Optimal Control by varying the Length of the String

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The Problem

Let $T = 2$. We control on the time interval $(0, T)$.

Let $D \in (0, 1)$. D is strictly less than the wave speed that equals 1.

Let $Lip = \{\phi : [0, T] \to (0, \infty),\ \phi$ is Lipschitz continuous.$\}$

Define the set of admissible ϕ:

$\Phi = \{\phi \in Lip : \phi$ has a Lipschitz constant $\leq D,\ \phi(0) = 1 = \phi(T)\}$.

Define the set of initial states

$B = \{(y_0, y_1) : y_0' \in L^2(0, 1),\ y_1 \in L^2(0, 1),\ y_0(0) = 0 = y_0(1)\}$.

Let $(y_0, y_1) \in B$ be given.

We define the Problem to move the boundary in such a way that at the time T, the energy is as small as possible.

Problem P :

$$\min_{\phi \in \Phi} W(T) = \int_0^1 v_x(x,T)^2 + v_t(x,T)^2 \, dx$$

such that $v(x,0) = y_0(x)$, $v_t(x,0) = y_1(x)$, $x \in (0,1)$,

$v(0,t) = 0$, $v(\phi(t),t) = 0$, $t \in (0,T)$

$v_{tt} = v_{xx}$ on $\Omega = \{(x,t) : t \in (0,T),\ x \in (0,\phi(t))\}$.

With the obvious definition of the set $\mathcal{A}$ of the admissible shapes Ω and the objective function $J(\Omega)$, this can be seen as a shape optimization problem

$$\boxed{\min_{\Omega \in \mathcal{A}} J(\Omega).}$$

Note that due to the upper bound $D < 1$ for the Lipschitz constant, the length of the string does not change faster than the wave speed.

Thm[Existence] There exists $\phi \in \Phi$ that solves P.

Thm[Uniqueness] Let

$$A(x) = \begin{cases} y_0'(-x) - y_1(-x), & x \in [-1, 0) \\ y_0'(x) + y_1(x), & x \in [0, 1]. \end{cases}$$

Define the set

$$M_z = \{x \in [-1, 1] :\ A(x) = 0\}.$$

If M_z has measure zero, the solution of P is uniquely determined.

Thm[Representation of the solution of P]

a) If $A = 0$ on $[-1, 1]$, we have $W(T) = 0$ for all $\phi \in \Phi$.

b) If $A \not\equiv 0$, there exists a number $\lambda > 0$, such that

$$\int_{-1}^{1} \sqcap_{[\frac{1-D}{1+D},\, \frac{1+D}{1-D}]}(\lambda |A(y)|)\, dy = 2.$$

With this number λ, we can define a solution of P as follows:

Define the function $h : [-1, 1] \to [1, 3]$ as

$$h(x) = 1 + \int_{-1}^{x} \sqcap_{[\frac{1-D}{1+D},\, \frac{1+D}{1-D}]}(\lambda |A(y)|)\, dy.$$

Let

$$H_1(x) = \frac{h(x) - x}{2},\ H_2(x) = \frac{h(x) + x}{2}.$$

Then a solution of P is

$$\phi(t) = H_1(H_2^{-1}(t)),\ t \in (0, 2).$$

The corresponding value of the objective function is

$$W(t) = \int_{-1}^{1} \frac{|A(y)|^2}{h'(y)} \, dy.$$

Example 1

$y_0(x) = |x - \frac{1}{2}| - \frac{1}{2}$, $y_1(x) = 0$. This yields $|A(x)| = 1 \in [\frac{1-D}{1+D}, \frac{1+D}{1-D}]$ for almost all $x \in (-1, 1)$, hence we have a unique solution and $\lambda = 1$. Thus $h(x) = 1 + (x - (-1)) = 2 + x$. Hence $H_1(x) = 1$ which yields $\phi(t) = 1$.

In this example, it is optimal not to move the boundary. Every change of the length of the string, for example caused by vibrations causes an increase in energy.

Example 2

Let k be a natural number.

Let $\omega = k\pi$.

Let $\epsilon\omega \in (0, \frac{2D}{1+D})$.

Let $y_0(x) = \epsilon \sin(\omega x)$, $y_1(x) = 1$.

Then $(y_0, y_1) \in B$.

$y_0'(x) = \epsilon\omega \cos(\omega x)$.

For $x \in [-1, 0)$, $y_0'(-x) - y_1(-x) = \epsilon\omega \cos(\omega x) - 1$.

Hence for $x \in [-1, 0)$, $|A(x)| = 1 - \epsilon\omega \cos(\omega x)$.

For $x \in [0, 1]$, $y_0'(x) + y_1(x) = \epsilon\omega \cos(\omega x) + 1$.

Hence for $x \in [0, 1]$, $|A(x)| = 1 + \epsilon\omega \cos(\omega x)$.

Hence for all $x \in [-1, 1]$, $|A(x)| \in [\frac{1-D}{1+D}, \frac{1+D}{1-D}]$.

Moreover, we have $\int_{-1}^{1} |A(x)|\, dx = 2$, hence $\lambda = 1$.

Therefore, for $x \in [-1, 0]$, $h(x) = 2 + x - \epsilon \sin(\omega x)$, and for $x \in (0, 1]$, $h(x) = 2 + x + \epsilon \sin(\omega x)$.

Thus for $x \in [-1, 0]$, $H_1(x) = 1 - \frac{1}{2}\epsilon \sin(\omega x)$
and for $x \in (0, 1]$, $H_1(x) = 1 + \frac{1}{2}\epsilon \sin(\omega x)$.

As always, we have $H_2(x) = H_1(x) + x$.

We can plot the graph of $\phi = H_1 \circ H_2^{-1}$ since

$$\{(t, \phi(t)) : t \in (0, T)\} = \{(H_2(x), H_1(x)),\ x \in [-1, 1]\}.$$

The result and the proofs will appear in SICON.