THE CONSTRUCTION OF ASYMPTOTIC THEORIES BY Γ-CONVERGENCE

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Benasque, August 28, 2007

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Γ-convergence: a tool for the asymptotic description of variational problems.

Underlying method: the study of complex minimum problems involving a (small) parameter ε is approximated by a minimum problem where the dependence on this parameter has been averaged out.

The notion of Γ-*convergence* of energies is designed to guarantee the **convergence of minimum problems**; i.e.,

$$F_\varepsilon \xrightarrow{\Gamma} F^{(0)} \implies \min F_\varepsilon =: m_\varepsilon \to m^{(0)} := \min F^{(0)},$$

and (almost)minimizers of $\min F_\varepsilon$ converge to minimizers of $F^{(0)}$.

(Note: compactness of minimizers is given for granted in the talk)

Technical definition:

(i) $x_\varepsilon \to x \Longrightarrow F^{(0)}(x) \le F_\varepsilon(x_\varepsilon) + o(1)$ (*ansatz-free* lower bound)

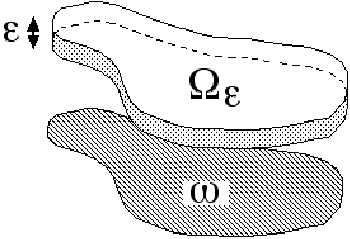
(ii) $F_\varepsilon(x_\varepsilon) \to F^{(0)}(x)$ for some $x_\varepsilon \to x$ (sharpness of lower bound)

Important property: Γ-convergence is **stable** with respect to addition of continuous perturbations: if $F_\varepsilon \xrightarrow{\Gamma} F^{(0)}$ then $(F_\varepsilon + G) \xrightarrow{\Gamma} (F^{(0)} + G)$. This means that once the Γ-limit $F^{(0)}$ is computed, that computation can be used to describe a whole class of problems (a *theory*).

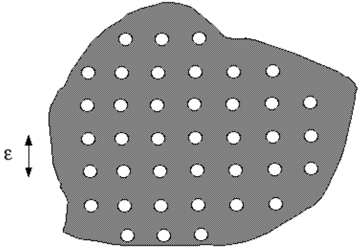
(Actually, joint stability and convergence of minima are equivalent to Γ-convergence)

Examples:

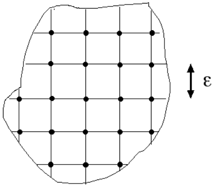
– dimensionally-reduced theories of thin structures



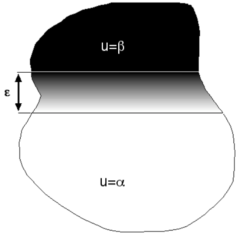
– effective theories of composites



– continuum elasticity as limit of lattice theories



– phase-transition models with sharp interfaces from gradient theories



– more... (B. Γ-convergence for Beginners, Oxford, 2002 Handbook of Γ-convergence, Elsevier, 2006)

ITERATION OF Γ-CONVERGENCE

If the description given by $F^{(0)}$ is too coarse, further information can be obtained by **iteration** of the Γ-limit procedure; e.g., if some $\alpha > 0$ exists such that

$$F_\varepsilon^{(\alpha)} := \frac{F_\varepsilon - m^{(0)}}{\varepsilon^\alpha} \xrightarrow{\Gamma} F^\alpha,$$

then, using again the convergence of minimum problems, we deduce that

$$m_\varepsilon^{(\alpha)} := \min F_\varepsilon^{(\alpha)} \to m^{(\alpha)} := \min F^{(\alpha)}.$$

Since $m_\varepsilon^{(\alpha)} = \dfrac{m_\varepsilon - m^{(0)}}{\varepsilon^\alpha}$ we have the **more accurate development**

$$m_\varepsilon = m^{(0)} + \varepsilon^\alpha m^{(\alpha)} + o(\varepsilon^\alpha).$$

(note the *simplified dependence* on ε)

This process of *development by Γ-convergence* (Anzellotti-Baldo) is resumed in the equality

$$F_\varepsilon \stackrel{\Gamma}{=} F^{(0)} + \varepsilon^\alpha F^{(\alpha)} + o(\varepsilon^\alpha)$$

(this is just a *formal equality* since the domains of the functionals may be different).

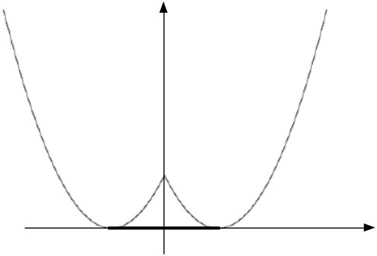
Note that in this process some **scale analysis** must be performed to understand what is the relevant scaling ε^α (in general $f(\varepsilon)$).

Example 1 (*Gradient theory of phase transitions*)

Let

$$F_\varepsilon(u) = \int_\Omega (W(u) + \varepsilon^2 |\nabla u|^2)\, dx, \qquad u \in H^1(\Omega)$$

with $W : \mathbb{R} \to [0, +\infty)$ a *double-well potential* (with wells in ± 1; e.g., $W(s) = \min\{(s+1)^2, (s-1)^2\}$).



Then $\alpha = 1$ and

$$\begin{aligned} F^{(0)}(u) &= \int_\Omega W^{**}(u)\,dx,\ u \in L^1(\Omega) \ \ (W^{**}\text{convex envelope}) \\ F^{(1)}(u) &= c_W \mathcal{H}^{n-1}(S(u)),\ u \in \{\pm 1\} \text{ piecewise constant,} \end{aligned}$$

$c_W = 2\int_{-1}^{1} \sqrt{W}\,ds$ the *surface tension*
$S(u)$ = *interface* between phases $\{u = \pm 1\}$
$\mathcal{H}^{n-1}$ $(n-1)$-dimensional measure.

Example 2 (*Theories of thin structures*)

In this case

$$F_\varepsilon(u) = \int_{\omega\times(0,\varepsilon)} f(\nabla u)\,dx,\ u \in W^{1,p}(\omega\times(0,\varepsilon);\mathbb{R}^3),$$

with f a nonlinear elastic energy with a minimum in the identity and ω an open subset of $\mathbb{R}^2$.
Γ-limits $F^{(\alpha)}$ at different scales ε^α can be computed, giving, e.g.,
– membrane theory ($\alpha = 1$),
– bending theory ($\alpha = 3$),
– von Karman theory ($\alpha = 4$), etc.
(see Le Dret and Raoult, Friesecke, James and Müller).

Theories 'justified by Γ-convergence'

In the examples above the computation of the Γ-limit suggests:
1) the use of a *sharp-interface theory* for phase transitions;
2) the use of one of the limit *low-dimensional theories* in the second case.

This **general paradigm** may be **in contrast with the use of other (successful) theories** by practitioners, or may provide a **poor approximation** of the original functional in certain regimes (further examples below).

Our **goal** is to **overcome this drawback in the use of Γ-convergence.**

Inaccuracy of convergence for *parametrized functionals*: how much can we trust our approximation?

Minimum problems are often **parametrized** by **lower-order terms**, whose form does not greatly affect the Γ-limit (continuous or ‘compatible’ perturbations; e.g., boundary conditions or volume constraints).

However, the **overall dependence** of the limit process on these parameters may be **inaccurately described** by the Γ-limit.

Example 3 (*Gradient theory of phase transitions with a volume constraint*)

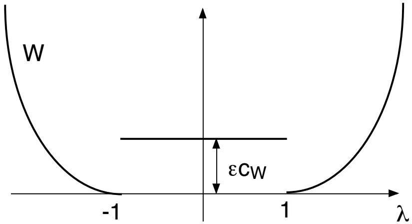
The volume constraint $\int u\,dt = \lambda$ is *compatible* with the Γ-limit procedure within the gradient theory of phase transitions.
This gives that for all $\lambda \in [-1, 1]$ the development of the minimum values

$$m_\varepsilon(\lambda) = \min\left\{\int_0^1 (W(u) + \varepsilon^2|u'|^2)\,dt : \int_0^1 u\,dt = \lambda\right\}$$

is given by $m^{(0)}(\lambda) + \varepsilon m^{(1)}(\lambda) + o(\varepsilon)$,
where $m^{(0)}(\lambda) = W^{**}(\lambda) = 0$ and

$$m^{(1)}(\lambda) = c_W \min\left\{\#(S(u)) : u \in \{\pm 1\}, \int_0^1 u\,dt = \lambda\right\}$$

$$m^{(1)}(\lambda) = \begin{cases} c_W & \text{if } |\lambda| < 1 \\ 0 & \text{if } |\lambda| = 1. \end{cases}$$



picture of $m^{(0)} + \varepsilon\, m^{(1)}$

NOTE:

the previous example shows a situation where the approximation by the development by Γ-convergence provides a **discontinuity** in the dependence of the parameter (that is clearly not there for $\varepsilon > 0$).

This discontinuity corresponds to a **singular behaviour** of the Γ-development with respect to the parameter λ.

SINGULAR POINTS

Let F_ε^λ be a family of **parametrized functionals** , with $\lambda \in \Lambda$.
We say that λ_0 is a **singular point at scale** ε^α if there exist m_ε, $\lambda'_\varepsilon \to \lambda_0$ and $\lambda''_\varepsilon \to \lambda_0$ such that (up to subsequences)

$$\Gamma\text{-}\lim_{\varepsilon\to 0} \frac{F_\varepsilon^{\lambda'_\varepsilon} - m_\varepsilon}{\varepsilon^\alpha} \neq \Gamma\text{-}\lim_{\varepsilon\to 0} \frac{F_\varepsilon^{\lambda''_\varepsilon} - m_\varepsilon}{\varepsilon^\alpha}, \tag{1}$$

and one of the two limits is not trivial.
If λ_0 is not a singular point, we say that it is a **regular point**.

Theorem 1 (*uniform convergence of minimum problems at scale* ε^α)
If Λ is compact and is composed of regular points at scale ε^α, and if $m_\varepsilon^\alpha(\lambda)$ exist such that the limit

$$\Gamma\text{-}\lim_{\varepsilon\to 0} \frac{F_\varepsilon^\lambda - m_\varepsilon^\alpha(\lambda)}{\varepsilon^\alpha} =: F_\lambda^{(\alpha)}$$

exists and is not trivial, then we have

$$\sup_\Lambda \left| \min F_\varepsilon^\lambda - m_\varepsilon^\alpha(\lambda) - \varepsilon^\alpha \min F_\lambda^{(\alpha)} \right| = o(\varepsilon^\alpha).$$

Remarks

(1) if a Γ-development exists $F_\varepsilon^\lambda \overset{\Gamma}{=} F_\lambda^{(0)} + \cdots + \varepsilon^\beta F_\lambda^{(\beta)} + o(\varepsilon^\beta)$ up to some ε^β, with $\beta < \alpha$ and ε^α is the next meaningful order, we may take $m_\varepsilon^\alpha(\lambda) = m^{(0)}(\lambda) + \cdots + \varepsilon^\beta m^{(\beta)}(\lambda)$ defined by the Γ-development;

(2) at scale 1 ($\alpha = 0$) we deduce that if there exists $F_\lambda^{(0)} = \Gamma\text{-}\lim_\varepsilon F_\varepsilon^\lambda$ and is not trivial, then

$$\sup_\Lambda \left| \min F_\varepsilon^\lambda - \min F_\lambda^{(0)} \right| = o(1);$$

(3) if $F_\lambda^{(0)}$ exists and λ_0 is regular at scale 1 then $\lambda \mapsto \min F_\lambda^{(0)}$ is continuous at λ_0.

Example 4

In Example 3 (phase transitions) the points ± 1 are singular at scale ε (from (3) above).

The previous theorem does not hold and we have (taking e.g. Λ a compact set containing a neighbourhood of 1)

$$\sup_{\Lambda} \left| \min F_{\varepsilon}^{\lambda} - m^{(0)}(\lambda) - \varepsilon \min F_{\lambda}^{(1)} \right| \geq C\varepsilon,$$

even though the limit $F_{\lambda}^{(1)}$ exists.

Analysis at singular points

At singular points λ_0 the computation of the Γ-limit or Γ-development with fixed λ_0 is not sufficient to accurately describe the behaviour of minimum problems. Then we have to look at the different limits that we may obtain as $\lambda_\varepsilon \to \lambda_0$.

Table of Γ-limits at λ_0
The *table of Γ-limits at scale* 1 for F_ε^λ at λ_0 are all sequences $(\varepsilon_j, \lambda_j)$, and functionals $F^{(0)}_{(\varepsilon_j,\lambda_j)}$ with $\varepsilon_j \to 0$, $\lambda_j \to \lambda_0$, and

$$F^{(0)}_{(\varepsilon_j,\lambda_j)} = \Gamma\text{-}\lim_j F^{\lambda_j}_{\varepsilon_j}\,.$$

The *table of Γ-limits at scale* ε^α is likewise defined (in the same spirit of the development by Γ-convergence).

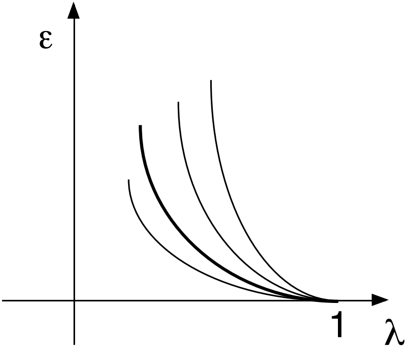
Example 5 (*Gradient theory of phase transitions – continued*)

Case $\lambda_0 = 1$. Note that the functionals $F^{(1)}_{(\varepsilon_j,\lambda_j)}$ are finite only at the constant function $u = 1$, so that it suffices to compute the limit

$$\lim_j \min \left\{ \int_0^1 \left(\frac{W(v)}{\varepsilon_j} + \varepsilon_j |v'|^2 \right) dt : \int_0^1 u\, dt = \lambda_j \right\}$$
$$= \lim_j \min \left\{ c_W, \frac{1}{2} W''(1) \frac{(1-\lambda_j)^2}{\varepsilon_j} \right\}.$$

Existence of the Γ-limit $F^{(1)}_{(\varepsilon_j,\lambda_j)}$ is equivalent to the existence of the last limit depending on the ratio $(1-\lambda_j)^2/\varepsilon_j$.

– if $\frac{1}{2}W''(1)\lim_j \frac{(1-\lambda_j)^2}{\varepsilon_j} < c_W$ uniform states are preferred
– if $\frac{1}{2}W''(1)\lim_j \frac{(1-\lambda_j)^2}{\varepsilon_j} > c_W$ sharp transitions are preferred.

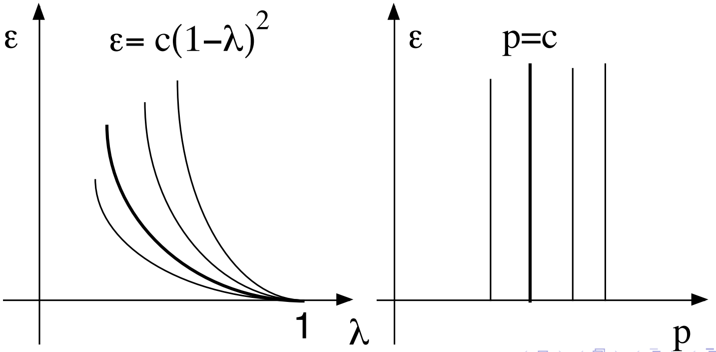


$W''(1)(1-\lambda)^2 = 2\varepsilon c_W$ **nucleation threshold**

Blow up at singular points

In the previous examples the behaviour of parametrized energies at singular point may be analyzed in terms of curves in the ε–λ space, along which a regular Γ-development exists. This is not the general case, but it is frequent in applications.

IDEA: change variables $(\varepsilon, \lambda) \mapsto (\varepsilon, p)$ so that the behavior is regular in the p variable



Rectifiability. Let λ_0 be a singular point for F_ε^λ at scale 1. We say that F_ε^λ is **rectifiable at** λ_0 at order 1 if energies H_ε^p exist and a function $p = p(\lambda, \varepsilon)$ such that
(i) H_ε^p Γ-converge to H^p, and all p are regular points;
(ii) $F_\varepsilon^\lambda = H_\varepsilon^{p(\lambda,\varepsilon)}$ for (λ, ε) in a neighbourhood of $(0, \lambda_0)$.
The definition can be easily extended to order ε^α.

Example 6 (*'rectification' for phase transitions*)

We may 'rectify' the functionals F_ε^λ at the point $\lambda_0 = 1$ at order ε, which means that the functionals $\frac{1}{\varepsilon}(F_\varepsilon^\lambda - \min F_\lambda^{(0)})$ are rectifiable at order 1.
We set $p = (1-\lambda)^2/\varepsilon$, and then we may simply define

$$H_\varepsilon^p(u) = \int_0^1 \left(\frac{W(u)}{\varepsilon} + \varepsilon |u'|^2 \right) dt,$$

with $\int_0^1 u\,dt = 1 - \sqrt{\varepsilon p}$, so that

$$H^p(u) = \begin{cases} \min\left\{ c_W, \frac{1}{2} W''(1) p \right\} & \text{if } u = 1 \\ +\infty & \text{otherwise} \end{cases}$$

Uniformly-equivalent functionals.

We say that two families of parametrized functionals F_ε^λ and G_ε^λ are *uniformly equivalent at scale ε^α at λ_0* if for every sequence $(\varepsilon_j, \lambda_j)$ converging to $(0, \lambda_0)$ there exist m_j such that the limits

$$\Gamma\text{-}\lim_j \frac{F_{\varepsilon_j}^{\lambda_j} - m_j}{\varepsilon^\alpha}, \qquad \Gamma\text{-}\lim_j \frac{G_{\varepsilon_j}^{\lambda_j} - m_j}{\varepsilon^\alpha}$$

exists, are not trivial and are equal.

Theorem 3 Let Λ be compact, and let F_ε^λ and G_ε^λ be uniformly equivalent at scale ε^α at all $\lambda \in \Lambda$. Then we have

$$\sup_\Lambda \left| \min F_\varepsilon^\lambda - \min G_\varepsilon^\lambda \right| = o(\varepsilon^\alpha). \tag{2}$$

The following theorem states that for rectifiable F_ε^λ a simple uniformly-equivalent family is given by H^p computed for $p = p(\lambda, \varepsilon)$.

Theorem 2 Let F_ε^λ be rectifiable at λ_0; then F_ε^λ is uniformly equivalent to $G_\varepsilon^\lambda = H^{p(\lambda,\varepsilon)}$ at λ_0

Example 7 (Trivial) uniform-equivalent energy at order ε for phase transitions ($|\lambda| \le 1$)

$$G_\varepsilon^\lambda(u) = \min\Big\{\varepsilon c_W, \frac{1}{2}W''(1)(1-|\lambda|)^2\Big\}\#S(u),\ u \in \{\pm 1\}.$$

Note: these uniform-equivalent energies may be over-simplified. More meaningful energies are given by extending these ones to piecewise-constant functions,

$$G_\varepsilon^\lambda(u) = \frac{1}{2}W''(1)\int (1-|u|)^2\,dt + \varepsilon c_W\#(S(u)),$$

or to SBV functions

$$G_\varepsilon^\lambda(u) = \int W(u)\,dt + \sum_{S(u)} g_\varepsilon(u^+, u^-)$$

(suitable conditions on g_ε), etc.

A general method: construction of equivalent theories

From the analysis of the previous sections we can sketch a procedure to construct equivalent families of parametrized functionals from a family F_ε^λ (functionals are scaled so that the analysis can be performed at scale 1):

1. **Identify singular points** of F_ε^λ;
2. **Compute the table of Γ-limits** at singular points;
3. **Rectify** the energies at singular points;
4. **Match asymptotics**; i.e., construct energies that are equivalent to the Γ-limit (or Γ-development) far from singular points, and to the 'rectified' energies close to singular points.

Of course, the last point has not a unique answer, and additional criteria (simplicity, computability, closeness to well-known theories, etc) can drive it.

More examples:

- homogenization with boundary effects ($\Lambda =$ boundary data)
- homogenization with concentrated forces ($\Lambda =$ size of the support of the forces)
- finite-scale microstructure ($\Lambda =$ elastic constants)
- discrete-to continuous problems ($\Lambda =$ macroscopic deformation)
- etc.

(see B-Truskinovsky. Asymptotic expansions by Γ-convergence @ `http://cvgmt.sns.it`)

MAIN OPEN ISSUES:

• test this procedure with more complex energies, where parameters cannot be reduced to a one-dimensional set

• include in the process additional 'non-energetic' criteria (convergence of critical points, improved convergence of minimizers, etc.)