\Omega}) \times \mathrm{V}$ of the following problem:

$$\begin{aligned} -\Delta p - k^2 p &= 0 && \text{in } B_R \setminus \overline{\Omega}, \\ -\operatorname{div}(\boldsymbol{A}\, \mathbf{grad}\, \hat{p}) - \gamma\hat{\gamma}k^2\hat{p} &= 0 && \text{in } B_{R^\star} \setminus \overline{B_R}, \\ p &= f && \text{on } \Gamma, \\ \frac{\partial p}{\partial \boldsymbol{\nu}} &= \boldsymbol{A}\, \mathbf{grad}\, \hat{p} \cdot \boldsymbol{\nu} && \text{in } \mathrm{H}^{-\frac{1}{2}}(S_R), \\ p &= \hat{p} && \text{on } S_R, \end{aligned}$$

where $\boldsymbol{A} = (\hat{\gamma}/\gamma)\boldsymbol{e}_r \otimes \boldsymbol{e}_r + (\gamma/\hat{\gamma})\boldsymbol{e}_\theta \otimes \boldsymbol{e}_\theta$.

Moreover, p coincides with the solution of the scattering problem.

Sketch of the proof

- **Existence:**

$$p = \text{Restriction to } B_R \setminus \overline{\Omega} \text{ of the classical scattering solution,}$$

$$\hat{p} = \sum_{n=-\infty}^{\infty} \frac{(p|_{S_R})_n}{H_n^{(1)}(kR)} H_n^{(1)}(k\hat{r}_{\boldsymbol{x}}) \, \mathrm{e}^{in\theta_{\boldsymbol{x}}}.$$

- **Uniqueness:**
  Using local regularity results (McLean 2000): the unique solution is the **smooth** one.

- **Recovering the classical scattering solution:**
  The classical DtN (denoted by G) is equal to the PML-DtN operator $\hat{G}$:

$$\hat{G}g = \frac{1}{\gamma}\frac{\partial \hat{p}}{\partial r} = \frac{\partial p}{\partial r} = Gg.$$

Conclusions

- We propose a new construction of the radial PML using a singular absorbing function σ.
- With this choice, we prove that the exact solution of the scattering problem is recovered by the PML technique.

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