# Heat equations with singular potentials: Hardy & Carleman inequalities, well-posedness & control

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Analysis and control of partial differential equations
JP$^2$'s $X$-th birthday
Pont à Mousson, June, 2007

# Outline

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1. Preliminaries
2. The Cauchy pbm
3. Control
4. Waves
5. Conclusion

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  PDE with singular potentials arising in combustion theory and quantum mechanics.
- Goal:
  Revise the existing theory of well-posedness and control when replacing $-\Delta$ by $-\Delta - \frac{\lambda}{|x|^2}$.

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# Origins

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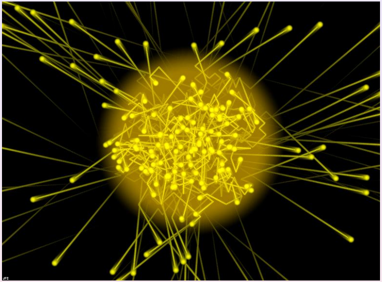

- And then....
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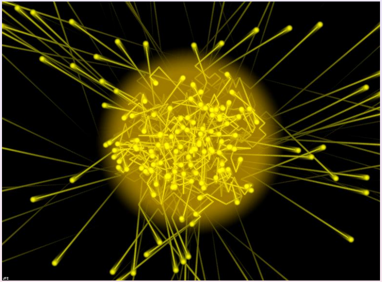

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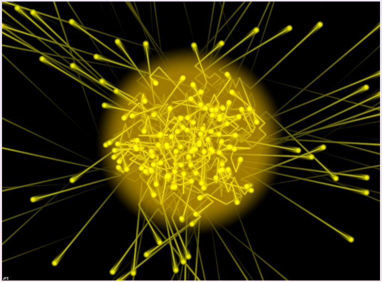

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Part of the literature on singular elliptic and parabolic problems:


- S. Chandrasekhar, An introduction to the study of stellar structure, New York, Dover, 1957.
- I. M. Gelfand, Some problems in the theory of quasilinear equations, Amer. Math. Soc. Transl., **29** (1963), 295-381.
- J. Serrin, *Pathological solution of an elliptic differential equation*, Ann. Scuola Norm. Sup. Pisa, **17** (1964), 385–387.
- D. S. Joseph & T. S. Lundgren, Quasilinear Dirichlet problems driven by positive sources, Arch. Rat. Mech. Anal., **49** (1973), 241-269.
- F. Mignot, F. Murat, J.-P. Puel, Variation d'un point de retournement par rapport au domaine, Comm. P. D. E. **4** (1979), 1263-1297.
- P. Baras, J. Goldstein, *The heat equation with a singular potential,* Trans. Amer. Math. Soc. **284** (1984), 121–139.
- T. Gallouet, F. Mignot & J. P. Puel, Quelques résultats sur le problème $-\Delta u = \lambda e^u$. C. R. Acad. Sci. Paris Sér. I, Math. **307** (7) (1988), 289-292.

# Examples:

Example 1:

$$-\Delta u - \mu(1+u)^p = 0,$$

$$p > n/(n-2),\ \mu = \frac{2}{p-1}(n - \frac{2p}{p-1}).$$

$$u(x) = |x|^{-2/(p-1)} - 1$$

After "linearization":

$$-\Delta v - \frac{\lambda}{|x|^2} v = f.$$

with

$$\lambda = \frac{2p}{p-1}(n - \frac{2p}{p-1}).$$

Example 2:

$$-\Delta u - \lambda e^u = 0, \ \lambda = 2(N-2)$$

$$u(x) = -2\log(|x|).$$

After "linearization":

$$-\Delta v - \frac{\lambda}{|x|^2} v = f.$$

Warning! Linearization is formal in these examples. Indeed, the complex behavior of solutions with respect the parameter $\lambda$ shows that Inverse Function Theorem fails to apply because of the lack of an appropriate functional setting.

Dirichlet Problems 245

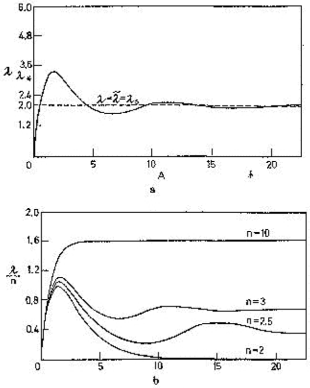


Fig. 1a. Bifurcation diagram for solutions of problem (II.3) with $n=3$. This curve was computed by numerical integration of (II.3). It gives the values of $\lambda=\lambda(A)$ for which solutions of (II.3) or the equivalent problem (I.2) are possible. For a fixed $\lambda$ it is possible to have different solutions $u(r,\lambda)$ having different values of $u(0,\lambda)=A$. When $\lambda>\lambda_*\approx 3.35$ there are no solutions. When $\lambda=\lambda_s=\tilde{\lambda}=2$ there are infinitely many solutions having different values of $A$.

Fig. 1b. Bifurcation diagrams for the solutions of problem (II.3). This figure is constructed by numerical integration of (II.3) for different values of $n$. When $n=2$ and $\lambda \neq \lambda_*$ there are either two solutions ($\lambda<\lambda_s$) or no solutions $\lambda>\lambda^*$. When $2<n<10$ there are infinitely many solutions for $\lambda=\tilde{\lambda}=\lambda_s$. When $n \gtrsim 10$ and $\lambda<\lambda_*=\lambda_s$ the solutions are unique.

D. Joseph et al., 1973.

# The Cauchy problem

$$\begin{cases} u_t - \Delta u - \frac{\lambda}{|x|^2} u = 0 & \text{in} \quad Q \\ u = 0 & \text{on} \quad \Sigma \\ u(x,0) = u^0(x) & \text{in} \quad \Omega. \end{cases}$$

Baras-Goldstein (1984), $N \geq 3$:

- Global existence for $\lambda \leq \lambda_* = (N-2)^2/4$;
- Instantaneous blow-up if $\lambda > \lambda_* = (N-2)^2/4$.

Explanation: Hardy's inequality:

$$\lambda_* \int_\Omega \frac{\varphi^2}{|x|^2} dx \leq \int_\Omega |\nabla \varphi|^2 dx.$$

True for any domain, optimal constant, not achieved: $\varphi = |x|^{-(N-2)/2}$.

Warning! In dimension $N = 2$ this inequality fails....$\lambda_* = 0$

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# Hardy-Poincaré inequality

H. Brézis-J. L. Vázquez, 1997:

$$\lambda_* \int_\Omega \frac{\varphi^2}{|x|^2} dx + C(\Omega) \int_\Omega \varphi^2 dx \leq \int_\Omega |\nabla \varphi|^2 dx, \; \forall \varphi \in H_0^1(\Omega).$$

Later improved [1]: $0 < s < 1$,

$$\lambda_* \int_\Omega \frac{\varphi^2}{|x|^2} dx + C(\Omega) ||\varphi||_s^2 \leq \int_\Omega |\nabla \varphi|^2 dx, \; \forall \varphi \in H_0^1(\Omega).$$

$-\Delta - \frac{\lambda_*}{|x|^2} I$ is almost as coercive as $-\Delta$.

The elliptic theory would be the same by replacing $H_0^1(\Omega)$ by $\mathcal{H}(\Omega)$, the closure of $\mathcal{D}(\Omega)$ with respect to the norm

$$||\varphi||_{\mathcal{H}} = \Big[ \int_\Omega \Big[ |\nabla \varphi|^2 - \lambda_* \int_\Omega \frac{\varphi^2}{|x|^2} \Big] dx \Big]^{1/2}.$$

[1] J. L. Vázquez & E. Z. The Hardy inequality and the asymptotic behavior of the heat equation with an inverse square potential. J. Funct. Anal., 173 (2000), 103–153.

# Three cases:

- $0 < \lambda < \lambda_*$: $u^0 \in L^2 \Rightarrow u \in C\left([0,T]\,;L^2\right) \cap L^2\left(0,T;H_0^1\right)$.
- $\lambda = \lambda_*$: $u^0 \in L^2 \Rightarrow u \in C\left([0,T]\,;L^2\right) \cap L^2\left(0,T;\mathcal{H}\right)$.
- $\lambda > \lambda_*$: Lack of well-posedness.

Solutions have to be interpreted in the semigroup sense. Uniqueness does not hold in the distributional one. For instance, for

$$\lambda = \lambda_*,\ u(x) = |x|^{-(N-2)/2}\log(1/|x|),$$

is a singular stationary solution. It is not the semigroup solution.

# A closer look:

$$\Omega = B(0,1); \quad \varphi = \varphi(r) \to \psi(r) = r^{(N-2)/2}\varphi(r).$$

$$||\varphi||_{\mathcal{H}} = \Big[\int_0^1 |\varphi'(r)|^2 r\, dr\Big]^{1/2}.$$

Over the space of radially symmetric functions

$$-\Delta - \frac{\lambda_*}{|x|^2} I \quad \text{in } \mathbf{R}^3 \sim -\Delta \ \text{ in } \mathbf{R}^2.$$

This guarantees coercivity in $H^s$, for $0 < s < 1$. But no further regularity/integrability. Note that

$$-\Delta\phi - \frac{\lambda}{|x|^2}\phi = 0$$

with

$$\phi = \frac{1}{|x|^{\alpha(\lambda)}}, \ \alpha = \frac{N-2}{2} - \Big[\Big[\frac{N-2}{2}\Big]^2 - \lambda\Big]^{1/2}.$$

- This function $\phi$ has the generic singularity of solutions at $x = 0$.
  In the critical case $\lambda = \lambda_*$

$$\phi = |x|^{-(N-2)/2} \Rightarrow \int \frac{|\phi|^2}{|x|^2} dx = \infty; \int |\nabla \phi|^2 dx = \infty.$$

  This is a further explanation of the fact that $H_0^1$-regularity may not be achieved.
- This does not happen when $\lambda < \lambda_*$. In that case $\phi \in H_0^1$.
- When $\lambda > \lambda_*$ this transformation yields

$$-\psi'' - \frac{\psi'}{r} - c\frac{\psi}{r^2} = f,$$

  with $c > 0$. Consequently we have a non-admissible perturbation of the 2-d Laplacian. The equation does not make sense in the distributions....

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# Control:

Let $N \geq 1$ and $T > 0$, $\Omega$ be a simply connected, bounded domain of $\mathbb{R}^N$ with smooth boundary $\Gamma$, $Q = (0, T) \times \Omega$ and $\Sigma = (0, T) \times \Gamma$:

$$\begin{cases} u_t - \Delta u - \lambda \frac{u}{|x|^2} = f 1_\omega & \text{in} \quad Q \\ u = 0 & \text{on} \quad \Sigma \\ u(x,0) = u^0(x) & \text{in} \quad \Omega. \end{cases}$$

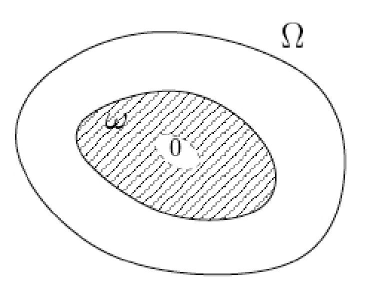
$1_\omega$ denotes the characteristic function of the subset $\omega$ of $\Omega$ where the control is active.
We assume that $u^0 \in L^2(\Omega)$ and $f \in L^2(Q)$:

$$\lambda < \lambda_* \Rightarrow u \in C\left([0, T]\, ; L^2(\Omega)\right) \cap L^2\left(0, T; H_0^1(\Omega)\right).$$

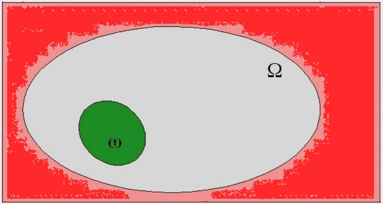
$$\lambda = \lambda_* \Rightarrow u \in C\left([0, T]\, ; L^2(\Omega)\right) \cap L^2\left(0, T; \mathcal{H}(\Omega)\right).$$

$u = u(x,t) = solution = state,\ f = f(x,t) = control$

We assume that the control subdomain contains and annulus:




Open problem: Obtain the same results for general subdomains $\omega$ as in the context of the heat equation: $\lambda = 0$.



We address the problem of null controllability: For all $u^0 \in L^2(\Omega)$ show the existence of $f \in L^2(\omega \times (0, T)$ such that:

$$u(T) \equiv 0.$$

Only makes sense if $\lambda \leq \lambda_*$.

The main result (J. Vancostenoble & E. Z., 2007):

**Theorem**

*For all $T > 0$, annular domain $\omega$ and $\lambda \leq \lambda_*$ null controllability holds.*

Note that, due to the regularizing effect, the subtle change in the functional setting between the cases $\lambda < \lambda_*$ and $\lambda = \lambda_*$ does not affect the final control result.

The control, $f = \tilde{\varphi}$, where $\tilde{\varphi}$ minimizes:

$$J_0(\varphi^0) = \frac{1}{2}\int_0^T \int_\omega \varphi^2 dxdt + \int_\Omega \varphi(0) u^0 dx$$

among the solutions of the adjoint system:

$$\begin{cases} -\varphi_t - \Delta\varphi - \lambda \frac{\varphi}{|x|^2} = 0 & \text{in } Q, \\ \varphi = 0 & \text{on } \Sigma, \\ \varphi(0,x) = \varphi^0(x) & \text{in } \Omega. \end{cases}$$

The key ingredient, needed to prove its coercivity, is the observability inequality:

$$\| \varphi(0) \|^2_{L^2(\Omega)} \leq C \int_0^T \int_\omega \varphi^2 dxdt, \quad \forall \varphi^0 \in L^2(\Omega).$$

The main tool for obtaining such estimates are the Carleman inequalities as developed by Fursikov and Imanuvilov (1996).[2] For heat equations with a bounded potentials $V = V(x)$ the following holds:

$$\| \varphi(0) \|^2_{(L^2(\Omega))^N} \leq \exp\left(C\left(1 + \frac{1}{T} + T \| V \|_\infty + \| V \|_\infty^{2/3}\right)\right) \int_0^T \int_\omega |\varphi|^2 dxdt. \quad (1)$$

It does not apply for singular potentials $V = \lambda |x|^{-2}$.

Goal: Combine, as done in the well-posedness of the Cauchy and boundary value problems, Hardy and Carleman inequalities.

[2] A. V. Fursikov and O. Yu. Imanuvilov, *Controllability of evolution equations*, Lecture Notes Series # 34, Research Institute of Mathematics, Global Analysis Research Center, Seoul National University, 1996.

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## Sketch of the proof:

Step 1. *Heat equation.*

Introduce a function $\eta^0 = \eta^0(x)$ such that:

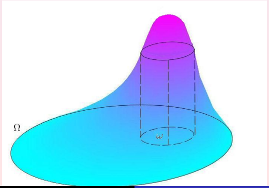
$$\begin{cases} \eta^0 \in C^2(\bar{\Omega}) & \\ \eta^0 > 0 & \text{in} \quad \Omega,\ \eta^0 = 0 \quad \text{in} \quad \partial\Omega \\ \nabla \eta^0 \neq 0 & \text{in} \quad \overline{\Omega \backslash \omega}. \end{cases} \tag{2}$$

Let $k > 0$ such that $k \geq 5 \max_{\bar{\Omega}} \eta^0 - 6 \min_{\bar{\Omega}} \eta^0$ and let

$$\beta^0 = \eta^0 + k, \bar{\beta} = \frac{5}{4} \max \beta^0,\ \rho^1(x) = e^{\lambda \bar{\beta}} - e^{\lambda \beta^0}$$

with $\lambda, \bar{\beta}$ sufficiently large. Let be finally

$$\gamma = \rho^1(x)/(t(T-t)); \rho(x,t) = \exp(\gamma(x,t)).$$



There exist positive constants $C_*, s_1 > 0$ such that

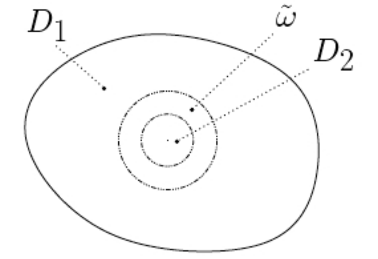
$$s^3 \int_Q \rho^{-2s} t^{-3}(T-t)^{-3} q^2 dxdt \\ \leq C_* \int_Q \rho^{-2s} \left[ |\partial_t q - \Delta q|^2 + s^3 t^{-3}(T-t)^{-3} 1_\omega q^2 \right] dxdt$$

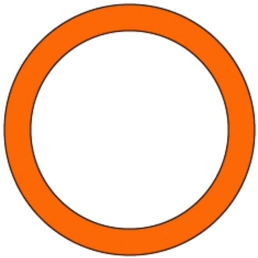
for all smooth $q$ vanishing on the lateral boundary and $s \geq s_1$.

Step 2. *Cut-off.*

Cutting-off the domain, we may:

- The previous estimate in the exterior domain $|x| \geq r$ where the potential $\lambda|x|^{-2}$ is bounded;
- Concentrate in the case where $\Omega = B_1$ and $\omega$ is a neighborhood of the boundary.



Step 3. *Spherical harmonics.* To fix ideas $N = 3$, $\lambda = \lambda_* = \frac{1}{4}$.
The most singular component is the one corresponding to radially symmetric solutions:

$$-\varphi_t - \varphi_{rr} - 2\frac{\varphi_r}{r} - \frac{\varphi}{4r^2} = 0.$$

After the change of variables $\psi = r^{1/2}\varphi$,

$$-\psi_t - \psi_{rr} - \frac{\psi_r}{r} = 0.$$

This is the $2 - d$ heat equation for $\psi$.
The standard Carleman inequality can be applied getting:

$$\int_0^1 \psi^2(r,0) r\, dr \leq C \int_0^T \int_a^1 \psi^2 r\, drdt$$

Going back to $\varphi$ we recover the observability inequality for $\varphi$ too, in its corresponding norm:

$$\int_0^1 \varphi^2(r,0) r^2\, dr \leq C \int_0^T \int_a^1 \varphi^2 r^2\, drdt.$$

Step 4. *Higher order harmonics.*
Even though for higher order harmonics the elliptic operator involved is more coercive, the potential is still singular and the existing Carleman inequalities can not be derived:

$$-\varphi_t - \varphi_{rr} - 2\frac{\varphi_r}{r} - \frac{\varphi}{4r^2} + c_j\frac{\varphi}{r^2} = 0,$$

$c_j$ being the eigenvalues of the Laplace-Beltrami operator.
This can be done by making a careful choice of the Carleman weight, exploiting the monotonicity properties of the potential. [3]

In the Carleman inequality obtained this way, there is an extra weight factor $|x|^2$ which compensates the presence of the singularity at $x = 0$.

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[3] Argument inspired in works by P. Cannarsa, P. Martinez, J. Vancostenoble, *Carleman estimates for a class of degenerate parabolic operators*, SIAM J. Control Optim., to be published.

# The supercritical case in the ball:

Similarly, for $\lambda > \lambda_*$ one can get, well-posedness and observability inequalities for sufficiently high frequency spherical harmonics such that $\lambda - \lambda_* < c_j$.

This allows to control to zero those highly oscillatory initial data.

Note that, this is not in contradiction with the instantaneous blow-up result by Baras-Goldstein that is mainly concerned with positive solutions that this argument does not address.

# Wave equation:

Under the condition $\lambda \leq \lambda_*$:

$$\begin{cases} \varphi_{tt} - \Delta\varphi - \lambda\frac{\varphi}{|x|^2} = 0 & \text{in } Q, \\ \varphi = 0 & \text{on } \Sigma, \\ \varphi(0,x) = \varphi^0(x), \varphi_t(0,x) = \varphi^1(x) & \text{in } \Omega. \end{cases}$$

The energy

$$E_\lambda(t) = \frac{1}{2}\int_\Omega \left[|\varphi_t|^2 + |\nabla\varphi|^2 - \lambda\frac{\varphi^2}{|x|^2}\right]dx,$$

is conserved, and it is coercive either in $H_0^1 \times L^2$ for $\lambda < \lambda_*$, or in $\mathcal{H} \times L^2$ for $\lambda = \lambda_*$.

Multipliers $(x \cdot \nabla\varphi)$:

$$TE_\lambda(0) + \int_\Omega \varphi_t\left(x\cdot\varphi + \frac{N-1}{2}\varphi\right)dx\Big|_0^T \leq \frac{R}{2}\int_\Sigma \left|\frac{\partial\varphi}{\partial\nu}\right|^2 d\Sigma,$$

$$\left|\int_\Omega \varphi_t\left(x\cdot\varphi + \frac{N-1}{2}\varphi\right)dx\Big|_0^T\right| \leq 2RE_0.$$

In principle:

- If $\lambda < \lambda_*$ this yields the observability inequality if

$$T > \frac{2R}{\left[1-\lambda/\lambda_*\right]^{1/2}} : E_\lambda(0) \leq C\int_\Sigma \left|\frac{\partial\varphi}{\partial\nu}\right|^2 d\Sigma.$$

- For $\lambda = \lambda_*$ observability fails, apparently, since the energy $E_\lambda$ is not coercive in $H_0^1 \times L^2$ and the term $2RE_0$ may not be estimated.

But things are better: $N = 3$, $\lambda = \lambda_* = 1/4$.
Again using spherical harmonics decomposition the most singular component is the radial one and, after the change of variables $\psi(r,t) = r^{1/2}\varphi(r,t)$, the problem reduces to

$$\psi_{tt} - \psi_{rr} - r\psi_r = 0,$$

which is the wave equation in $2 - d$ in radial coordinates.
Then observability holds and we recover:

$$E_\lambda(0) \leq \int_\Sigma \left| \frac{\partial \varphi}{\partial \nu} \right|^2 d\Sigma$$

for $T > 2R$.

# Open problems:

- Other singular problems (cf. Murat's lecture):

$$-\Delta u = |\nabla u|^q, \; u(x) = c_q(|x|^{-(2-q)/(q-1)} - 1)$$

Linearization:

$$-\Delta v = q|\nabla u|^{q-2}\nabla u \cdot \nabla v \sim -\Delta v = \mu \frac{1}{|x|^2} x \cdot \nabla v.$$

In radial coordinates:

$$-v_{rr} - N - 1\frac{v_r}{r} - \mu \frac{v_r}{r} = 0.$$

Coercivity?

$$\left| \int \frac{1}{|x|^2} x \cdot \nabla v \, v dx \right| = \frac{1}{2} \int \frac{1}{|x|^2} x \cdot \nabla (v^2) dx = \frac{N-2}{2} \int \frac{v^2}{|x|^2} dx$$

$$\leq \left( \frac{N-2}{2} \right)^3 \int |\nabla v|^2 dx.$$

Hardy constant

$$\left(\frac{N-2}{2}\right)^2.$$

Hardy-Murat constant

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**Theorem**

*Hardy + Murat > Hardy if and only if $N \geq 4$.*

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**Theorem**

*Hardy + Murat > Hardy if and only if $N \geq 4$.*

- Heat equation with inverse square Laplacian and arbitrary $\omega$.
- Incorporate positivity to Carleman estimates not to have to deal with all potentials of the form $\frac{\mu}{|x|^2}$, with $\mu > 0$...
- Nonlinear problems:
    - This may be done in a standard way around bounded stationary solutions (that act as atractors in dimensions $3 \leq N \leq 9$);
    - Not for singular ones, which do exist in dimensions $N \geq 10$.

    $$-\Delta u - \lambda e^u = 0, \ \lambda = 2(N-2), \ u(x) = -2\log(|x|).$$

    And

    $$2(N-2) \leq \lambda_* \sim N \geq 10.$$

    Recall that the Inverse Function Theorem works badly even when linearizing the elliptic problem.
    $u \sim v$ in $H_0^1$ does not mean that $e^u \sim e^v$ in $H^{-1}$.
- Wave equation: Better explain the propagation phenomena using bicharacteristic rays (semi-classical, Wigner, $H$-measures,...).

- Heat equation with inverse square Laplacian and arbitrary $\omega$.
- Incorporate positivity to Carleman estimates not to have to deal with all potentials of the form $\frac{\mu}{|x|^2}$, with $\mu > 0$...
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To learn more on this topic:

F. Mignot, J.-P. Puel, *Solution radiale singulière de* $-\Delta u = \lambda e^u$, C. R. Acad. Sci. Paris Sr. I Math. 307 (1988), no. 8, 379–382.

And if you still want more...

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