# STRICTLY HYPERBOLIC OPERATORS WITH NON-REGULAR COEFFICIENTS

BENASQUE, AUGUST 2007

JOINT WORKS

E. DE GIORGI - S. SPAGNOLO - F.C. ('79)

N. LERNER - F.C. ('95)

G. MÉTIVIER - F.C. ('07)

D. DEL SANTO - M. REISSIG - F.C. ('03)

D. DEL SANTO - G. MÉTIVIER - F.C. (in progress)

Let us consider Cauchy problem

$$\text{CP} \begin{cases} Lu := \partial_t^2 - \sum_{i,j=1}^{n} \partial_{x_i}(a_{ij}(t,x)\,\partial_{x_j} u) = 0 \\ u(0,x) = u_0 \qquad \partial_t u(0,x) = u_1 \end{cases}$$

In general

$$Lu = f \quad , \text{but } f \equiv 0 \text{ here}$$

$$\lambda |\xi|^2 \leq \sum a_{ij}\,\xi_i \xi_j \leq \Lambda |\xi|^2 \qquad 0 < \lambda \leq \Lambda$$

$$a_{ij} = a_{ji}$$

Classical assumptions

$t \mapsto a_{ij}(t,x)$ Lipschitz, uniformly in $x$

$x \to a_{ij}(t,x) \in C^\infty$

C.P. well posed in $C^\infty$, in $H^s$:

$u_0 \in H^{s+1}$, $u_1 \in H^s$ $\exists_1\, u \in C^1([0,T], H^s) \cap$

$\cap\, C([0,T], H^{s+1})$

C. DG. S. '79

$$a_{ij} = a_{ij}(t) \in \text{LogLip}$$

LL $\quad |a_{ij}(t+\tau) - a_{ij}(t)| \leq C |\tau| |\log|\tau|| \qquad |\tau| \leq 1/2$

Under assumption LL, CP well posed in $C^\infty$ ②

Th $\exists\ \alpha = \alpha(\Lambda, \lambda)$ : $\forall s$, $\forall t \in [0,T]$

$$\|u(t,\cdot)\|_{H^{s+1-\alpha t}} + \|\partial_t u\|_{H^{s-\alpha t}} \leq$$

$$\leq C_0 \left( \|u_0\|_{H^{s+1}} + \|u_1\|_s \right)$$

loss of derivatives

LL is optimal

CE $\exists\ a$, $\frac{1}{2} \leq a \leq \frac{3}{2}$, $a \in \bigcap_{\alpha<1} C^\alpha([0,T])$

$a \in C^\infty(]0,T])$

$\exists\ u_0, u_1$ : CP no solution in $C([0,T], \mathcal{D}')$

More precisely (C-LERNER)

Th $\forall \omega$ $\quad \omega(\tau) \xrightarrow[\tau\to 0^+]{} +\infty$ $\quad \left( \omega(\tau)\tau^\alpha \xrightarrow[\tau\to 0^+]{} 0 \quad \forall \alpha>0 \right)$

$\exists\ a$, $\frac{1}{2} \leq a \leq \frac{3}{2}$

$$|a(t+\tau) - a(t)| \leq C\, |\tau|\, |\lg|\tau||\ \omega(\tau)$$

$\exists\ u_0, u_1 \in C^\infty \cap H^\infty$ $\quad$ CP no solution in $C([0,T], \mathcal{D}')$

Idea of the proof of W. P.

$$\begin{cases} u_{tt} - a(t) u_{xx} = 0 \\ u_0 , \quad u_1 \end{cases} \qquad n = 1 \text{ for semplicity}$$

Fourier tranform in $x$

$$v(t,\xi) = \mathcal{F}_x u(t,\xi) = \hat{u}(t,\xi)$$

$$* \qquad v'' + a(t)\xi^2 v = 0$$

$$\rho(\tau) \geq 0 \quad \int \rho = 1 \quad \operatorname{supp} \rho \subset [-1,1] \qquad \rho_\varepsilon(\tau) = \frac{1}{\varepsilon}\rho\left(\frac{x}{\varepsilon}\right)$$

$$a_\varepsilon = a * \rho_\varepsilon$$

$$\text{I} \qquad \int_0^T |a - a_\varepsilon| \, dt \leq C \varepsilon \log \frac{1}{\varepsilon}$$

$$\text{II} \qquad \int_0^T |a'_\varepsilon| \leq C \log \frac{1}{\varepsilon}$$

Energy depending on $\varepsilon$:

$$E_\varepsilon(t,\xi) = |v'|^2 + a_\varepsilon(t)\xi^2|v|^2$$

$$E'_\varepsilon(t,\xi) = 2\,\mathrm{Re}(\bar{v}', v'') + a'_\varepsilon \xi^2 |v|^2 + a_\varepsilon \xi^2 2\,\mathrm{Re}(\bar{v}', v)$$

from $*$ $\quad E_\varepsilon(t,\xi) \leq E_\varepsilon(0,\xi) \exp\left[ \int \frac{|a'_\varepsilon|}{a_\varepsilon} + |\xi| \int \frac{|a - a_\varepsilon|}{\sqrt{a_\varepsilon}} \right]$

$$E_\varepsilon(t, \xi) \leq E_\varepsilon(0, \xi)\, e^{c\left[\lg \frac{1}{\varepsilon} + |\xi|\, \varepsilon \lg \frac{1}{\varepsilon}\right]}$$

now we choose $\varepsilon = \frac{1}{|\xi|}$ $\quad (|\xi| \geq 1)$

$$E_\varepsilon(t, \xi) \leq E_\varepsilon(0, \xi)\, e^{cT \lg |\xi|} \Rightarrow$$

$\Rightarrow$ W. p in $C^\infty$, loss of derivatives encreasing with $t$.

C. L (95) depending on $x$

$$\begin{cases} Lu = u_{tt} - \sum \partial_{x_i}(a_{ij}(t,x)\, \partial_{x_j} u) = 0 \\ u_0 \in H^{s+1},\ u_1 \in H^s \end{cases}$$

Assumption: Hyperbolicity, and

$$a \in \mathrm{Log\,Lip}(\mathbb{R}^{n+1})$$

Th. $\exists T^* \ (\leq T)$, $\exists C$, $\exists \beta > 0$: $\forall t \leq T^*$

$$\sup_{0 \leq s \leq t} \|u_t(s)\|_{H^{-\beta(s)}(\mathbb{R}^n)} + \|u(s)\|_{H^{1-\beta s}} \leq$$

(*)

$$\leq C\left[\|u_0\|_{H^1} + \|u_1\|_{H^0} + \int_0^t \|Lu\|_{H^{-\beta s}}\, ds\right]$$

From $*$ C.P. w.p. for $t \le T^*$ ($T^*$ depending on $\|a_{ij}\|_{LL}$)

Now $a_{ij}(t,x) \in LL$ in $t$, $C^\infty(\mathbb{R}^n_x)$

$$|a_{ij}(t,x) - a_{ij}(t+\tau,x)| \le C|\tau|\,|\log|\tau||$$

Th. $\exists\, \beta > 0$ $(\beta(\lambda, \Lambda, \|a\|_{LL}))$, $\exists\, T^*$

$u$ solution of C.P.: $\forall\, m \ge 0$ $\exists\, C_m$

$$\sup_{0 \le t \le T^*}\left[\|u(t)\|_{H^{m+1-\beta t}} + \|\partial_t u(t)\|_{H^{m-\beta t}}\right] \le$$

$$\le \left[\|u_0\|_{H^{m+1}} + \|u_1\|_{H^m} + \int_0^{T^*}\|Lu(s)\|_{H^{m-\beta s}}\right]$$

$T^*$ independent of $m$.

To prove these theorems we use "approximate energy" and paradifferential calculus of Bony.

Let $\varphi_0 \in C_0^\infty(\mathbb{R}^n)$ $\quad 0 \leq \varphi(\xi) \leq 1$

$$\varphi_0(\xi) = 1 \quad \text{if } |\xi| \leq 1$$

$$\varphi_0(\xi) = 0 \quad \text{if } |\xi| \geq 2$$

$\varphi_0$ radial decreasing of $|\xi|$

$$\varphi(\xi) = \varphi_0(\xi) - \varphi_0(2\xi)$$

so that

$$\varphi(\xi) = 0 \quad \begin{cases} |\xi| \leq 1/2 \\ |\xi| \geq 2 \end{cases}$$

$$\varphi_\nu(\xi) = \varphi\left(\frac{\xi}{2^\nu}\right)$$

$\varphi_\nu(D_x) = \varphi\left(\frac{D_x}{2^\nu}\right)$ defined by

$$\varphi\left(\frac{D_x}{2^\nu}\right) u = \int e^{2\pi i \xi x} \varphi\left(\frac{\xi}{2^\nu}\right) \hat{u}(\xi)\, d\xi$$

$$S_\nu(\xi) = \varphi_0\left(\frac{\xi}{2^\nu}\right)$$

Theorem 1

$u \in LL([0,T] \times \mathbb{R}^n_x) \iff u \in L^\infty$ and

$$\overline{\lim_{\nu \to \infty}} \|\nabla S_\nu(D_x) u\| \, \nu^{-1} < +\infty$$

Theorem 2

$a \in LL(\mathbb{R}^n)$ $\quad s \in \mathbb{R}$, $\quad |s| < 1$

Then $\quad u \mapsto au$ is continuous $H^s \to H^s$, and

$$\|au\|_{H^s} \leq C(s,n)\,\|a\|_{LL}\,\|u\|_{H^s}$$

Energy estimate:

$$u_\nu(t,x) = \varphi_\nu(D_x)(u(t,\cdot))(x) \quad , \nu \in \mathbb{N}$$

for $u \in C^2(\mathbb{R}_t, L^2(\mathbb{R}^n_x))$ $\quad , \|\cdot\| = \|\cdot\|_{L^2}$

$$\|\dot{u}_\nu(T)\|^2 e^{-\lambda_\nu T} + \|u_\nu(T)\|^2 e^{-\lambda_\nu T} 2^{2\nu}$$

$$+ \beta \int_0^T e^{-\lambda_\nu t} \left\{ (\nu+1)\left[\|\dot{u}_\nu(t)\|^2 + 2^{2\nu}\|u_\nu(t)\|^2\right\} dt \leq\right.$$

$$\leq 2\,\mathrm{Re} \int_0^T \langle L u_\nu, \dot{u}_\nu(t)\rangle e^{-\lambda_\nu t}\, dt + \|\dot{u}_\nu(0)\|^2 + 2^{2\nu}\|u_\nu(0)\|$$

$\langle\ \rangle$ scalar product in $L^2$

In C.-LERNER homogeneous assumption in $(t,x)$,

but only global in $x$ result.

No results of local existence and local uniqueness.

C-MÉTIVIER (2007)

$$Lu := \sum_{i,k=0}^{n} \partial_{y_i}(a_{ik}\,\partial_{y_k} u)$$

$$+ \sum_{j=0}^{n} \{ b_j\, \partial_{y_j} u + \partial_{y_j}(c_j u) \} + d u = f$$

$y \in \mathbb{R}^{n+1}$

Assumption: $L$ defined in a neighborhood of $\underline{y}$ $\Omega$

$a_{ik} \in LL(\Omega)$, $b_j$ and $c_j \in C^{\alpha}(\Omega)$, $\frac{1}{2} < \alpha < 1$, $d \in L^{\infty}$

$\Sigma$ smooth surface, $\underline{y} \in \Sigma$, $L$ strictly hyperbolic in the direction conormal to $\Sigma$.

We can suppose $\Sigma$ given, near $\underline{y}$, by $\{\varphi = 0\}$

We consider the Cauchy problem on a component, say on $\Omega_{+} = \Omega \cap \{\varphi > 0\}$

Lemma i) $\forall s \in ]1-\alpha, \alpha[$ and.

$$u \in H^s_{loc}(\Omega \cap \{\varphi > 0\})$$

(that is $\forall \Omega_1 \subset\subset \Omega$, the restriction of $u$

to $\Omega_1 \cap \{\varphi > 0\}$ belongs to $H^s$),

all terms entering in $L$ are well defined as

distributions in $H^{s-2}_{loc}(\Omega \cap \{\varphi \geq 0\})$

ii) if $u \in H^s_{loc}(\Omega \cap \{\varphi > 0\})$ and $Lu \in L^2_{loc}(\Omega \cap \{\varphi > 0\})$

then the traces $u_{|\Sigma}$ and $(X_\Sigma u)_{|\Sigma}$ are

well defind in $H^{s-\frac{1}{2}}_{loc}(\Sigma \cap \Omega)$ and $H^{s-\frac{3}{2}}_{loc}(\Sigma \cap \Omega)$

respectively.

Thanks to this lemma, the Cauchy problem

with source term in $L^2$ and solution in $H^s$, $s > 1-\alpha$,

makes sense.

Theorem <u>local existence</u>

let $s > 1-\alpha$ and $\omega$ a neighborhood of $\underline{y}$, $\underline{y} \in \Sigma$

then there are $s' \in ]1-\alpha, \alpha[$ and a neighborhood $\Omega'$ of $\underline{y}$ s.t. $\forall (u_0, u_1) \in H^s \times H^{s-1}(\omega)$ and $f \in L^2(\Omega' \cap \{\varphi > 0\})$, the Cauchy problem

$$L u = f, \quad u_{|\Sigma} = u_0 \quad (X_\Sigma u)_{|\Sigma} = u_1$$

has a solution $u \in H^{s'}(\Omega' \cap \{\varphi > 0\})$.

Theorem <u>local uniqueness</u> If $s > 1-\alpha$ and $u \in H^s(\Omega \cap \{\varphi > 0\})$ satisfies

$$L u = 0, \quad u_{|\Sigma} = 0 \quad (X_\Sigma u)_{|\Sigma} = 0$$

then $u \equiv 0$ near $\underline{y}$.

Until now, condition only on the first derivative of $a_{ij}(t)$.

Now we use second derivative:

Theorem: Let us consider    C.-Del Santo-Reissig 2003

$$Lu = (\partial_t^2 - a(t)\,\partial_x^2)\,u$$

$$\lambda < a < \Lambda, \quad a \in C[0,T] \cap C^2(0,T]$$

$$* \quad |\partial_t^k a(t)| \leq A_k \left(\frac{1}{t} \lg \frac{1}{t}\right)^k \quad k = 1, 2$$

then Cauchy problem for $L$ is $C^\infty$ well posed.

Again one has a loss of derivatives.

Problem: $(*)$ with $k=1$ only, implies W. p.?

Tif

$$|\partial_t a| \leq \frac{1}{t} \quad \text{yes, but} \quad |\partial_t a| \leq \frac{1}{t} \lg \frac{1}{t} \ ?$$

In order to prove this theorem we used another energy, using $a$, $\partial_t a$ and $\partial_t^2 a$.

We have counterexamples for

$$|\partial_t a| \leq C\,\frac{1}{t} \lg \frac{1}{t}\, \omega\!\left(\frac{1}{t}\right)$$

$$\omega(\tau) \xrightarrow[\tau \to \infty]{} +\infty$$

Estimates by $a$, $\partial_t a$, $\partial_t^2 a$

Tarama 2007

$$Lu := \left(\partial_t^2 - a(t)\,\partial_x^2\right) u$$

assumption:

$$(*) \qquad |a(t+\tau) + a(t-\tau) - 2a(t)| \leq C|\tau|\,|\log|\tau|| \qquad |\tau| \leq \tfrac{1}{2}$$

Remark: if $a \in LL \Rightarrow (*)$ verified

$\not\Leftarrow$ easy examples.

Tarama uses energy:

$$E(u) = \frac{1}{a(t)}\left|\partial_t u + \frac{a'(t)}{2a(t)}\,u\right|^2 + a(t)|u|^2$$

with assumption $(*)$

In C - Del Santo - Métivier (2007) we consider

$$L(u) := \left(\partial_t^2 - \partial_x\left(a(t,x)\,\partial_x\right)\right) u$$

$$0 < \lambda_0 \leq a(t) \leq \Lambda$$

with assumption

$$\boxtimes \qquad |a(t+\tau, x) + a(t-\tau, x) - 2a(t,x)| \leq C|\tau|\,|\log|\tau||$$

Theorem Under assumption $\boxtimes$ and

$$x \mapsto a(t,x) \in C^\infty$$

the Cauchy problem for $L$ is well-posed in $H^\infty$.

Energy estimate

There exist $T, \beta > 0$ and, $\forall m \in \mathbb{R}$, $\exists C_m$ :

$$\sup_{0\le t\le T} \left( \|u(t,\cdot)\|_{H^{m+1-\beta t}} + \|\partial_t u(t,\cdot)\|_{H^{m-\beta t}} \right) \le$$

$$\le C_m \|u(0,\cdot)\|_{H^{m+1}} + \|\partial_t u(0,\cdot)\|_{H^m} + \int_0^T \|Lu(t,\cdot)\|_{H^{m-\beta t}} dt$$

$\forall u \in C^2([0,T], H^\infty)$

We use again dyadic decomposition; with same notation

$$w_\nu(x) = \varphi_\nu(D) w(x) = \frac{1}{2\pi} \int e^{ix\xi} \varphi_\nu(\xi) \hat{w}(\xi) d\xi =$$

$$= \frac{1}{2\pi} \int \hat{\varphi}(2^\nu y) 2^\nu w(x-y) dy$$

again $a_\varepsilon(t,x) = \int \rho_\varepsilon(t-s) a(s,x) ds$

$\rho_\varepsilon(s) = \frac{1}{\varepsilon} \rho\left(\frac{s}{\varepsilon}\right)$ ....

We have

$$|a_\varepsilon(t,x) - a(t,x)| \le C|\varepsilon| \log \left|\frac{1}{\varepsilon}\right|$$

and

$$|\partial_t^2 a_\varepsilon(t,x) + \partial_t a_\varepsilon(t,x)|^2 \le C \frac{1}{|\varepsilon|} \log \frac{1}{|\varepsilon|} \qquad 0 < |\varepsilon| < \frac{1}{2}$$

Approximate energy

Let $u(t,x) \in C^2([0,T], H^\infty(\mathbb{R}^n_x))$

$$u_\nu(t,x) = \varphi_\nu(D)\, u(t,x)$$

We obtain

$$\partial_t^2 u_\nu = \partial_x (a(t,x)\partial_x u_\nu) + \partial_x([\varphi_\nu, a]\partial_x u) + (Lu)_\nu$$

here $[\varphi_\nu, a]$ commutator

We introduce approximate energy of $u_\nu$:

$$e_{\nu,\varepsilon}(t) = \int \frac{1}{a_\varepsilon(t,x)} \left|\partial_t u_\nu + \frac{\partial_t a_\varepsilon(t,x)}{2a_\varepsilon(t,x)} u_\nu\right|^2 + a_\varepsilon(t,x)|\partial_x u_\nu|^2 \, dx$$

Next we choose $\varepsilon = 2^{-\nu}$.

The important fact is the estimate of the commutator

$$\| [\varphi_\nu, a] \varphi_\mu \|_{\mathcal{L}(l^2, l^2)} \leq \begin{cases} C\, 2^{-\nu} & \text{if } |\mu - \nu| \leq 2 \\ C_N\, 2^{-N \max(\nu, \mu)} & \text{if } |\mu - \nu| \geq 3. \end{cases}$$

Idea of the construction of the counterexample.

Theorem Let $\psi : [M, +\infty) \to \mathbb{R}$
$\psi$ increasing and concave, $\psi(+\infty) = +\infty$.
Then there exist $a : [0, +\infty) \to \mathbb{R}$, $\frac{1}{2} \leq a \leq \frac{3}{2}$
such that

$$|a(t+\tau) - a(t)| \leq C |\tau| |\lg|\tau|| \ \psi(|\lg|\tau||)$$

and $u_0, u_1 \in C^\infty(\mathbb{R})$ such that:

$$\begin{cases} \partial_t^2 u - a(t)\, \partial_x^2 u = 0 \\ u(0, x) = u_0, \quad \partial_t u(0, x) = u_1 \end{cases}$$

has no solution in $C([0, T], \mathcal{D}'(\mathbb{R}))$.

We take

$$b(\varepsilon, \tau) = 1 - 4\varepsilon \sin 2\tau - \varepsilon^2 (1 - \cos 2\tau)^2$$

and

$$\varepsilon_k = \eta_k^{-1} (\lg \eta_k) \,\psi(\lg \eta_k)$$

where $\eta_k \to +\infty$, $\varepsilon_k \to 0$, $\varepsilon_k \eta_k \to +\infty$, $\varepsilon_k \eta_k \geq 2^k$

Writing

$$[0, 1[ = \bigcup_{k=1}^{\infty} I_k \qquad I_k = [1 - 2^{-k}, 1 - 2^{-k-1}[$$

we define

$$a(t) = \begin{cases} b(\varepsilon_k, \eta_k t) & t \in I_k \\ 1 & \text{elsewhere.} \end{cases}$$

We have

$$|a(t+h) - a(t)| \leq C\, |h|\, \lg |h|\, \psi(\lg |h|)$$

Now we pose

$$v_j''(t) + \eta_j^2\, a(t)\, v_j(t) = 0$$

$$v_j(t_j) = 0 \qquad v_j'(t_j) = 1$$

where $t_j = 1 - 3\,2^{-j-2}$ is the center of $I_j$

We have

$$v_j(t) = \frac{1}{\eta_j} w(\varepsilon_j, \eta_j(t-t_j)) \quad \text{for } t \in I_j$$

where

$$w(\varepsilon,\tau) = \sin\tau \exp\left(\varepsilon\left(\tau - \frac{1}{2}\sin 2\tau\right)\right)$$

Noting $I_j = [t_j', t_j''[$ , we have

$$\begin{cases} v_j(t_j') = 0 & v_j'(t_j') = e^{-\varepsilon_j \eta_j} \\ v_j(t_j'') = 0 & v_j'(t_j'') = e^{\varepsilon_j \eta_j} \end{cases}$$

So, $v_j$ is "little" at $t = t_j'$, big at $t = t_j''$

Moreover, for $0 \leq t \leq t_j'$

$$\eta_j |v_j(t)| + |v_j'(t)| \leq C\left(\eta_j v_j(t_j') + v_j'(t_j')\right) e^{\int_0^{t_j'} \frac{|a'(s)|}{a(s)} ds}$$

But $a \geq \frac{1}{2}$ and, for $0 \leq t \leq t_j'$ $|a'(t)| \leq \varepsilon_{j-1}\eta_{j-1}$

And so we get, for $0 \leq t \leq t_j'$

$$\eta_j |v_j(t)| + |v_j'(t)| \leq e^{-\varepsilon_j \eta_j + \varepsilon_{j-1}\eta_{j-1}} \leq e^{-\frac{1}{2}\varepsilon_j \eta_j}$$

Finally we define

$$v(t,x) = \sum_{j=0}^{\infty} v_j(t) \sin(\eta_j x)$$

$v_j$ defined. Then

$$\left(\partial_t^2 - a(t)\,\partial_x^2\right) v = 0$$

and $v \in C^2([0,1[\,, C^\infty(\mathbb{R}))$.

Moreover $v_j'(t_j'')$ increases faster than any power of $\eta_j$, so that

$$\sum v_j(t) \sin(\eta_j x)$$

cannot be extended as a distribution beyond 1.