A regularity result for a solid-fluid system associated to the compressible Navier-Stokes system

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Abstract

In this paper we deal with a fluid-structure interaction problem for a compressible fluid and a rigid structure immersed in a regular bounded domain in dimension 3. The fluid is modelled by the compressible Navier-Stokes system in the barotropic regime with no-slip boundary conditions and the motion of the structure is described by the usual law of balance of linear and angular moment.

The main result of the paper states that, for small initial data, we have the existence and uniqueness of global smooth solutions as long as no collisions occur. This result is proved in two steps; first, we prove the existence and uniqueness of local solution and then we establish some a priori estimates independently of time.

Résumé

Dans cet article, nous considérons un problème d'interaction fluide-structure entre un fluide compressible et une structure rigide évoluant à l'intérieur d'un domaine borné et régulier en dimension 3. Le fluide est décrit par le système de Navier-Stokes compressible barotrope avec des conditions de non-glissement sur le bord et le mouvement de la structure est régi par les lois de conservation des moments linéaire et angulaire.

Nous montrons, pour des données initiales petites, l'existence et l'unicité de solutions globales régulières tant qu'il n'y a pas de chocs. Ce résultat est obtenu en deux temps; tout d'abord, nous prouvons l'existence et l'unicité de solutions locales puis nous démontrons des estimations a priori indépendamment du temps.

1 Introduction

1.1 Statement of problem

We consider a rigid structure immersed in a viscous compressible fluid. At time t, we denote by $\Omega_S(t)$ the domain occupied by the structure. The structure and the fluid are contained in a fixed bounded domain $\Omega \subset \mathbb{R}^3$. We suppose that the boundaries of $\Omega_S(0)$ and Ω are smooth (C^3 for instance) and that

$$\Omega_S(0) \text{ is convex}, \quad d(\Omega_S(0), \partial\Omega) > 0. \tag{1}$$

For any $t > 0$, we denote by $\Omega_F(t) = \Omega \setminus \overline{\Omega_S(t)}$ the region occupied by the fluid. The time evolution of the eulerian velocity u and the density ρ in the fluid are governed by the compressible Navier-Stokes equations and the continuity equation: $\forall\, t > 0$, $\forall\, x \in \Omega_F(t)$

$$\begin{cases} \big(\rho_t + \nabla\cdot(\rho u)\big)(t,x) = 0, \\ \big(\rho u_t + \rho(u\cdot\nabla)u\big)(t,x) - \nabla\cdot\big(2\mu\epsilon(u) + \mu'(\nabla\cdot u)Id\big)(t,x) + \nabla p(t,x) = 0, \end{cases} \tag{2}$$

where $\epsilon(u) = \frac{1}{2}(\nabla u + \nabla u^t)$ denotes the symmetric part of the gradient. The viscosity coefficients μ and μ' are real constants which are supposed to satisfy

$$\mu > 0,\ \mu + \mu' \geq 0. \tag{3}$$

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We suppose that we are in a barotropic regime where a constitutive law gives the relation between the pressure of the fluid p and the density ρ. Thus, we suppose that

$$p = P(\rho) \text{ where } P \in C^\infty(\mathbb{R}_+^*),\ P(\rho) > 0 \text{ and } P'(\rho) > 0,\ \forall\, \rho > 0. \tag{4}$$

For instance, $P(\rho) := \rho^\gamma$ with $\gamma > 0$ is admissible.

Concerning the compressible fluids, a local in time result of existence and uniqueness of a smooth solution was proved in [21]. In [18], the authors proved the existence and uniqueness of a regular solution for small initial data and external forces.

Next, for isentropic fluids ($P(\rho) = \rho^\gamma$, $\gamma > 0$), the global existence of a weak solution for small initial data was proved in [13] (for $\gamma = 1$) and in [14] (for $\gamma > 1$). Also for an isentropic fluid, the first global result for large data was proved in [15] (see also [16]) (with $\gamma \geq 9/5$ for dimension $N = 3$ and with $\gamma > N/2$ for $N \geq 4$). Finally, this last result was improved in [9] (see also [11]) (with $\gamma > N/2$ for $N \geq 3$).

At time t, the motion of the rigid structure is given by the position $a(t) \in \mathbb{R}^3$ of the center of mass and by a rotation (orthogonal) matrix $Q(t) \in \mathbb{M}_{3\times 3}(\mathbb{R})$. Without loss of generality, we can suppose that

$$a(0) = 0 \text{ and } Q(0) = Id. \tag{5}$$

At time t, the domain occupied by the structure $\Omega_S(t)$ is defined by

$$\Omega_S(t) = \chi_S(t, \Omega_S(0)), \tag{6}$$

where χ_S denotes the flow associated to the motion of the structure:

$$\chi_S(t, x) = a(t) + Q(t)x,\ \forall\, x \in \Omega_S(0),\ \forall\, t > 0. \tag{7}$$

We notice that, for each $t > 0$, $\chi_S(t,\cdot) : \Omega_S(0) \to \Omega_S(t)$ is invertible and

$$\chi_S(t,\cdot)^{-1}(x) = Q(t)^{-1}(x - a(t)),\ \forall\, x \in \Omega_S(t).$$

Thus, the eulerian velocity of the structure is given by

$$(\chi_S)_t(t,\cdot) \circ \chi_S(t,\cdot)^{-1}(x) = \dot{a}(t) + \dot{Q}(t)Q(t)^{-1}(x - a(t)),\ \forall\, x \in \Omega_S(t).$$

Since $\dot{Q}(t)Q(t)^{-1}$ is skew-symmetric, for each $t > 0$, we can represent this matrix by a unique vector $\omega(t) \in \mathbb{R}^3$ such that

$$\dot{Q}(t)Q(t)^{-1}y = \omega(t) \wedge y,\ \forall\, y \in \mathbb{R}^3.$$

Reciprocally, if ω belongs to $L^2(0,T)$, then there exists a unique matrix $Q \in H^1(0,T)$ such that $Q(0) = Id$ and which satisfies this formula.
Thus, the eulerian velocity u_S of the structure is given by

$$u_S(t,x) = \dot{a}(t) + \omega(t) \wedge (x - a(t)),\ \forall\, x \in \Omega_S(t). \tag{8}$$

For the equations of the structure, we denote by $m > 0$ the mass of the rigid structure and $J(t) \in \mathbb{M}_{3\times 3}(\mathbb{R})$ its tensor of inertia at time t. This tensor is given by

$$J(t)b \cdot \tilde{b} = \int_{\Omega_S(0)} \rho_{0,S}(x)\big(b \wedge Q(t)x\big) \cdot \big(\tilde{b} \wedge Q(t)x\big)\, dx \quad \forall b,\, \tilde{b} \in \mathbb{R}^3, \tag{9}$$

where $\rho_{0,S} > 0$ is the initial density of the structure. One can prove that

$$J(t)b \cdot b \geq C_J|b|^2 > 0 \text{ for all } b \in \mathbb{R}^3 \setminus \{0\}, \tag{10}$$

where C_J is independent of $t > 0$. The equations of the structure motion are given by the balance of linear and angular momentum. We have, for all $t \in (0,T)$

$$\begin{cases} m\ddot{a} = \displaystyle\int_{\partial\Omega_S(t)} \big(2\mu\epsilon(u) + \mu'(\nabla\cdot u)Id - pId\big)n\, d\sigma, \\ J\dot{\omega} = \big(J\omega\big) \wedge \omega + \displaystyle\int_{\partial\Omega_S(t)} (x - a) \wedge \Big(\big(2\mu\epsilon(u) + \mu'(\nabla\cdot u)Id - pId\big)n\Big)\, d\sigma. \end{cases} \tag{11}$$

In these equations, n is the outward unit normal to $\partial\Omega_S(t)$. On the boundary of the fluid, the eulerian velocity has to satisfy a no-slip boundary condition. Therefore, we have, for all $t > 0$

$$\begin{cases} u(t,x) = 0, \forall\, x \in \partial\Omega, \\ u(t,x) = \dot{a}(t) + \omega(t) \wedge (x - a(t)), \,\forall\, x \in \partial\Omega_S(t). \end{cases} \tag{12}$$

The system is completed by the following initial conditions:

$$u(0,\cdot) = u_0 \text{ in } \Omega_F(0),\, \rho(0,\cdot) = \rho_0 \text{ in } \Omega_F(0),\, a(0) = 0,\, \dot{a}(0) = a_0,\, \omega(0) = \omega_0, \tag{13}$$

which satisfy

$$a_0,\, \omega_0 \in \mathbb{R}^3,\;\; \rho_0,\, u_0 \in H^3(\Omega_F(0)),\;\; \rho_0(x) > 0, \forall\, x \in \Omega_F(0). \tag{14}$$

Since we will deal with smooth solutions, we will also need some compatibility conditions to be satisfied:

$$u_0 = a_0 + \omega_0 \wedge x \text{ on } \partial\Omega_S(0), \quad u_0 = 0 \text{ on } \partial\Omega \tag{15}$$

and

$$\begin{aligned} &\frac{1}{\rho_0}\nabla\cdot(2\mu\epsilon(u_0) + \mu'(\nabla\cdot u_0)Id) - \frac{1}{\rho_0}\nabla P(\rho_0) = \frac{1}{m}\int_{\partial\Omega_S(0)} (2\mu\epsilon(u_0) + \mu'(\nabla\cdot u_0)Id - P(\rho_0)Id)n\, d\sigma \\ &+(J(0)^{-1}(J(0)\omega_0)\wedge\omega_0)\wedge x + J(0)^{-1}\left(\int_{\partial\Omega_S(0)} x \wedge ((2\mu\epsilon(u_0) + \mu'(\nabla\cdot u_0)Id - P(\rho_0)Id)n)\, d\sigma\right)\wedge x \\ &+\omega_0\wedge(\omega_0\wedge x) \text{ on } \partial\Omega_S(0), \\ &\nabla\cdot(2\mu\epsilon(u_0) + \mu'(\nabla\cdot u_0)Id) - \nabla P(\rho_0) = 0 \text{ on } \partial\Omega \end{aligned} \tag{16}$$

These two conditions are formally obtained by differentiating system (12) with respect to time and taking $t = 0$. To do this, we consider the second equation of (12) on a fix domain by setting $x = \chi_S(t,y)$, with $y \in \partial\Omega_S(0)$.

Let us now recall some of the most relevant results in problems of fluid-structure interaction. In the below lines, when we refer to a global result, we mean before collision (in the case of a rigid solid) or before interpenetration of the structure (in the case of an elastic solid).

• Incompressible fluids:

As long as rigid solids are concerned, a local result was proved in [12], while the existence of global weak solutions is proved in [5] and [7] (with variable density) and [19] ($2D$, with variable density); in this last paper, the existence of a solution is proved even beyond collisions. Later, the existence and uniqueness of strong global solutions in $2D$ was proved in [20] as well as the local in time existence and uniqueness of strong solutions in $3D$.

When talking about elastic solids, a first existence result of weak solution was proved in [8], when the elastic deformation is given by a finite sum of modes. A local existence result of a strong solution for an elastic plate was proved in [1] ($2D$). The local existence of a strong solution is proved in [6]. In [4] and [3] (with variable density), the authors proved the global existence of a weak solution.

• Compressible fluids:

Concerning rigid solids, the global existence of a weak solution was proved in [7] for $\gamma \geq 2$ and in [10] for $\gamma > N/2$. For elastic solids, in [2] the author proved the global existence of a weak solution in $3D$ for $\gamma > 3/2$.

In this paper, we will prove the existence and uniqueness of smooth global solutions for small initial data (Theorem 3). We can also prove the same result for initial data close to a stationary solution $(\rho, u, a, \omega) = (\rho_e, 0, 0, 0)$ and for special right hand sides (see Remark 6 for more details).

We give a lemma which allows to extend the flow χ_S by a flow χ defined on the global domain Ω.

Lemma 1 *Let $T \in (0,+\infty)$ and $(a,\omega) \in H^3(0,T) \times H^2(0,T)$ be given. We suppose that (a,Q) satisfies (5) where $Q \in H^3(0,T)$ is the rotation matrix associated to ω. We consider the associated flow χ_S, the eulerian velocity u_S and the domain defined by (6) to (8). We suppose that there exists $\alpha > 0$ such that*

$$\forall\, t \in [0,T],\ d(\Omega_S(t), \partial\Omega) \geq \alpha > 0. \tag{17}$$

Then, we can extend the flow χ_S by a flow $\chi \in H^3(0,T; C^\infty(\Omega))$ such that

- $\chi(t,x) = a(t) + Q(t)x$, $\forall\, x \in \Omega_S(0)$, $\forall\, t \in (0,T)$.
- $\chi(t,x) = x$, $\forall\, x \in \partial\Omega$, $\forall\, t \in (0,T)$.
- *For all $t \in (0,T)$, $\chi(t,\cdot)$ is invertible from Ω onto Ω and from $\Omega_F(0)$ onto $\Omega_F(t)$ and*

  $$(t,x) \in (0,T) \times \Omega \to \chi(t,\cdot)^{-1}(x)$$

  belongs to $H^3(0,T; C^\infty(\Omega))$.
- *The eulerian velocity $v \in H^2(0,T; C_o^\infty(\Omega))$ associated to χ satisfies*

  $$v(t,x) = u_S(t,x),\ \forall\, t \in (0,T),\ \forall\, x \in \Omega_S(t).$$

- *For $1 \leqslant i \leqslant 3$ and $k \in \mathbf{N}$ there exists two positive functions C_0 and Θ such that*

  $$\|\chi - I\|_{H^i(0,T;C^k(\Omega))} \leqslant C_0(T)\Theta(\|a\|_{H^i(0,T)} + \|\omega\|^2_{H^{i-1}(0,T)} + \|a\|^2_{W^{i-1,\infty}(0,T)}). \tag{18}$$

  Here, C_0 is a function of α, Ω and T and is an increasing function of T. Moreover, $\Theta : \mathbb{R}_+ \to \mathbb{R}_+$ is a regular function which vanishes at 0 at least like a linear function, that is to say, there exists $\varepsilon > 0$ and $C > 0$ such that

  $$\Theta(x) \leqslant Cx \quad \forall x \leqslant \varepsilon. \tag{19}$$

Proof: The eulerian velocity defined by (8) can be extended by a velocity $v \in H^2(0,T; C_o^\infty(\Omega))$ such that, for $1 \leqslant i \leqslant 3$

$$\|v\|_{H^{i-1}(0,T;C_c^\infty(\Omega))} \leqslant C(\|a\|_{H^i(0,T)} + \|\omega\|^2_{H^{i-1}(0,T)} + \|a\|^2_{W^{i-1,\infty}(0,T)}), \tag{20}$$

for $C > 0$. For instance, we take v as the solution of the following elliptic equation:

$$\begin{cases} -\Delta v = 0 & \text{in } \Omega_F(t), \\ v = \dot{a} + \omega \wedge (x-a) & \text{on } \partial\Omega_S(t), \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{21}$$

Next, we can define the flow χ associated to v. For all $y \in \Omega$, $\chi(\cdot,y)$ is defined on $(0,T)$ as the solution of the ordinary differential equation

$$\begin{cases} \chi_t(t,y) &=& v(t,\chi(t,y)), \\ \chi(0,y) &=& y. \end{cases} \tag{22}$$

On $\partial\Omega$, since $v = 0$, $\chi(t,x) = x$. Moreover, according to the uniqueness of the flow, $\chi = \chi_s$ in $\Omega_S(0)$. The last point comes from (20) and the fact that

$$\chi(t,y) - y = \int_0^t v(s,\chi(s,y))\, ds. \quad \square$$

1.2 Main result

Definition 2 *: Let $\Psi : (0,T) \times \Omega \mapsto \Omega$ be such that, for all $t \in (0,T)$, $\Psi(t,\cdot)$ is a C^2-diffeomorphism from Ω to Ω and belongs to $C^2(0,T;C^\infty(\Omega))$. Let $S \subset \Omega$ be a regular domain and assume that $\Psi(t,\cdot)$ is also a C^2-diffeomorphism from S to $S(t) := \Psi(t,S)$. For a function $u(t,\cdot) : S(t) \mapsto \mathbb{R}$, we consider*

$$\tilde{u}(t,y) = u(t,\Psi(t,y)),\ \forall\, t \in (0,T),\ \forall\, y \in S.$$

Then, we define, for all $k \in \mathbb{N}$ and for $l = 0,1,2$,

$$C^l\big(0,T;H^k\big(S(t)\big)\big) = \Big\{u \,/\, \tilde{u} \in C^l\big(0,T;H^k(S)\big)\Big\},$$

$$H^l\big(0,T;H^k\big(S(t)\big)\big) = \Big\{u \,/\, \tilde{u} \in H^l\big(0,T;H^k(S)\big)\Big\}.$$

The main goal of our paper is to prove the following theorem:

Theorem 3 *Let $\overline{\rho}$ be the mean-value of ρ_0 in $\Omega_F(0)$. We suppose that (1), (3) and (4) are satisfied. We suppose that the initial conditions satisfy (5), (14) and the compatibility conditions (15)-(16). Then there exists a constant $\delta > 0$ such that, if*

$$\|\rho_0 - \overline{\rho}\|_{H^3(\Omega_F(0))} + \|u_0\|_{H^3(\Omega_F(0))} + |a_0| + |\omega_0| < \delta, \tag{23}$$

the system of equations (2), (11), (12) and (13) admits a unique solution (ρ,u,a,ω) defined on $(0,T)$ for all T such that (17) is satisfied. Moreover, this solution belongs to the following space

$$\begin{aligned}
&\rho \in C([0,T];H^3(\Omega_F(t))) \cap C^1([0,T];H^2(\Omega_F(t))) \cap H^2(0,T;L^2(\Omega_F(t))),\\
&u \in L^2(0,T;H^4(\Omega_F(t))) \cap C([0,T];H^3(\Omega_F(t))) \cap C^1([0,T];H^1(\Omega_F(t))) \cap H^2(0,T;L^2(\Omega_F(t))),\\
&a \in H^3(0,T),\ \omega \in H^2(0,T)
\end{aligned}$$

and there exists a positive constant C_1 independent of T such that

$$\begin{aligned}
&\|\rho - \overline{\rho}\|_{L^\infty(0,T;H^3(\Omega_F(t)))} + \|\rho_t\|_{L^\infty(0,T;H^2(\Omega_F(t)))} + \|\rho_{tt}\|_{L^2(0,T;L^2(\Omega_F(t)))} + \|\rho_t\|_{L^2(0,T;H^2(\Omega_F(t)))}\\
&+\|u\|_{L^2(0,T;H^4(\Omega_F(t)))} + \|u\|_{L^\infty(0,T;H^3(\Omega_F(t)))} + \|u_t\|_{L^\infty(0,T;H^1(\Omega_F(t)))} + \|u_{tt}\|_{L^2(0,T;L^2(\Omega_F(t)))}\\
&+\|a\|_{W^{2,\infty}(0,T)} + \|a\|_{H^3(0,T)} + \|\omega\|_{W^{1,\infty}(0,T)} + \|\omega\|_{H^2(0,T)}\\
&\qquad\qquad \leqslant C_1(\|\rho_0 - \overline{\rho}\|_{H^3(\Omega_F(0))} + \|u_0\|_{H^3(\Omega_F(0))} + |a_0| + |\omega_0|).
\end{aligned} \tag{24}$$

Remark 4 *In Theorem 3, we prove the existence and uniqueness of a regular solution of the system of equations (2), (11), (12) and (13) provided that the structure does not touch $\partial\Omega$ (see (17)). Observe that this condition will always be satisfied on an interval $(0,T_{min})$, where $T_{min} > 0$ only depends on C_1, δ and $d(\partial\Omega, \overline{\Omega}_S(0))$, as easily seen from estimate (24).*

Remark 5 *This condition on T is natural for fluid-structure interaction problems (see, for instance, [7], [5] and [20]). For results concerning the existence of weak solutions of this type of systems after a collision has occurred, we refer to [19] and [10].*

Remark 6 *As proved in [18], there exists a unique stationary solution $(\rho_e(x),0,0,0)$ of (2), (12), where, on the fluid, the compressible Navier-Stokes equation is completed with a right hand side of the form $f_i(x) = \partial_i\phi(x)\,(i = 1,2,3)$ satisfying (23) for the H^3-norm. Then, we can prove that Theorem 3 also holds with $\overline{\rho}$ replaced by ρ_e and for a right hand side of the above kind.*

2 Rewriting of the problem and intermediate results

2.1 Statement of the problem in a fixed domain

The system of equations can be written on the reference domains $\Omega_S(0)$ and $\Omega_F(0)$ with the help of the flow defined by Lemma 1. We consider (ρ, u, a, ω) which satisfies the system (2)-(11)-(12) with the hypothesis (3)-(4). We suppose that (17) is satisfied for some $\alpha > 0$ and that the functions ρ, u, a and ω are regular enough. Let us define the functions $\tilde{u}$, $\tilde{\rho}$ and $\tilde{p}$ on $(0,T)\times\Omega_F(0)$ by

$$\tilde{u}(t,x) = u(t,\chi(t,x)),\ \tilde{\rho}(t,x) = \rho(t,\chi(t,x)) - \overline{\rho},\ \tilde{p}(t,x) = P(\tilde{\rho}(t,x)+\overline{\rho}),\ \forall\, t\in(0,T),\ \forall\, x\in\Omega_F(0). \tag{25}$$

Moreover, we also define $\tilde{v}$ by

$$\tilde{v}(t,x) = v(t,\chi(t,x)) = \chi_t(t,x),\ \forall\, t\in(0,T),\ \forall\, x\in\Omega_F(0)$$

where v is given by lemma 1. We have the following formulas, $\forall\,(t,x)\in(0,T)\times\Omega_F(0)$,

$$\tilde{\rho}_t(t,x) = \rho_t(t,\chi(t,x)) + \left(\nabla\chi(t,x)^{-1}\tilde{v}(t,x)\right)\cdot\nabla\tilde{\rho}(t,x) = \rho_t(t,\chi(t,x)) + \sum_{j,k=1}^{3}\left(((\nabla\chi)^{-1})_{jk}\tilde{v}_k\partial_{x_j}\tilde{\rho}\right)(t,x)$$

$$(\nabla\tilde{u})_{ij}(t,x) = \partial_{x_j}\tilde{u}_i = (\nabla u(t,\chi(t,x))\nabla\chi(t,x))_{ij} = \sum_{k=1}^{3}\partial_{x_k}u_i(t,\chi(t,x))\partial_{x_j}\chi_k(t,x).$$

Thus we get, for the first equation of system (2),

$$\tilde{\rho}_t + ((\nabla\chi)^{-1}(\tilde{u}-\tilde{v}))\cdot\nabla\tilde{\rho} + (\tilde{\rho}+\overline{\rho})\,\mathrm{tr}\big(\nabla\tilde{u}(\nabla\chi)^{-1}\big) = 0,\ \text{ on } (0,T)\times\Omega_F(0). \tag{26}$$

Next, the second equation of (2) becomes, $\forall\, i=1,2,3$, on $(0,T)\times\Omega_F(0)$

$$\begin{aligned}&(\tilde{\rho}+\overline{\rho})(\tilde{u}_i)_t + (\tilde{\rho}+\overline{\rho})(\tilde{u}_j-\tilde{v}_j)(\nabla\tilde{u}(\nabla\chi)^{-1})_{ij} - \mu\partial_{x_l}(\partial_{x_k}\tilde{u}_i(\nabla\chi)^{-1}_{kj})(\nabla\chi)^{-1}_{lj}\\ &-(\mu+\mu')\partial_{x_l}(\partial_{x_k}\tilde{u}_j(\nabla\chi)^{-1}_{kj})(\nabla\chi)^{-1}_{li} + (\nabla\chi)^{-1}_{ki}\partial_{x_k}\tilde{p} = 0.\end{aligned} \tag{27}$$

In this equation and in what follows, we implicitly sum over repeated indexes. Equations (11) become:

$$\left\{\begin{aligned}&m\ddot{a} = \int_{\partial\Omega_S(0)}\Big(\mu\big(\nabla\tilde{u}(\nabla\chi)^{-1} + (\nabla\chi)^{-t}(\nabla\tilde{u})^t\big) + \mu'\mathrm{tr}(\nabla\tilde{u}(\nabla\chi)^{-1})Id - \tilde{p}Id\Big)Qn\,d\sigma,\\ &J\dot{\omega} = \big(J\omega\big)\wedge\omega\\ &+\int_{\partial\Omega_S(0)}(Qx)\wedge\Big(\big(\mu\big(\nabla\tilde{u}(\nabla\chi)^{-1} + (\nabla\chi)^{-t}(\nabla\tilde{u})^t\big) + \mu'\mathrm{tr}(\nabla\tilde{u}(\nabla\chi)^{-1})Id - \tilde{p}Id\big)Qn\Big)\,d\sigma,\end{aligned}\right. \tag{28}$$

where we have denoted $A^{-t} = (A^t)^{-1}$ At last, the boundary conditions (12) become

$$\left\{\begin{aligned}&\tilde{u}(t,x) = 0, \forall\, x\in\partial\Omega,\\ &\tilde{u}(t,x) = \dot{a}(t) + \omega(t)\wedge(Qx),\ \forall\, x\in\partial\Omega_S(0).\end{aligned}\right. \tag{29}$$

Let us define p^0 by

$$p^0 := \frac{P'(\overline{\rho})}{\overline{\rho}},$$

and we recall the definition of $\overline{\rho}$:

$$\overline{\rho} = \frac{1}{V(\Omega_F(0))}\int_{\Omega_F(0)}\rho_0,$$

where $V(\Omega_F(0))$ stands for the volume of $\Omega_F(0)$. Furthermore, we change $m/\overline{\rho}$ into m, $J/\overline{\rho}$ into J, $\mu/\overline{\rho}$ into μ and $\mu'/\overline{\rho}$ into μ'. Thus, the system of equations (26) to (28) can be written as follows:

$$\begin{cases} \tilde{\rho}_t + ((\nabla\chi)^{-1}(\tilde{u}-\tilde{v}))\cdot\nabla\tilde{\rho} + \overline{\rho}\,\nabla\cdot\tilde{u} = f_0(\tilde{\rho},\tilde{u},a,\omega) & \text{in } (0,T)\times\Omega_F(0),\\ \tilde{u}_t - 2\mu\nabla\cdot(\epsilon(\tilde{u})) - \mu'\nabla\cdot((\nabla\cdot\tilde{u})Id) + p^0\nabla\tilde{\rho} = f_1(\tilde{\rho},\tilde{u},a,\omega) & \text{in } (0,T)\times\Omega_F(0),\\ m\ddot{a} = \displaystyle\int_{\partial\Omega_S(0)} \Big(2\mu\epsilon(\tilde{u}) + \mu'(\nabla\cdot\tilde{u})Id - p^0\tilde{\rho}Id\Big)n\,d\sigma + f_2(\tilde{\rho},\tilde{u},a,\omega) & \text{in } (0,T),\\ J\dot{\omega} = \displaystyle\int_{\partial\Omega_S(0)} (Qx)\wedge\Big(\big(2\mu\epsilon(\tilde{u}) + \mu'(\nabla\cdot\tilde{u})Id - p^0\tilde{\rho}Id\big)n\Big)\,d\sigma & \\ \qquad + \big(J\omega\big)\wedge\omega + f_3(\tilde{\rho},\tilde{u},a,\omega) & \text{in } (0,T),\\ \tilde{u} = 0 & \text{on } (0,T)\times\partial\Omega,\\ \tilde{u} = \dot{a} + \omega\wedge(Qx) & \text{on } (0,T)\times\partial\Omega_S(0),\\ \tilde{\rho}(0,\cdot) = \rho_0 - \overline{\rho},\ \ \tilde{u}(0,\cdot) = u_0 & \text{in } \Omega_F(0),\\ a(0) = 0,\ \ \dot{a}(0) = a_0,\ \ \omega(0) = 0, & \end{cases} \tag{30}$$

where

$$\begin{cases} f_0(\tilde{\rho},\tilde{u},a,\omega) & = \overline{\rho}\,\nabla\cdot\tilde{u} - (\tilde{\rho}+\overline{\rho})\,\mathrm{tr}\big(\nabla\tilde{u}(\nabla\chi)^{-1}\big),\\ (f_1)_i(\tilde{\rho},\tilde{u},a,\omega) & = -(\tilde{u}_j-\tilde{v}_j)(\nabla\tilde{u}(\nabla\chi)^{-1})_{ij} + \mu\left(\dfrac{\overline{\rho}}{\tilde{\rho}+\overline{\rho}}\partial_{x_l}(\partial_{x_k}\tilde{u}_i(\nabla\chi)^{-1}_{kj})(\nabla\chi)^{-1}_{lj} - \partial^2_{x_j}\tilde{u}_i\right)\\ & \quad +(\mu+\mu')\left(\dfrac{\overline{\rho}}{\tilde{\rho}+\overline{\rho}}\partial_{x_l}(\partial_{x_k}\tilde{u}_j(\nabla\chi)^{-1}_{kj})(\nabla\chi)^{-1}_{li} - \partial^2_{x_ix_j}\tilde{u}_j\right)\\ & \quad -\left((\nabla\chi)^{-1}_{ki}\dfrac{\partial_{x_k}\tilde{p}}{\tilde{\rho}+\overline{\rho}} - p^0\partial_{x_i}\tilde{\rho}\right),\\ f_2(\tilde{\rho},\tilde{u},a,\omega) & = \mu\displaystyle\int_{\partial\Omega_S(0)}\Big((\nabla\tilde{u}(\nabla\chi)^{-1}Q - \nabla\tilde{u}) + ((\nabla\chi)^{-t}(\nabla\tilde{u})^tQ - (\nabla\tilde{u})^t)\Big)n\,d\sigma\\ & \quad + \displaystyle\int_{\partial\Omega_S(0)}\Big(\mu'(\mathrm{tr}(\nabla\tilde{u}(\nabla\chi)^{-1})Q - (\nabla\cdot\tilde{u})Id) + (p^0\tilde{\rho}Id - \frac{\tilde{p}}{\overline{\rho}}Q)\Big)n\,d\sigma,\\ f_3(\tilde{\rho},\tilde{u},a,\omega) & = \mu\displaystyle\int_{\partial\Omega_S(0)}(Qx)\wedge\Big((\nabla\tilde{u}(\nabla\chi)^{-1}Q - \nabla\tilde{u}) + ((\nabla\chi)^{-t}(\nabla\tilde{u})^tQ - (\nabla\tilde{u})^t)\Big)n\,d\sigma\\ & \quad + \displaystyle\int_{\partial\Omega_S(0)}(Qx)\wedge\Big(\big(\mu'(\mathrm{tr}(\nabla\tilde{u}(\nabla\chi)^{-1})Q - (\nabla\cdot\tilde{u})Id) + (p^0\tilde{\rho}Id - \frac{\tilde{p}}{\overline{\rho}}Q)\big)n\Big)\,d\sigma. \end{cases} \tag{31}$$

Thanks to the assumptions (of smallness) on $\tilde{\rho}$, on $Q-Id$ and on $\nabla\chi - Id$, we can rewrite the previous expressions of f_0, f_1, f_2 and f_3 as some 'small' functions multiplying the density $\tilde{\rho}$ and the velocity vector field $\tilde{u}$ this way:

$$f_0(\tilde{\rho},\tilde{u},a,\omega) = \overline{\rho}\,\mathrm{tr}(\nabla\tilde{u}(Id - (\nabla\chi)^{-1})) - \tilde{\rho}\,\mathrm{tr}\big(\nabla\tilde{u}(\nabla\chi)^{-1}\big), \tag{32}$$

$$\begin{aligned} (f_1)_i(\tilde{\rho},\tilde{u},a,\omega) &= -(\tilde{u}_j-\tilde{v}_j)(\nabla\tilde{u}(\nabla\chi)^{-1})_{ij} + \mu\left(\frac{\overline{\rho}}{\tilde{\rho}+\overline{\rho}} - 1\right)\partial_{x_l}(\partial_{x_k}\tilde{u}_i(\nabla\chi)^{-1}_{kj})(\nabla\chi)^{-1}_{lj}\\ &\quad +\mu\big[\partial_{x_l}(\partial_{x_k}\tilde{u}_i((\nabla\chi)^{-1}_{kj} - \delta_{kj}))(\nabla\chi)^{-1}_{lj} + \partial^2_{x_lx_j}\tilde{u}_i((\nabla\chi)^{-1}_{lj} - \delta_{lj})\big]\\ &\quad +(\mu+\mu')\left(\frac{\overline{\rho}}{\tilde{\rho}+\overline{\rho}} - 1\right)\partial_{x_l}(\partial_{x_k}\tilde{u}_j(\nabla\chi)^{-1}_{kj})(\nabla\chi)^{-1}_{li}\\ &\quad +(\mu+\mu')\big[\partial_{x_l}(\partial_{x_k}\tilde{u}_j((\nabla\chi)^{-1}_{kj} - \delta_{kj}))(\nabla\chi)^{-1}_{li} + \partial^2_{x_lx_j}\tilde{u}_j((\nabla\chi)^{-1}_{li} - \delta_{li})\big]\\ &\quad -((\nabla\chi)^{-1}_{ki} - \delta_{ki})\frac{P'(\tilde{\rho}+\overline{\rho})\partial_{x_k}\tilde{\rho}}{\tilde{\rho}+\overline{\rho}} - \left(\frac{P'(\tilde{\rho}+\overline{\rho})}{\tilde{\rho}+\overline{\rho}} - \frac{P'(\overline{\rho})}{\overline{\rho}}\right)\partial_{x_i}\tilde{\rho}, \end{aligned} \tag{33}$$

$$
\begin{aligned}
f_2(\tilde\rho,\tilde u,a,\omega) \quad &= \mu\int_{\partial\Omega_S(0)}\Big[\nabla\tilde u((\nabla\chi)^{-1}-Id)Q+\nabla\tilde u(Q-Id)\Big]n\,d\sigma\\
&+\mu\int_{\partial\Omega_S(0)}\Big[((\nabla\chi)^{-t}-Id)(\nabla\tilde u)^tQ+(\nabla\tilde u)^t(Q-Id)\Big]n\,d\sigma\\
&+\mu'\int_{\partial\Omega_S(0)}\Big[\mathrm{tr}(\nabla\tilde u((\nabla\chi)^{-1}-Id))Q+(\nabla\cdot\tilde u)(Q-Id)\Big]n\,d\sigma\\
&-\int_{\partial\Omega_S(0)}\frac{P(\tilde\rho+\overline\rho)-P(\overline\rho)}{\overline\rho}(Q-Id)n\,d\sigma-\int_{\partial\Omega_S(0)}(P(\tilde\rho+\overline\rho)-P(\overline\rho)-P'(\overline\rho)\tilde\rho)\frac{n}{\overline\rho}\,d\sigma
\end{aligned}
\tag{34}
$$

and

$$
\begin{aligned}
f_3(\tilde\rho,\tilde u,a,\omega) \quad &= \mu\int_{\partial\Omega_S(0)}(Qx)\wedge\Big[\nabla\tilde u((\nabla\chi)^{-1}-Id)Q+\nabla\tilde u(Q-Id)\Big]n\,d\sigma\\
&+\mu\int_{\partial\Omega_S(0)}(Qx)\wedge\Big[((\nabla\chi)^{-t}-Id)(\nabla\tilde u)^tQ+(\nabla\tilde u)^t(Q-Id)\Big]n\,d\sigma\\
&+\mu'\int_{\partial\Omega_S(0)}(Qx)\wedge\Big[\Big(\mathrm{tr}(\nabla\tilde u((\nabla\chi)^{-1}-Id))Q+\mathrm{tr}(\nabla\tilde u)(Q-Id)\Big)\Big]n\,d\sigma\\
&+\int_{\partial\Omega_S(0)}\Big[(Qx)\wedge\big(p^0\tilde\rho(Id-Q)n\big)\Big]\,d\sigma\\
&+\int_{\partial\Omega_S(0)}(P(\overline\rho)+P'(\overline\rho)\tilde\rho-P(\tilde\rho+\overline\rho))(Qx)\wedge Q\frac{n}{\overline\rho}\,d\sigma.
\end{aligned}
\tag{35}
$$

For the last two integrals in the expression of f_2, we have respectively used that

$$
\frac{P(\overline\rho)}{\overline\rho}\int_{\partial\Omega_S(0)}(Q-Id)n\,d\sigma=0 \quad\text{and}\quad \frac{P(\overline\rho)}{\overline\rho}\int_{\partial\Omega_S(0)}n\,d\sigma=0.
$$

We have also used that

$$
\frac{P(\overline\rho)}{\overline\rho}\int_{\partial\Omega_S(0)}(Qx)\wedge(Qn)\,d\sigma=0
$$

for the last term of f_3.

2.2 Two intermediate results

Let us now introduce some notation which we will employ all along the paper. First, for $t>0$, we define

$$
Q_t:=(0,t)\times\Omega_F(0),\ \Sigma_t:=(0,t)\times\partial\Omega_S(0)
$$

and

$$
H_t^r(H^s):=H^r(0,t;H^s(\Omega_F(0))),\ C_t^r(H^s):=C^r([0,t];H^s(\Omega_F(0))),
$$

where r, $s\geq 0$ are natural numbers.

For $0\leqslant h\leqslant+\infty$, we define the space

$$
\begin{aligned}
X(0,h)=\{(\rho,u,a,\omega)\in(C_h^0(H^3)\cap L_h^2(H^3)\cap C_h^1(H^2)\cap H_h^1(H^2)\cap H_h^2(L^2))\\
\times(L_h^2(H^4)\cap C_h^0(H^3)\cap C_h^1(H^1)\cap H_h^2(L^2))\times(C_h^2\cap H_h^3)\times(C_h^1\cap H_h^2)\}
\end{aligned}
$$

endowed with the following norm:

$$
\begin{aligned}
N_{0,h}(\rho,u,a,\omega)=\Big(&\|\rho\|^2_{L_h^\infty(H^3)}+\|\rho\|^2_{L_h^2(H^3)}+\|\rho\|^2_{W_h^{1,\infty}(H^2)}+\|\rho\|^2_{H_h^1(H^2)}+\|\rho\|^2_{H_h^2(L^2)}+\|u\|^2_{L_h^\infty(H^3)}\\
+&\|u\|^2_{L_h^2(H^4)}+\|u\|^2_{W_h^{1,\infty}(H^1)}+\|u\|^2_{H_h^2(L^2)}+\|a\|^2_{W_h^{2,\infty}}+\|a\|^2_{H_h^3}+\|\omega\|^2_{W_h^{1,\infty}}+\|\omega\|^2_{H_h^2}\Big)^{1/2}.
\end{aligned}
\tag{36}
$$

The proof of Theorem 3 is divided, as usual, in two steps: first, a local existence result and next a priori estimates for the system. A suitable combination of both will yield the desired global result.

Thus, we first formulate a local existence and uniqueness result for small initial data.

Proposition 7 *We suppose that there exists $h \geq 0$ and $E_0 > 0$ such that (30) admits a unique solution $(\tilde{\rho}, \tilde{u}, a, \omega)$ in $X(0,h)$ satisfying $N_{0,h}(\tilde{\rho}, \tilde{u}, a, \omega) \leqslant E_0$ and $d(\partial\Omega_S(h), \partial\Omega) > 0$.*
Then there exist constants $0 < \varepsilon_0 \leqslant E_0$, $\tau > 0$ and $C_2 > 0$ independent of h such that, if $N_{h,h}(\tilde{\rho}, \tilde{u}, a, \omega) \leqslant \varepsilon_0$, problem (30) has a unique solution $(\tilde{\rho}, \tilde{u}, a, \omega)$ defined on $(h, h+\tau)$ satisfying

$$(\tilde{\rho}, \tilde{u}, a, \omega) \in X(h, h+\tau), \ N_{h,h+\tau}(\tilde{\rho}, \tilde{u}, a, \omega) \leqslant C_2 N_{h,h}(\tilde{\rho}, \tilde{u}, a, \omega).$$

We present now the a priori estimates:

Proposition 8 *We suppose that there exists $T > 0$ such that the problem (30) admits a solution $(\tilde{\rho}, \tilde{u}, a, \omega)$ in $X(0,T)$. Then there exist constants $0 < \delta_1 \leqslant \delta_0$ and $C_1 > 0$ independent of t such that if $N_{0,T}(\tilde{\rho}, \tilde{u}, a, \omega) \leqslant \delta_1$ then*

$$N_{0,T}(\tilde{\rho}, \tilde{u}, a, \omega) \leqslant C_1 N_{0,0}(\tilde{\rho}, \tilde{u}, a, \omega).$$

These two propositions will be proved in the next sections. They allow to prove theorem 3 in the following way:
Proof of theorem 3: We will apply iteratively propositions 7 and 8. We suppose that

$$N_{0,0}(\tilde{\rho}, \tilde{u}, a, \omega) \leqslant \min\left(\delta_0, \frac{\delta_1}{C_0}, \frac{\delta_1}{C_1\sqrt{1+C_0^2}}\right).$$

According to proposition 7 for $h = 0$, one can define on $(0,\tau)$ a solution $(\tilde{\rho}, \tilde{u}, a, \omega) \in X(0,\tau)$ such that

$$N_{0,\tau}(\tilde{\rho}, \tilde{u}, a, \omega) \leqslant C_0 N_{0,0}(\tilde{\rho}, \tilde{u}, a, \omega) \leqslant \delta_1 \leqslant \delta_0.$$

Since $N_{\tau,\tau}(\tilde{\rho}, \tilde{u}, a, \omega) \leqslant N_{0,\tau}(\tilde{\rho}, \tilde{u}, a, \omega)$, we can apply again proposition 7. Our solution can be extended on $(\tau, 2\tau)$ and $N_{\tau,2\tau}(\tilde{\rho}, \tilde{u}, a, \omega) \leqslant C_0 N_{0,\tau}(\tilde{\rho}, \tilde{u}, a, \omega)$. Thus

$$N_{0,2\tau}^2(\tilde{\rho}, \tilde{u}, a, \omega) = (N_{0,\tau}^2 + N_{\tau,2\tau}^2)(\tilde{\rho}, \tilde{u}, a, \omega) \leqslant (1 + C_0^2) N_{0,\tau}^2(\tilde{\rho}, \tilde{u}, a, \omega).$$

Thanks to Proposition 8 for $T = \tau$ and the choice of $N_{0,0}(\tilde{\rho}, \tilde{u}, a, \omega)$, we deduce

$$N_{0,2\tau}(\tilde{\rho}, \tilde{u}, a, \omega) \leqslant C_1\sqrt{1+C_0^2} N_{0,0}(\tilde{\rho}, \tilde{u}, a, \omega) \leqslant \delta_1.$$

This allows to repeat this process and obtain the existence of a regular solution as long as (17) is satisfied.

3 A local existence result: proof of Proposition 7

This section is devoted to the proof of proposition 7. We only consider the case $h = 0$. To prove this result, we will first prove the existence of solution for a linearized problem. Let $R > 0$ be small enough and $s > 0$ be given in terms of R (to be chosen later on). We define the affine space $Y((0,s);R)$ by

$$Y((0,s);R) = \Big\{(\tilde{\rho}, \tilde{u}_F, a, \omega) \in X(0,s)/a(0) = 0, \ Q(0) = Id, \ \tilde{u}_F = 0 \text{ on } \partial\Omega_F(0), \ N_{0,s}(\tilde{\rho}, \tilde{u}_F, a, \omega) \leqslant R\Big\}.$$

In this definition, Q is the rotation matrix associated to ω.

3.1 Linearized system

Let $(\hat\rho, \hat u_F, \hat a, \hat\omega)$ be given in $Y((0,s);R)$ where $R<1$. Since $\Omega_S(0)$ satisfies (1), we can suppose that s is small enough to get the following property: there exists $\alpha>0$ such that

$$\forall\, t\in(0,s),\ d(\widehat\Omega_S(t),\partial\Omega)\geq\alpha>0,$$

where $\widehat\Omega_S(t)=\hat a(t)+\widehat Q(t)\Omega_S(0)$. Then, thanks to lemma 1, we can define a flow $\hat\chi$ and a velocity $\hat v$ associated to $\hat a$ and $\hat\omega$. This allows to construct a continuous velocity at the interface $\hat u$ given by

$$\hat u=\hat u_F+\hat v, \tag{37}$$

where $\hat u_F$ is extended by 0 in $\Omega_S(0)$. We then look for $(\tilde\rho,\tilde u,a,\omega)$ solution of the linearized problem

$$\left\{\begin{array}{ll}
\tilde\rho_t+((\nabla\hat\chi)^{-1}(\hat u-\hat v))\cdot\nabla\tilde\rho=F_0(\hat\rho,\hat u,\hat a,\hat\omega) & \text{in } (0,T)\times\Omega_F(0),\\
\tilde u_t-2\mu\nabla\cdot(\epsilon(\tilde u))-\mu'\nabla\cdot((\nabla\cdot\tilde u)Id)+p^0\nabla\tilde\rho=f_1(\hat\rho,\hat u,\hat a,\hat\omega) & \text{in } (0,T)\times\Omega_F(0),\\
m\ddot a=\displaystyle\int_{\partial\Omega_S(0)}\Big(2\mu\epsilon(\tilde u)+\mu'(\nabla\cdot\tilde u)Id-p^0\tilde\rho Id\Big)n\,d\sigma+f_2(\hat\rho,\hat u,\hat a,\hat\omega) & \text{in } (0,T),\\
\hat J\dot\omega=\displaystyle\int_{\partial\Omega_S(0)}(\widehat Qx)\wedge\Big(\big(2\mu\epsilon(\tilde u)+\mu'(\nabla\cdot\tilde u)Id-p^0\tilde\rho Id\big)n\Big)\,d\sigma & \\
\qquad+\big(\hat J\hat\omega\big)\wedge\omega+f_3(\hat\rho,\hat u,\hat a,\hat\omega) & \text{in } (0,T),\\
\tilde u=0 & \text{on } (0,T)\times\partial\Omega,\\
\tilde u=\dot a+\omega\wedge(\widehat Qx) & \text{on } (0,T)\times\partial\Omega_S(0),\\
\tilde\rho(0,\cdot)=\rho_0-\overline\rho,\ \ \tilde u(0,\cdot)=u_0 & \text{in } \Omega_F(0),\\
a(0)=0,\ \ \dot a(0)=a_0,\ \ \omega(0)=\omega_0. &
\end{array}\right. \tag{38}$$

In this system, $\hat J$ is defined by (9) where we replace Q by $\hat Q$ and F_0 is given by

$$F_0(\hat\rho,\hat u,\hat a,\hat\omega)=f_0(\hat\rho,\hat u,\hat a,\hat\omega)-\overline\rho(\nabla\cdot\hat u)=-(\hat\rho+\overline\rho)tr(\nabla\hat u(\nabla\hat\chi)^{-1}). \tag{39}$$

Estimates for $\tilde\rho$

Observe that the first equation is decoupled from the others. We have that $F_0(\hat\rho,\hat u,\hat a,\hat\omega)$ belongs to $L^2_s(H^3)\cap C^0_s(H^2)\cap H^1_s(H^1)$ and there exists $C>0$ such that

$$\|f_0(\hat\rho,\hat u,\hat a,\hat\omega)\|_{L^2_s(H^3)}+\|f_0(\hat\rho,\hat u,\hat a,\hat\omega)\|_{L^\infty_s(H^2)}+\|f_0(\hat\rho,\hat u,\hat a,\hat\omega)\|_{H^1_s(H^1)}\leqslant CR^2,$$
$$\|\overline\rho(\nabla\cdot\hat u)\|_{L^2_s(H^3)}+\|\overline\rho(\nabla\cdot\hat u)\|_{L^\infty_s(H^2)}+\|\overline\rho(\nabla\cdot\hat u)\|_{H^1_s(H^1)}\leqslant CR.$$

Using that $\hat u-\hat v=0$ on $\partial\Omega_F(0)$, standard arguments allow to prove that $\tilde\rho$ is defined on $(0,s)$ and belongs to $C^0_s(H^3)\cap C^1_s(H^2)\cap H^2_s(H^1)$. Moreover,

$$\begin{aligned}
\|\tilde\rho\|_{L^\infty_s(H^3)}+\|\tilde\rho\|_{W^{1,\infty}_s(H^2)}+\|\tilde\rho\|_{H^2_s(H^1)} &\leqslant Ce^{Cs/R}(R^{1/2}(\|F_0(\hat\rho,\hat u,\hat a,\hat\omega)\|_{L^2_s(H^3)}+\|F_0(\hat\rho,\hat u,\hat a,\hat\omega)\|_{L^\infty_s(H^2)}\\
&\quad+\|F_0(\hat\rho,\hat u,\hat a,\hat\omega)\|_{H^1_s(H^1)})+\|\rho_0-\overline\rho\|_{H^3})\\
&\leqslant Ce^{Cs/R}(R^2+\|\rho_0-\overline\rho\|_{H^3}),
\end{aligned} \tag{40}$$

for $R>0$ small enough.

Estimates for $\tilde u$, a and ω

Taking $R>0$ small enough, we can suppose that $|\tilde\rho|<\overline\rho/2$ and so every single term of the expression of f_1 (see (33)) makes sense.

In the rest of this subsection, we denote f_i instead of $f_i(\hat\rho,\hat u,\hat a,\hat\omega)$ for $i=1,2,3$.

• For the coupled system on $(\tilde u,\,a,\,\omega)$, we first notice that f_1 belongs to $L^2_s(H^2)\cap C^0_s(H^1)\cap H^1_s(L^2)$ and f_2 and f_3 belong to H^1_s. Moreover, since $(\hat\rho,\hat u,\hat a,\hat\omega)$ belongs to $Y((0,s);R)$ and $\hat\chi$ satisfies (18), we have

$$\|f_1\|_{L^2_s(H^2)}+\|f_1\|_{L^\infty_s(H^1)}+\|f_1\|_{H^1_s(L^2)}+\|f_2\|_{H^1_s}+\|f_3\|_{H^1_s}\leqslant CR^2. \tag{41}$$

Next, we multiply the equation of $\tilde{u}$ by $\tilde{u}$, we integrate on $Q_r = (0,r)\times\Omega_F(0)$ for all $r\in(0,s)$ and we integrate by parts. Taking the supremum in r, this gives

$$\begin{aligned}
&\frac{1}{2}\sup_{r\in(0,s)}\left(|\dot{a}|^2(r)+|\omega|^2(r)+\int_{\Omega_F(0)}|\tilde{u}|^2(r)\,dx\right)+\iint_{Q_s}(2\mu|\epsilon(\tilde{u})|^2+\mu'|\nabla\cdot\tilde{u}|^2)\,dx\,dr\\
&\leqslant C\left(\iint_{Q_s}(|f_1|^2+|\nabla\tilde{\rho}|^2+\delta|\tilde{u}|^2)\,dx\,dr+\int_{\Omega_F(0)}|u_0|^2dx+|a_0|^2+|\omega_0|^2\right.\\
&\left.+(\sup_{r\in(0,s)}|\dot{a}|)\left(\int_0^s|f_2|\,dr+\iint_{\Sigma_s}|\tilde{\rho}|\,d\sigma\,dr\right)+(\sup_{r\in(0,s)}|\omega|)\left(\int_0^s|f_3|\,dr+\iint_{\Sigma_s}|\tilde{\rho}|\,d\sigma\,dr\right)\right),
\end{aligned}\tag{42}$$

for δ small enough. Here, we have used that $\hat{J}$ is coercive (see 10) and satisfies

$$\|\hat{J}\|_{L^\infty_t}\leqslant C,\ \left(\dot{\hat{J}}\omega\right)\cdot\omega=0.$$

Now, observe that from the fact that $\Omega_S(0)$ is convex (see (1)) and thanks to $\tilde{u}=0$ on $\partial\Omega\times(0,s)$, we have that

$$\iint_{Q_s}|\epsilon(\tilde{u})|^2dx\,dr\geq C\|\tilde{u}\|^2_{L^2_s(H^1)},\tag{43}$$

for some $C>0$. Indeed, first we have $\sum_{i=1}^3|\partial_{x_i}\tilde{u}_i|^2\leqslant|\epsilon(\tilde{u})|^2$. Then, for instance for $\tilde{u}_1$ we notice that

$$\tilde{u}_1(x)=\tilde{u}_1(x_1,x_2,x_3)=\int_{x^*}^{x_1}\partial_{\xi_1}\tilde{u}_1(\xi_1,x_2,x_3)\,d\xi_1\ \ \forall x\in\Omega_F(0),$$

where $x^*\in\partial\Omega\cap\{x+\lambda e_1:\lambda\in\mathbb{R}\}$; this is possible since $\Omega_S(0)$ is convex. Of course, the same is also true for $\tilde{u}_2$ and $\tilde{u}_3$, so we deduce that

$$\int_{\Omega_F(0)}|\tilde{u}|^2dx\leqslant C\sum_{i=1}^3\int_{\Omega_F(0)}|\partial_{x_i}\tilde{u}_i|^2dx\leqslant C\int_{\Omega_F(0)}|\epsilon(\tilde{u})|^2dx.$$

Finally, since $\tilde{u},\epsilon(\tilde{u})\in L^2(\Omega_F(0))$ and $\tilde{u}=0$ on a part of the boundary, Korn's inequality tells that $\tilde{u}\in H^1(\Omega_F(0))$ so we obtain (43).

Thus, using estimate (40) on $\tilde{\rho}$, we deduce from (42) that $(\tilde{u},\,a,\,\omega)$ belongs to $C^0_s(L^2)\cap L^2_s(H^1)\times C^1_s\times C^0_s$ and there exists $C>0$ such that

$$\|\tilde{u}\|_{L^\infty_s(L^2)}+\|\tilde{u}\|_{L^2_s(H^1)}+\|a\|_{W^{1,\infty}_s}+\|\omega\|_{L^\infty_s}\leqslant Ce^{Cs/R}(R^{3/2}+\|\rho_0-\overline{\rho}\|_{H^3}+\|u_0\|_{L^2}+|a_0|+|\omega_0|).\tag{44}$$

- Let us multiply the equation of $\tilde{u}$ by $\tilde{u}_t$. Arguing as before, this yields

$$\begin{aligned}
&\frac{1}{2}\sup_{r\in(0,s)}\left(\int_{\Omega_F(0)}(2\mu|\epsilon(\tilde{u})|^2(r)+\mu'|\nabla\cdot\tilde{u}|^2(r))\,dx\right)+\iint_{Q_s}|\tilde{u}_t|^2dx\,dr\\
&+\iint_{\Sigma_s}\left((2\mu\epsilon(\tilde{u})+\mu'(\nabla\cdot\tilde{u})Id)n\right)\cdot\tilde{u}_t\,d\sigma dr\leqslant C\left(\iint_{Q_s}(|f_1|^2+|\nabla\tilde{\rho}|^2)dx\,dr+\int_{\Omega_F(0)}|\nabla u_0|^2dx\right).
\end{aligned}\tag{45}$$

Using the boundary condition of $\tilde{u}$ on $\partial\Omega_S(0)$, we obtain the following for the boundary term:

$$\iint_{\Sigma_s}((2\mu\epsilon(\tilde{u})+\mu'(\nabla\cdot\tilde{u})Id)n)\cdot\tilde{u}_t\,d\sigma dr=\iint_{\Sigma_s}((2\mu\epsilon(\tilde{u})+\mu'(\nabla\cdot\tilde{u})Id)n)\cdot(\ddot{a}+\dot{\omega}\wedge(\widehat{Q}x)+\omega\wedge(\dot{\widehat{Q}}x))\,d\sigma dr.$$

For the two first terms we use the equations of a and ω in (38). This produces the norms $\|\ddot{a}\|^2_{L^2_s}$ and $\|\dot{\omega}\|^2_{L^2_s}$. For the third term, we have

$$\begin{aligned}
\left|\iint_{\Sigma_s}\left((2\mu\epsilon(\tilde{u})+\mu'(\nabla\cdot\tilde{u})Id)n\right)\cdot(\omega\wedge\dot{\widehat{Q}}x)\,d\sigma dr\right| &\leqslant CR\|\omega\|_{L^2_s}\|\tilde{u}\|_{L^2_s(H^2)}\\
&\leqslant R^2\|\tilde{u}\|^2_{L^2_s(H^2)}+C\|\omega\|^2_{L^2_s}.
\end{aligned}$$

Using (44) to estimate $\|\omega\|^2_{L^2_s}$ and (40), we obtain from (45)

$$\begin{aligned}&\|\tilde u\|_{H^1_s(L^2)} + \|\tilde u\|_{L^\infty_s(H^1)} + \|a\|_{H^2_s} + \|\omega\|_{H^1_s}\\ &\leqslant R\|\tilde u\|_{L^2_s(H^2)} + Ce^{Cs/R}\big(R^{3/2} + \|\rho_0 - \overline{\rho}\|_{H^3} + \|u_0\|_{H^1} + |a_0| + |\omega_0|\big).\end{aligned} \tag{46}$$

Now, we regard the equation of $\tilde u$ as a stationary system:

$$\begin{cases} -\mu\Delta\tilde u - (\mu+\mu')\nabla(\nabla\cdot\tilde u) = f_1 - p^0\nabla\tilde\rho - \tilde u_t & \text{in } \Omega_F(0),\\ \tilde u = (\dot a + \omega\wedge(\widehat{Q}x))1_{\partial\Omega_S(0)} & \text{on } \partial\Omega_F(0).\end{cases} \tag{47}$$

The solution of this system belongs to H^2 and we have

$$\|\tilde u\|_{H^2} \leqslant C(\|f_1\|_{L^2} + \|\tilde\rho\|_{H^1} + \|\tilde u_t\|_{L^2} + |\dot a| + |\omega|).$$

Taking here the L^2_s norm and using (44) to estimate $\|\dot a\|_{L^2_s} + \|\omega\|_{L^2_s}$ and (40) to estimate $\|\tilde\rho\|_{L^2_s(H^1)}$, we obtain from (46)

$$\begin{aligned}&\|\tilde u\|_{H^1_s(L^2)} + \|\tilde u\|_{L^2_s(H^2)} + \|\tilde u\|_{L^\infty_s(H^1)} + \|a\|_{H^2_s} + \|\omega\|_{H^1_s}\\ &\leqslant Ce^{Cs/R}\big(R^{3/2} + \|\rho_0 - \overline{\rho}\|_{H^3} + \|u_0\|_{H^1} + |a_0| + |\omega_0|\big)\end{aligned} \tag{48}$$

(recall that $R>0$ is small enough).

- If we differentiate with respect to time the system satisfied by $(\tilde u,\, a,\, \omega)$, we obtain

$$\begin{cases} \tilde u_{tt} - 2\mu\nabla\cdot(\epsilon(\tilde u_t)) - \mu'\nabla\cdot((\nabla\cdot\tilde u_t)Id) = f_{1,t} - p^0\nabla\tilde\rho_t & \text{in } (0,T)\times\Omega_F(0),\\ m\dddot a = f_{2,t} + \displaystyle\int_{\partial\Omega_S(0)} \Big(2\mu\epsilon(\tilde u_t) + \mu'(\nabla\cdot\tilde u_t)Id - p^0\tilde\rho_t Id\Big)n\,d\sigma & \text{in } (0,T),\\ \hat J\ddot\omega = -\dot{\hat J}\dot\omega + \big(\dot{\hat J}\hat\omega\big)\wedge\omega + \big(\hat J\dot{\hat\omega}\big)\wedge\omega + \big(\hat J\hat\omega\big)\wedge\dot\omega + f_{3,t} & \\ \qquad + \displaystyle\int_{\partial\Omega_S(0)} (\dot{\widehat{Q}}x)\wedge\Big(\big(2\mu\epsilon(\tilde u) + \mu'(\nabla\cdot\tilde u)Id - p^0\tilde\rho Id\big)n\Big)\,d\sigma & \\ \qquad + \displaystyle\int_{\partial\Omega_S(0)} (\widehat{Q}x)\wedge\Big(\big(2\mu\epsilon(\tilde u_t) + (\mu'\nabla\cdot\tilde u_t - p^0\tilde\rho_t)Id\big)n\Big)\,d\sigma & \text{in } (0,T),\\ \tilde u_t = 0 & \text{on } (0,T)\times\partial\Omega,\\ \tilde u_t = \ddot a + \dot\omega\wedge(\widehat{Q}x) + \omega\wedge(\dot{\widehat{Q}}x) & \text{on } (0,T)\times\partial\Omega_S(0).\end{cases} \tag{49}$$

Let us multiply the first equation by $\tilde u_t$ and integrate in space and in time as before. We obtain

$$\begin{aligned}&\frac12\sup_{r\in(0,s)}\int_{\Omega_F(0)}|\tilde u_t|^2\,dx + \iint_{Q_s}(2\mu|\epsilon(\tilde u_t)|^2 + \mu'|\nabla\cdot\tilde u_t|^2)\,dx\,dr + \iint_{\Sigma_s}\Big(\big(2\mu\epsilon(\tilde u_t) + \mu'(\nabla\cdot\tilde u_t)Id\big)n\Big)\cdot\tilde u_t\,d\sigma\,dr\\ &= \frac12\int_{\Omega_F(0)}|\tilde u_t(0)|^2\,dx + \iint_{Q_s} f_{1,t}\cdot\tilde u_t\,dx\,dr - \iint_{Q_s}p^0\nabla\tilde\rho_t\cdot\tilde u_t\,dx\,dr.\end{aligned}$$

According to the boundary conditions satisfied by $\tilde u_t$ on $\partial\Omega_S(0)$, we have

$$\begin{aligned}&\iint_{\Sigma_s}\Big(\big(2\mu\epsilon(\tilde u_t) + \mu'(\nabla\cdot\tilde u_t)Id\big)n\Big)\cdot\tilde u_t\,d\sigma\,dr = \int_0^s \ddot a\cdot(m\dddot a - f_{2,t})dr + \int_0^s\ddot a\cdot\int_{\partial\Omega_S(0)}p^0\tilde\rho_t n\,d\sigma\,dr\\ &+ \int_0^s\dot\omega\cdot\Big(\hat J\ddot\omega + \dot{\hat J}\dot\omega - \big(\dot{\hat J}\hat\omega\big)\wedge\omega - \big(\hat J\dot{\hat\omega}\big)\wedge\omega - \big(\hat J\hat\omega\big)\wedge\dot\omega - f_{3,t}\Big)dr\\ &- \int_0^s\dot\omega\cdot\int_{\partial\Omega_S(0)}(\dot{\widehat{Q}}x)\wedge\Big(\big(2\mu\epsilon(\tilde u) + \mu'(\nabla\cdot\tilde u)Id - p^0\tilde\rho Id\big)n\Big)\,d\sigma\,dr + \int_0^s\dot\omega\cdot\int_{\partial\Omega_S(0)}(\widehat{Q}x)\wedge p^0\tilde\rho_t n\,d\sigma\,dr\\ &+ \iint_{\Sigma_s}\Big(\big(2\mu\epsilon(\tilde u_t) + \mu'(\nabla\cdot\tilde u_t)Id\big)n\Big)\cdot(\omega\wedge(\dot{\widehat{Q}}x))\,d\sigma\,dr.\end{aligned}$$

Thus, since $\dot\omega\cdot\hat J\ddot\omega=\dfrac12\dfrac{d}{dt}(\dot\omega\cdot\hat J\dot\omega)-\dfrac12\dot\omega\cdot\dot{\hat J}\dot\omega$ and $\|\dot{\hat J}\|_{L^\infty_t}\leqslant C\|\hat\omega\|_{L^\infty_t}\leqslant CR$, we obtain that

$$\sup_{r\in(0,s)}\left(|\ddot a|^2(r)+|\dot\omega|^2(r)+\int_{\Omega_F(0)}|\tilde u_t|^2(r)\,dx\right)+\iint_{Q_s}(2\mu|\epsilon(\tilde u_t)|^2+\mu'|\nabla\cdot\tilde u_t|^2)\,dx\,dr$$
$$\leqslant C\left(\iint_{Q_s}(|f_{1,t}|^2+|\tilde u_t|^2+|\nabla\tilde\rho_t|^2)\,dx\,dr+\int_{\Omega_F(0)}|\tilde u_t(0)|^2dx+|\ddot a(0)|^2+|\dot\omega(0)|^2\right.$$
$$+\big(\sup_{r\in(0,s)}|\ddot a|\big)\left(\int_0^s|f_{2,t}|dr+\iint_{\Sigma_s}|\tilde\rho_t|\,d\sigma\,dr\right)+\big(\sup_{r\in(0,s)}|\dot\omega|\big)\left(\int_0^s|f_{3,t}|dr+\iint_{\Sigma_s}(|\tilde\rho|+|\tilde\rho_t|)\,d\sigma\,dr\right)$$
$$\left.+\int_0^s(|\omega|^2+|\dot\omega|^2)\,dr+\|\tilde u\|^2_{L^2_s(H^2)}\right)+R^2\|\tilde u_t\|^2_{L^2_s(H^2)}.$$

Using (40) and (48), this implies that

$$\|\tilde u\|_{W^{1,\infty}_s(L^2)}+\|\tilde u\|_{H^1_s(H^1)}+\|a\|_{W^{2,\infty}_s}+\|\omega\|_{W^{1,\infty}_s}\leqslant C\big(\|f_1\|_{H^1_s(L^2)}+\|f_2\|_{H^1_s}+\|f_3\|_{H^1_s}$$
$$+\|\tilde u_t(0)\|_{L^2}+|\ddot a(0)|+|\dot\omega(0)|+Ce^{Cs/R}\big(R^{3/2}+\|u_0\|_{H^1}+\|\rho_0-\overline\rho\|_{H^3}+|a_0|+|\omega_0|\big)\big)+R\|\tilde u_t\|_{L^2_s(H^2)}. \tag{50}$$

From (41), we have that

$$\|\tilde u_t(0)\|_{L^2}+|\ddot a(0)|+|\dot\omega(0)|\leqslant C(\|u_0\|_{H^2}+\|\rho_0-\overline\rho\|_{H^1}+R|\omega(0)|+R^2).$$

Then, thanks to (40) and (48), there exists C such that

$$\begin{aligned}&\|\tilde u\|_{W^{1,\infty}_s(L^2)}+\|\tilde u\|_{H^1_s(H^1)}+\|a\|_{W^{2,\infty}_s}+\|\omega\|_{W^{1,\infty}_s}\\&\leqslant Ce^{Cs/R}\big(R^{3/2}+\|\rho_0-\overline\rho\|_{H^3}+\|u_0\|_{H^2}+|a_0|+|\omega_0|\big)+R\|\tilde u_t\|_{L^2_s(H^2)}.\end{aligned} \tag{51}$$

- Next, if we multiply the first equation of (49) by $\tilde u_{tt}$, we obtain that

$$\iint_{Q_s}|\tilde u_{tt}|^2\,dx\,dr+\frac12\sup_{r\in(0,s)}\int_{\Omega_F(0)}(2\mu|\epsilon(\tilde u_t)|^2(r)+\mu'|\nabla\cdot\tilde u_t|^2(r))\,dx$$
$$+\iint_{\Sigma_s}\Big(\big(2\mu\epsilon(\tilde u_t)+\mu'(\nabla\cdot\tilde u_t)Id\big)n\Big)\cdot\tilde u_{tt}\,d\sigma\,dr=\frac12\int_{\Omega_F(0)}(2\mu|\epsilon(\tilde u_t)|^2(0)+\mu'|\nabla\cdot\tilde u_t|^2(0))\,dx$$
$$+\iint_{Q_s}f_{1,t}\cdot\tilde u_{tt}\,dx\,dr-\iint_{Q_s}p^0\nabla\tilde\rho\cdot\tilde u_{tt}\,dx\,dr.$$

On $(0,T)\times\partial\Omega_S(0)$, we have

$$\tilde u_{tt}=\dddot a+\ddot\omega\wedge(\widehat Qx)+2\dot\omega\wedge(\dot{\widehat Q}x)+\omega\wedge(\ddot{\widehat Q}x). \tag{52}$$

Thus, we deduce

$$\iint_{\Sigma_s}\Big(\big(2\mu\epsilon(\tilde u_t)+\mu'(\nabla\cdot\tilde u_t)Id\big)n\Big)\cdot\tilde u_{tt}\,d\sigma\,dr=\int_0^s\dddot a\cdot(m\dddot a-f_{2,t})dr+\int_0^s\dddot a\cdot\int_{\partial\Omega_S(0)}p^0\tilde\rho_tn\,d\sigma\,dr$$
$$+\int_0^s\ddot\omega\cdot\Big(\hat J\ddot\omega+\dot{\hat J}\dot\omega-\big(\dot{\hat J}\hat\omega\big)\wedge\omega-\big(\hat J\dot{\hat\omega}\big)\wedge\omega-\big(\hat J\hat\omega\big)\wedge\dot\omega-f_{3,t}\Big)dr+\int_0^s\ddot\omega\cdot\int_{\partial\Omega_S(0)}(\widehat Qx)\wedge\big(p^0\tilde\rho_tn\big)\,d\sigma\,dr$$
$$-\int_0^s\ddot\omega\cdot\int_{\partial\Omega_S(0)}(\dot{\widehat Q}x)\wedge\Big(\big(2\mu\epsilon(\tilde u)+\mu'(\nabla\cdot\tilde u)Id-p^0\tilde\rho Id\big)n\Big)\,d\sigma\,dr$$
$$+\iint_{\Sigma_s}\Big(\big(2\mu\epsilon(\tilde u_t)+\mu'(\nabla\cdot\tilde u_t)Id\big)n\Big)\cdot\big(2\dot\omega\wedge(\dot{\widehat Q}x)+\omega\wedge(\ddot{\widehat Q}x)\big)\,d\sigma\,dr.$$

Then, proceeding as before and using (51), (48) and (40), we obtain that there exists C such that

$$\begin{aligned} &\|\tilde{u}_{tt}\|_{L^2_s(L^2)} + \|\tilde{u}_t\|_{L^\infty_s(H^1)} + \|\dddot{a}\|_{L^2_s} + \|\ddot{\omega}\|_{L^2_s} \\ &\leqslant Ce^{Cs/R}\big(R^{3/2} + \|\rho_0 - \overline{\rho}\|_{H^3} + \|u_0\|_{H^3} + |a_0| + |\omega_0|\big) + R\|\tilde{u}_t\|_{L^2_s(H^2)}. \end{aligned} \tag{53}$$

According to the elliptic equation satisfied by $\tilde{u}_t$ (see (47)), we have

$$\|\tilde{u}_t\|_{L^2_s(H^2)} \quad \leqslant \quad C\big(\|f_{1,t}\|_{L^2_s(L^2)} + \|\tilde{\rho}_t\|_{L^2_s(H^1)} + \|\tilde{u}_{tt}\|_{L^2_s(L^2)} + \|\ddot{a}\|_{L^2_s} + \|\omega\|_{H^1_s}\big). \tag{54}$$

This allows to deduce from (40), (41), (51) and (53) that

$$\|\tilde{u}_{tt}\|_{L^2_s(L^2)} + \|\tilde{u}_t\|_{L^\infty_s(H^1)} + \|\dddot{a}\|_{L^2_s} + \|\ddot{\omega}\|_{L^2_s} \leqslant Ce^{Cs/R}\big(R^{3/2} + \|\rho_0 - \overline{\rho}\|_{H^3} + \|u_0\|_{H^3} + |a_0| + |\omega_0|\big).$$

Moreover, (51) implies that

$$\|\tilde{u}\|_{W^{1,\infty}_s(L^2)} + \|\tilde{u}\|_{H^1_s(H^1)} + \|a\|_{W^{2,\infty}_s} + \|\omega\|_{W^{1,\infty}_s} \leqslant Ce^{Cs/R}\big(R^{3/2} + \|\rho_0 - \overline{\rho}\|_{H^3} + \|u_0\|_{H^3} + |a_0| + |\omega_0|\big).$$

At last, if we consider system (47) satisfied by $\tilde{u}$, since the right-hand side $f_1 - p^0\nabla\tilde{\rho} - \tilde{u}_t$ belongs to $L^2_s(H^2)\cap L^\infty_s(H^1)$ and the boundary condition belongs to $H^1_s(C^\infty(\partial\Omega_F(0)))$, $\tilde{u}$ belongs to $L^2_s(H^4)\cap L^\infty_s(H^3)$. Thus, we finally obtain that for a fixed s of the form

$$s = CR, \ \ C > 0, \tag{55}$$

there exists $C_2 > 0$ such that

$$N_{0,s}(\tilde{\rho}, \tilde{u}, a, \omega) \leqslant \frac{C_2}{2}\big(R^{3/2} + N_{0,0}(\tilde{\rho}, \tilde{u}, a, \omega)\big). \tag{56}$$

This allows to assert that, if R satisfies

$$R \leqslant (N_{0,0}(\tilde{\rho}, \tilde{u}, a, \omega))^{2/3}, \tag{57}$$

then

$$N_{0,s}(\tilde{\rho}, \tilde{u}, a, \omega) \leqslant C_2 N_{0,0}(\tilde{\rho}, \tilde{u}, a, \omega).$$

Let us now define $\tilde{u}_F \in L^2_s(H^1_0)$ by

$$\tilde{u}_F = \tilde{u} - \tilde{u}_{S,e}, \tag{58}$$

where $\tilde{u}_{S,e}$ is an extension to the fluid domain $\Omega_F(0)$ of the solid velocity $\dot{a} + \omega \wedge (\widehat{Q}x)$ such that

$$\tilde{u}_{S,e} = \dot{a} + \omega \wedge (\widehat{Q}x) \text{ on } \partial\Omega_S(0),\ \tilde{u}_{S,e} = 0 \text{ on } \partial\Omega,\ \|\tilde{u}_{S,e}\|_{H^i_s(H^4)} \leqslant C(\|a\|_{H^{i+1}_s} + \|\omega\|_{H^i_s})\ (i = 0, 1, 2). \tag{59}$$

For instance, we define $\tilde{u}_{S,e}$ by $\tilde{u}_{S,e} = E(\tilde{u}_S)$ where E associates to $u_{\partial\Omega_S(0)}$ the solution u of

$$\begin{cases} -\Delta u &= 0 & \text{in } \Omega_F(0), \\ \quad u &= u_{\partial\Omega_S(0)} & \text{on } \partial\Omega_S(0), \\ \quad u &= 0 & \text{on } \partial\Omega. \end{cases}$$

Then, thanks to (59) and (56), there exists C such that

$$N_{0,s}(\tilde{\rho}, \tilde{u}_F, a, \omega) \leqslant C\big(R^{3/2} + N_{0,0}(\tilde{\rho}, \tilde{u}, a, \omega)\big) \leqslant 2CN_{0,0}(\tilde{\rho}, \tilde{u}, a, \omega),$$

since R satisfies (57). Thus, if R also satisfies

$$2CN_{0,0}(\tilde{\rho}, \tilde{u}, a, \omega) \leqslant R \tag{60}$$

(which implies that $N_{0,0}(\tilde{\rho}, \tilde{u}, a, \omega)$ is small enough), then $(\tilde{\rho}, \tilde{u}_F, a, \omega)$ belongs to $Y((0, s); R)$.

3.2 Fixed point argument

Let us fix R such that (57) and (60) are satisfied. We define the mapping Λ by

$$\begin{array}{rccc}\Lambda: & Y((0,s);R) & \rightarrow & Y((0,s);R) \\ & (\hat{\rho},\hat{u}_F,\hat{a},\hat{\omega}) & \rightarrow & (\tilde{\rho},\tilde{u}_F,a,\omega),\end{array}$$

where $\tilde{u}_F$ is given by (58) and $(\tilde{\rho},\tilde{u},a,\omega)$ is the solution of (38).

We will prove that, for R small enough, Λ is a contraction in $Y((0,s);R)$ for the norm $\|\cdot\|$ defined by

$$\|(\rho,u_F,a,\omega)\| = \|\rho\|_{L^\infty_s(H^1)} + \|u_F\|_{L^2_s(H^2)} + \|a\|_{H^1_s} + \|\omega\|_{L^2_s}.$$

Let us consider $(\hat{\rho}_1,\hat{u}_{F,1},\hat{a}_1,\hat{\omega}_1)$ and $(\hat{\rho}_2,\hat{u}_{F,2},\hat{a}_2,\hat{\omega}_2)$ in $Y((0,s);R)$. We define

$$(\tilde{\rho}_1,\tilde{u}_{F,1},a_1,\omega_1) := \Lambda(\hat{\rho}_1,\hat{u}_{F,1},\hat{a}_1,\hat{\omega}_1) \ \text{ and } \ (\tilde{\rho}_2,\tilde{u}_{F,2},a_2,\omega_2) := \Lambda(\hat{\rho}_2,\hat{u}_{F,2},\hat{a}_2,\hat{\omega}_2).$$

Estimates for $\tilde{\rho}_1-\tilde{\rho}_2$.

First, we have

$$\begin{aligned}(\tilde{\rho}_1-\tilde{\rho}_2)_t + ((\nabla\hat{\chi}_1)^{-1}(\hat{u}_1-\hat{v}_1))\cdot\nabla(\tilde{\rho}_1-\tilde{\rho}_2) &= \big((\nabla\hat{\chi}_2)^{-1}(\hat{u}_2-\hat{v}_2)-(\nabla\hat{\chi}_1)^{-1}(\hat{u}_1-\hat{v}_1)\big)\cdot\nabla\tilde{\rho}_2 \\ &\quad + F_0(\hat{\rho}_1,\hat{u}_1,\hat{a}_1,\hat{\omega}_1) - F_0(\hat{\rho}_2,\hat{u}_2,\hat{a}_2,\hat{\omega}_2).\end{aligned} \tag{61}$$

Let us multiply this equation by $\tilde{\rho}_1-\tilde{\rho}_2$ and integrate in space. For the second term of this equality, we obtain

$$\begin{aligned}\left|\int_{\Omega_F(0)}((\nabla\hat{\chi}_1)^{-1}(\hat{u}_1-\hat{v}_1))\cdot\nabla(\tilde{\rho}_1-\tilde{\rho}_2)(\tilde{\rho}_1-\tilde{\rho}_2)\,dx\right| &= \frac{1}{2}\left|\int_{\Omega_F(0)}|\tilde{\rho}_1-\tilde{\rho}_2|^2\nabla\cdot((\nabla\hat{\chi}_1)^{-1}(\hat{u}_1-\hat{v}_1))\,dx\right| \\ &\leqslant C\|\tilde{\rho}_1-\tilde{\rho}_2\|^2_{L^2},\end{aligned}$$

where we have used that $\|\hat{u}_1\|_{L^\infty_s(W^{1,\infty})} + \|\hat{v}_1\|_{L^\infty_s(W^{1,\infty})} \leqslant R \leqslant 1$. Moreover, since $\|\nabla\tilde{\rho}_2\|_{L^\infty} \leqslant R$ in $(0,s)$,

$$\begin{aligned}&\left|\int_{\Omega_F(0)}\Big(\big((\nabla\hat{\chi}_2)^{-1}(\hat{u}_2-\hat{v}_2)-(\nabla\hat{\chi}_1)^{-1}(\hat{u}_1-\hat{v}_1)\big)\cdot\nabla\tilde{\rho}_2\Big)(\tilde{\rho}_1-\tilde{\rho}_2)\,dx\right| \\ &= \left|\int_{\Omega_F(0)}\Big(\big((\nabla\hat{\chi}_2)^{-1}-(\nabla\hat{\chi}_1)^{-1}\big)(\hat{u}_2-\hat{v}_2)+(\nabla\hat{\chi}_1)^{-1}(\hat{u}_2-\hat{u}_1-\hat{v}_2+\hat{v}_1)\Big)\cdot\nabla\tilde{\rho}_2(\tilde{\rho}_1-\tilde{\rho}_2)\,dx\right| \\ &\leqslant C(R^2\|(\nabla\hat{\chi}_1)^{-1}-(\nabla\hat{\chi}_2)^{-1}\|_{L^2}+R(\|\hat{u}_1-\hat{u}_2\|_{L^2}+\|\hat{v}_1-\hat{v}_2\|_{L^2}))\|\tilde{\rho}_1-\tilde{\rho}_2\|_{L^2} \\ &\leqslant C(R^4\|(\nabla\hat{\chi}_1)^{-1}-(\nabla\hat{\chi}_2)^{-1}\|^2_{L^2}+R^2(\|\hat{u}_1-\hat{u}_2\|^2_{L^2}+\|\hat{v}_1-\hat{v}_2\|^2_{L^2})+\|\tilde{\rho}_1-\tilde{\rho}_2\|^2_{L^2}).\end{aligned}$$

Moreover, according to the definition of F_0 (given by (39)),

$$\begin{aligned}F_0(\hat{\rho}_1,\hat{u}_1,\hat{a}_1,\hat{\omega}_1)-F_0(\hat{\rho}_2,\hat{u}_2,\hat{a}_2,\hat{\omega}_2) &= (\hat{\rho}_2-\hat{\rho}_1)\mathrm{tr}\big(\nabla\hat{u}_2(\nabla\hat{\chi}_2)^{-1}\big)+(\hat{\rho}_1+\overline{\rho})\mathrm{tr}\big(\nabla(\hat{u}_2-\hat{u}_1)(\nabla\hat{\chi}_2)^{-1}\big) \\ &\quad +(\hat{\rho}_1+\overline{\rho})\mathrm{tr}\big(\nabla\hat{u}_1((\nabla\hat{\chi}_2)^{-1}-(\nabla\hat{\chi}_1)^{-1})\big),\end{aligned}$$

which implies that

$$\begin{aligned}&\|F_0(\hat{\rho}_1,\hat{u}_1,\hat{a}_1,\hat{\omega}_1)-F_0(\hat{\rho}_2,\hat{u}_2,\hat{a}_2,\hat{\omega}_2)\|_{L^2_s(L^2)} \\ &\leqslant C(R\big(\|\hat{\rho}_1-\hat{\rho}_2\|_{L^2_s(L^2)}+\|(\nabla\hat{\chi}_1)^{-1}-(\nabla\hat{\chi}_2)^{-1}\|_{L^2_s(L^2)})+\|\hat{u}_1-\hat{u}_2\|_{L^2_s(H^1)}\big).\end{aligned}$$

Thus, thanks to Gronwall's inequality, we obtain

$$\begin{aligned}\|\tilde{\rho}_1-\tilde{\rho}_2\|_{L^\infty_s(L^2)} \leqslant Ce^{Cs/R}\Big(&R\|\hat{v}_1-\hat{v}_2\|_{L^2_s(L^2)}+\sqrt{R}\|\hat{u}_1-\hat{u}_2\|_{L^2_s(H^1)}+R^{3/2}\|\hat{\rho}_1-\hat{\rho}_2\|_{L^2_s(L^2)} \\ &+R^{3/2}\|(\nabla\hat{\chi}_1)^{-1}-(\nabla\hat{\chi}_2)^{-1}\|_{L^2_s(L^2)}\Big).\end{aligned}$$

Let us now see that

$$\|(\nabla\hat\chi_1)^{-1}-(\nabla\hat\chi_2)^{-1}\|_{L^2_s(L^2)}+\|\hat v_1-\hat v_2\|_{L^2_s(L^2)}\leqslant C(\|\hat a_1-\hat a_2\|_{H^1_s}+\|\hat\omega_1-\hat\omega_2\|_{L^2_s}). \tag{62}$$

First, since $|(\nabla\hat\chi_1)^{-1}-(\nabla\hat\chi_2)^{-1}|=|(\nabla\hat\chi_1)^{-1}(\nabla\hat\chi_1-\nabla\hat\chi_2)(\nabla\hat\chi_2)^{-1}|\leqslant C|\nabla\hat\chi_1-\nabla\hat\chi_2|$ we have, thanks to Lemma 1,

$$\|(\nabla\hat\chi_1)^{-1}-(\nabla\hat\chi_2)^{-1}\|_{L^2_s(L^2)}\leqslant C(s)\|\hat v_1-\hat v_2\|_{L^2_s(H^1)},$$

where $C(s)>0$ is an increasing function of s. Then, from the elliptic equation satisfied by $v_1(t,\cdot):=(\hat v_1\circ\hat\chi_1^{-1})(t,\cdot)$ (see (21)) and performing the change of variables

$$\begin{cases} y:=\hat\chi_1^{-1}(t,x)\in\Omega_F(0),\\ x\in\Omega_F(t),\end{cases}$$

we have that $\hat v_1$ satisfies the following 'elliptic' equation:

$$\begin{cases} -\Delta_y\hat v_1=\hat g & \text{in } \Omega_F(0),\\ \hat v_1=\dot{\hat a}_1+\hat\omega_1\wedge\hat Q_1 y & \text{on } \partial\Omega_S(0),\\ \hat v_1=0 & \text{on } \partial\Omega,\end{cases} \tag{63}$$

where

$$\hat g=\partial_{y_ky_l}\hat v_1(\partial_{y_i}\hat\chi_{1,l}-\delta_{il})(\partial_{y_i}\hat\chi_{1,k}-\delta_{ki})+\partial_{y_ky_i}\hat v_1(\partial_{y_i}\hat\chi_{1,k}-\delta_{ki})+\partial_{y_iy_l}\hat v_1(\partial_{y_i}\hat\chi_{1,l}-\delta_{il})+\partial_{y_k}\hat v_1\partial^2_{y_iy_i}\hat\chi_{1,k}.$$

Using now Lemma 1 (to estimate $\partial_{y_i}\hat\chi_{1,l}-\delta_{il}$, $\partial_{y_i}\hat\chi_{1,k}-\delta_{ki}$ and $\partial^2_{y_iy_i}\hat\chi_{1,k}$) and the fact that R is small enough, one can prove that $\hat v_1\in L^2_s(H^k)\,\forall k\in\mathbb{N}$ and

$$\|\hat v_1\|_{L^2_s(H^k)}\leqslant C(\|\hat a_1\|_{H^1_s}+\|\hat\omega_1\|_{L^2_s}). \tag{64}$$

All the previous properties are also true for $\hat v_2$ in place of $\hat v_1$. Now, using (19) and (64) one can easily prove that

$$|\Delta(v_1-v_2)|\leqslant CR(|\Delta\hat\chi_1-\Delta\hat\chi_2|+|\nabla\hat\chi_1-\nabla\hat\chi_2|+|\nabla\hat v_1-\nabla\hat v_2|+R|D^2\hat v_1-D^2\hat v_2|).$$

Consequently,

$$\|\Delta\hat\chi_1-\Delta\hat\chi_2\|_{L^2_s(L^2)}+\|\nabla\hat\chi_1-\nabla\hat\chi_2\|_{L^2_s(L^2)}+\|\hat v_1-\hat v_2\|_{L^2_s(H^2)}\leqslant C(\|\hat a_1-\hat a_2\|_{H^1_s}+\|\hat\omega_1-\hat\omega_2\|_{L^2_s}).$$

Here, we have used that $\|\hat Q_1-\hat Q_2\|_{L^\infty_s}\leqslant C(s)\|\hat\omega_1-\hat\omega_2\|_{L^2_s}$.

Thus, we obtain

$$\|\tilde\rho_1-\tilde\rho_2\|_{L^\infty_s(L^2)}\leqslant C\sqrt{R}e^{Cs/R}\Big(\|\hat\rho_1-\hat\rho_2\|_{L^\infty_s(L^2)}+\|\hat u_1-\hat u_2\|_{L^2_s(H^1)}+\|\hat a_1-\hat a_2\|_{H^1_s}+\|\hat\omega_1-\hat\omega_2\|_{L^2_s}\Big). \tag{65}$$

Moreover, if we differentiate equation (61) with respect to space, we have

$$\begin{aligned}&(\nabla(\tilde\rho_1-\tilde\rho_2))_t+((\nabla\hat\chi_1)^{-1}(\hat u_1-\hat v_1))\cdot\nabla(\nabla(\tilde\rho_1-\tilde\rho_2))+\nabla((\nabla\hat\chi_1)^{-1}(\hat u_1-\hat v_1))\nabla(\tilde\rho_1-\tilde\rho_2)\\ &=\nabla\Big(\big((\nabla\hat\chi_2)^{-1}(\hat u_2-\hat v_2)-(\nabla\hat\chi_1)^{-1}(\hat u_1-\hat v_1)\big)\cdot\nabla\tilde\rho_2\Big)+\nabla\big(F_0(\hat\rho_1,\hat u_1,\hat a_1,\hat\omega_1)-F_0(\hat\rho_2,\hat u_2,\hat a_2,\hat\omega_2)\big).\end{aligned}$$

Let us multiply this equation by $\nabla(\tilde\rho_1-\tilde\rho_2)$ and integrate in $\Omega_F(0)$. Arguing as previously, we obtain

$$\begin{aligned}\|\nabla(\tilde\rho_1-\tilde\rho_2)\|_{L^\infty_s(L^2)}\leqslant Ce^{Cs/R}\Big(&R\|\hat v_1-\hat v_2\|_{L^2_s(H^1)}+\sqrt{R}\|\hat u_1-\hat u_2\|_{L^2_s(H^2)}\\ &+R^{3/2}\|(\nabla\hat\chi_1)^{-1}-(\nabla\hat\chi_2)^{-1}\|_{L^2_s(H^1)}+R^{3/2}\|\hat\rho_1-\hat\rho_2\|_{L^2_s(H^1)}\Big).\end{aligned}$$

Thus, concluding as before, we obtain

$$\|\tilde\rho_1-\tilde\rho_2\|_{L^\infty_s(H^1)}\leqslant C\sqrt{R}e^{Cs/R}\Big(\|\hat\rho_1-\hat\rho_2\|_{L^\infty_s(H^1)}+\|\hat u_1-\hat u_2\|_{L^2_s(H^2)}+\|\hat a_1-\hat a_2\|_{H^1_s}+\|\hat\omega_1-\hat\omega_2\|_{L^2_s}\Big). \tag{66}$$

Estimates for $\tilde u_1 - \tilde u_2$.

Next, we consider the equations satisfied by $(\tilde u_1 - \tilde u_2, a_1 - a_2, \omega_1 - \omega_2)$

$$\begin{cases} (\tilde u_1 - \tilde u_2)_t - 2\mu\nabla\cdot\big(\epsilon(\tilde u_1 - \tilde u_2)\big) - \mu'\nabla\cdot\big((\nabla\cdot(\tilde u_1 - \tilde u_2))Id\big) \\ \qquad = f_1(\hat\rho_1, \hat u_1, \hat a_1, \hat\omega_1) - f_1(\hat\rho_2, \hat u_2, \hat a_2, \hat\omega_2) - p^0\nabla(\tilde\rho_1 - \tilde\rho_2) \\ m(\ddot a_1 - \ddot a_2) = \displaystyle\int_{\partial\Omega_S(0)} \Big(2\mu\epsilon(\tilde u_1 - \tilde u_2) + \mu'(\nabla\cdot(\tilde u_1 - \tilde u_2))Id\Big) n\, d\sigma \\ \qquad + f_2(\hat\rho_1, \hat u_1, \hat a_1, \hat\omega_1) - f_2(\hat\rho_2, \hat u_2, \hat a_2, \hat\omega_2) - \displaystyle\int_{\partial\Omega_S(0)} p^0(\tilde\rho_1 - \tilde\rho_2) n\, d\sigma, \\ \hat J_1(\dot\omega_1 - \dot\omega_2) = \big(\hat J_2 - \hat J_1\big)\dot\omega_2 + \big(\hat J_1\hat\omega_1\big)\wedge(\omega_1 - \omega_2) + \hat J_1(\hat\omega_1 - \hat\omega_2)\wedge\omega_2 + \big(\hat J_1 - \hat J_2\big)\hat\omega_2\wedge\omega_2 \\ \qquad + \displaystyle\int_{\partial\Omega_S(0)} (\widehat Q_1 x)\wedge\Big(\big(2\mu\epsilon(\tilde u_1 - \tilde u_2) + \mu'(\nabla\cdot(\tilde u_1 - \tilde u_2))Id\big)n\Big)\, d\sigma \\ \qquad + \displaystyle\int_{\partial\Omega_S(0)} ((\widehat Q_1 - \widehat Q_2)x)\wedge\Big(\big(2\mu\epsilon(\tilde u_2) + \mu'(\nabla\cdot\tilde u_2)Id - p^0\tilde\rho_2 Id\big)n\Big)\, d\sigma \\ \qquad + f_3(\hat\rho_1, \hat u_1, \hat a_1, \hat\omega_1) - f_3(\hat\rho_2, \hat u_2, \hat a_2, \hat\omega_2) - \displaystyle\int_{\partial\Omega_S(0)} (\widehat Q_1 x)\wedge\big(p^0(\tilde\rho_1 - \tilde\rho_2)n\big)\, d\sigma, \\ \tilde u_1 - \tilde u_2 = (\dot a_1 - \dot a_2) + (\omega_1 - \omega_2)\wedge(\widehat Q_1 x) + \omega_2\wedge(\widehat Q_1 - \widehat Q_2)x \quad \text{on } \partial\Omega_S(0). \end{cases}$$

We multiply the first equation by $\tilde u_1 - \tilde u_2$ and integrate in space and next we multilpy the second and the third equations respectively by $\dot a_1 - \dot a_2$ and $\omega_1 - \omega_2$. We obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_{\Omega_F(0)} |\tilde u_1 - \tilde u_2|^2\, dx + \mu\int_{\Omega_F(0)} \big|\epsilon(\tilde u_1 - \tilde u_2)\big|^2\, dx + \mu'\int_{\Omega_F(0)} \big|\nabla\cdot(\tilde u_1 - \tilde u_2)\big|^2\, dx \\ &+ \frac{m}{2}\frac{d}{dt}|\dot a_1 - \dot a_2|^2 + \frac{1}{2}\frac{d}{dt}\big(\hat J_1(\omega_1 - \omega_2)\cdot(\omega_1 - \omega_2)\big) \\ &= \int_{\Omega_F(0)} \Big(f_1(\hat\rho_1, \hat u_1, \hat a_1, \hat\omega_1) - f_1(\hat\rho_2, \hat u_2, \hat a_2, \hat\omega_2)\Big)\cdot(\tilde u_1 - \tilde u_2)\, dx - \int_{\Omega_F(0)} p^0\nabla(\tilde\rho_1 - \tilde\rho_2)\cdot\big(\tilde u_1 - \tilde u_2\big)\, dx \\ &+ \big(f_2(\hat\rho_1, \hat u_1, \hat a_1, \hat\omega_1) - f_2(\hat\rho_2, \hat u_2, \hat a_2, \hat\omega_2)\big)\cdot\big(\dot a_1 - \dot a_2\big) - \left(\int_{\partial\Omega_S(0)} p^0(\tilde\rho_1 - \tilde\rho_2)n\, d\sigma\right)\cdot\big(\dot a_1 - \dot a_2\big) \\ &+ \big(\hat J_2 - \hat J_1\big)\dot\omega_2\cdot\big(\omega_1 - \omega_2\big) + \Big(\big(\hat J_1(\hat\omega_1 - \hat\omega_2)\wedge\omega_2\Big)\cdot\big(\omega_1 - \omega_2\big) + \big(\big(\hat J_1 - \hat J_2\big)\hat\omega_2\wedge\omega_2\big)\cdot\big(\omega_1 - \omega_2\big) \\ &+ \left(\int_{\partial\Omega_S(0)} ((\widehat Q_1 - \widehat Q_2)x)\wedge\Big(\big(2\mu\epsilon(\tilde u_2) + \mu'(\nabla\cdot\tilde u_2)Id - p^0\tilde\rho_2 Id\big)n\Big)\, d\sigma\right)\cdot\big(\omega_1 - \omega_2\big) \\ &+ \big(f_3(\hat\rho_1, \hat u_1, \hat a_1, \hat\omega_1) - f_3(\hat\rho_2, \hat u_2, \hat a_2, \hat\omega_2)\big)\cdot\big(\omega_1 - \omega_2\big) - \left(\int_{\partial\Omega_S(0)} (\widehat Q_1 x)\wedge\big(p^0(\tilde\rho_1 - \tilde\rho_2)n\big)\, d\sigma\right)\cdot\big(\omega_1 - \omega_2\big) \\ &+ \int_{\partial\Omega_S(0)} \Big(\big(2\mu\epsilon(\tilde u_1 - \tilde u_2) + \mu'\nabla\cdot(\tilde u_1 - \tilde u_2)Id\big)n\Big)\cdot\big(\omega_2\wedge((\widehat Q_1 - \widehat Q_2)x)\big)\, d\sigma + \frac{1}{2}\big(\dot{\hat J}_1(\omega_1 - \omega_2)\cdot(\omega_1 - \omega_2)\big). \end{aligned}$$

Thanks to Gronwall's inequality, we deduce that

$$\begin{aligned} &\|\tilde u_1 - \tilde u_2\|_{L^\infty_s(L^2)} + \|\tilde u_1 - \tilde u_2\|_{L^2_s(H^1)} + \|a_1 - a_2\|_{W^{1,\infty}_s} + \|\omega_1 - \omega_2\|_{L^\infty_s} \\ &\leqslant Ce^{Cs}(\|f_1(\hat\rho_1, \hat u_1, \hat a_1, \hat\omega_1) - f_1(\hat\rho_2, \hat u_2, \hat a_2, \hat\omega_2))\|_{L^2_s(L^2)} + \|\tilde\rho_1 - \tilde\rho_2\|_{L^2_s(H^1)} \\ &+ \|f_2(\hat\rho_1, \hat u_1, \hat a_1, \hat\omega_1) - f_2(\hat\rho_2, \hat u_2, \hat a_2, \hat\omega_2))\|_{L^2_s} + \sqrt{R}\|\hat\omega_1 - \hat\omega_2\|_{L^2_s} \\ &+ \|f_3(\hat\rho_1, \hat u_1, \hat a_1, \hat\omega_1) - f_3(\hat\rho_2, \hat u_2, \hat a_2, \hat\omega_2))\|_{L^2_s} + \sqrt{R}\|\tilde u_1 - \tilde u_2\|_{L^2_s(H^2)}). \end{aligned} \tag{67}$$

We recall that the definitions of f_1, f_2 and f_3 are given by (33), (34) and (35) respectively. Let us first consider the first term in the right-hand side of (67). We do not detail the whole computation but we explain how to estimate some terms. The remaining terms in $f_1(\hat\rho_1,\hat u_1,\hat a_1,\hat\omega_1)-f_1(\hat\rho_2,\hat u_2,\hat a_2,\hat\omega_2))$ can be estimated in the same way. We have

$$(\hat u_{1,j}-\hat v_{1,j})(\nabla\hat u_1(\nabla\hat\chi_1)^{-1})_{ij}-(\hat u_{2,j}-\hat v_{2,j})(\nabla\hat u_2(\nabla\hat\chi_2)^{-1})_{ij}=((\hat u_{1,j}-\hat u_{2,j})-(\hat v_{1,j}-\hat v_{2,j}))(\nabla\hat u_1(\nabla\hat\chi_1)^{-1})_{ij}$$
$$+(\hat u_{2,j}-\hat v_{2,j})(\nabla(\hat u_1-\hat u_2)(\nabla\hat\chi_1)^{-1})_{ij}+(\hat u_{2,j}-\hat v_{2,j})(\nabla\hat u_2((\nabla\hat\chi_1)^{-1}-(\nabla\hat\chi_2)^{-1}))_{ij}.$$

This implies that

$$\|(\hat u_{1,j}-\hat v_{1,j})(\nabla\hat u_1(\nabla\hat\chi_1)^{-1})_{ij}-(\hat u_{2,j}-\hat v_{2,j})(\nabla\hat u_2(\nabla\hat\chi_2)^{-1})_{ij}\|_{L^2_s(L^2)}\leqslant CR\big(\|\hat u_1-\hat u_2\|_{L^2_s(H^1)}$$
$$+\|\hat v_1-\hat v_2\|_{L^2_s(L^2)}+\|(\nabla\hat\chi)_1^{-1}-(\nabla\hat\chi)_2^{-1}\|_{L^2_s(L^2)}\big).$$

We also have

$$\left(\frac{\overline\rho}{\hat\rho_1+\overline\rho}-1\right)\partial_{x_l}(\partial_{x_k}\hat u_{1,i}(\nabla\hat\chi_1)^{-1}_{kj})(\nabla\hat\chi_1)^{-1}_{lj}-\left(\frac{\overline\rho}{\hat\rho_2+\overline\rho}-1\right)\partial_{x_l}(\partial_{x_k}\hat u_{2,i}(\nabla\hat\chi_2)^{-1}_{kj})(\nabla\hat\chi_2)^{-1}_{lj}$$
$$=\overline\rho\left(\frac{1}{\hat\rho_1+\overline\rho}-\frac{1}{\hat\rho_2+\overline\rho}\right)\partial_{x_l}(\partial_{x_k}\hat u_{1,i}(\nabla\hat\chi_1)^{-1}_{kj})(\nabla\hat\chi_1)^{-1}_{lj}+\left(\frac{\overline\rho}{\hat\rho_2+\overline\rho}-1\right)\partial_{x_l}(\partial_{x_k}(\hat u_{1,i}-\hat u_{2,i})(\nabla\hat\chi_1)^{-1}_{kj})(\nabla\hat\chi_1)^{-1}_{lj}$$
$$+\left(\frac{\overline\rho}{\hat\rho_2+\overline\rho}-1\right)\partial_{x_l}(\partial_{x_k}\hat u_{2,i}((\nabla\hat\chi_1)^{-1}_{kj}-(\nabla\hat\chi_2)^{-1}_{kj}))(\nabla\hat\chi_1)^{-1}_{lj}$$
$$+\left(\frac{\overline\rho}{\hat\rho_2+\overline\rho}-1\right)\partial_{x_l}(\partial_{x_k}\hat u_{2,i}(\nabla\hat\chi_2)^{-1}_{kj})((\nabla\hat\chi_1)^{-1}_{lj}-(\nabla\hat\chi_2)^{-1}_{lj}).$$

Since R has been chosen small enough,

$$\left\|\frac{1}{\hat\rho_1+\overline\rho}-\frac{1}{\hat\rho_2+\overline\rho}\right\|_{L^2_s(L^2)}\leqslant C\left\|\hat\rho_1-\hat\rho_2\right\|_{L^\infty_s(L^2)}.$$

Thus,

$$\left\|\left(\frac{\overline\rho}{\hat\rho_1+\overline\rho}-1\right)\partial_{x_l}(\partial_{x_k}\hat u_{1,i}(\nabla\hat\chi_1)^{-1}_{kj})(\nabla\hat\chi_1)^{-1}_{lj}-\left(\frac{\overline\rho}{\hat\rho_2+\overline\rho}-1\right)\partial_{x_l}(\partial_{x_k}\hat u_{2,i}(\nabla\hat\chi_2)^{-1}_{kj})(\nabla\hat\chi_2)^{-1}_{lj}\right\|_{L^2_s(L^2)}$$
$$\leqslant CR\big(\|\hat\rho_1-\hat\rho_2\|_{L^\infty_s(L^2)}+\|\hat u_1-\hat u_2\|_{L^2_s(H^2)}+\|(\nabla\hat\chi)_1^{-1}-(\nabla\hat\chi)_2^{-1}\|_{L^2_s(H^1)}\big).$$

For the last term in the expression of f_1, we notice that

$$\left(\frac{P'(\hat\rho_1+\overline\rho)}{\hat\rho_1+\overline\rho}-\frac{P'(\overline\rho)}{\overline\rho}\right)\partial_{x_i}\hat\rho_1-\left(\frac{P'(\hat\rho_2+\overline\rho)}{\hat\rho_2+\overline\rho}-\frac{P'(\overline\rho)}{\overline\rho}\right)\partial_{x_i}\hat\rho_2=P'(\hat\rho_1+\overline\rho)\left(\frac{1}{\hat\rho_1+\overline\rho}-\frac{1}{\hat\rho_2+\overline\rho}\right)\partial_{x_i}\hat\rho_1$$
$$+\left(\frac{P'(\hat\rho_1+\overline\rho)-P'(\hat\rho_2+\overline\rho)}{\hat\rho_2+\overline\rho}\right)\partial_{x_i}\hat\rho_1+\left(\frac{P'(\hat\rho_2+\overline\rho)}{\hat\rho_2+\overline\rho}-\frac{P'(\overline\rho)}{\overline\rho}\right)\partial_{x_i}(\hat\rho_1-\hat\rho_2).$$

Thus, we directly obtain that

$$\left\|\left(\frac{P'(\hat\rho_1+\overline\rho)}{\hat\rho_1+\overline\rho}-\frac{P'(\overline\rho)}{\overline\rho}\right)\partial_{x_i}\hat\rho_1-\left(\frac{P'(\hat\rho_2+\overline\rho)}{\hat\rho_2+\overline\rho}-\frac{P'(\overline\rho)}{\overline\rho}\right)\partial_{x_i}\hat\rho_2\right\|_{L^2_s(L^2)}\leqslant CR\|\hat\rho_1-\hat\rho_2\|_{L^2_s(H^1)}.$$

With the same kind of arguments for the other terms and using (62), we obtain that

$$\|f_1(\hat\rho_1,\hat u_1,\hat a_1,\hat\omega_1)-f_1(\hat\rho_2,\hat u_2,\hat a_2,\hat\omega_2))\|_{L^2_s(L^2)}\leqslant CR\big(\|\hat u_1-\hat u_2\|_{L^2_s(H^2)}+\|\hat a_1-\hat a_2\|_{H^1_s}$$
$$+\|\hat\omega_1-\hat\omega_2\|_{L^2_s}+\|\hat\rho_1-\hat\rho_2\|_{L^2_s(H^1)}\big). \tag{68}$$

Moreover, we can also prove that

$$\|f_2(\hat\rho_1,\hat u_1,\hat a_1,\hat\omega_1)-f_2(\hat\rho_2,\hat u_2,\hat a_2,\hat\omega_2))\|_{L^2_s}+\|f_3(\hat\rho_1,\hat u_1,\hat a_1,\hat\omega_1)-f_3(\hat\rho_2,\hat u_2,\hat a_2,\hat\omega_2))\|_{L^2_s}$$
$$\leqslant CR\big(\|\hat a_1-\hat a_2\|_{H^1_s}+\|\hat\omega_1-\hat\omega_2\|_{L^2_s}+\|\hat u_1-\hat u_2\|_{L^2_s(H^2)}+\|\hat\rho_1-\hat\rho_2\|_{L^2_s(H^1)}\big). \tag{69}$$

Thanks to (66), (68) and (69), inequality (67) finally becomes

$$\|\tilde u_1-\tilde u_2\|_{L^\infty_s(L^2)}+\|\tilde u_1-\tilde u_2\|_{L^2_s(H^1)}+\|a_1-a_2\|_{W^{1,\infty}_s}+\|\omega_1-\omega_2\|_{L^\infty_s}\leqslant Ce^{Cs}\sqrt{R}\|\tilde u_1-\tilde u_2\|_{L^2_s(H^2)}$$
$$+C\sqrt{R}e^{Cs/R}\Big(\|\hat\rho_1-\hat\rho_2\|_{L^\infty_s(H^1)}+\|\hat u_1-\hat u_2\|_{L^2_s(H^2)}+\|\hat a_1-\hat a_2\|_{H^1_s}+\|\hat\omega_1-\hat\omega_2\|_{L^2_s}\Big). \tag{70}$$

To get additional estimates, we multiply the first equation by $(\tilde u_1-\tilde u_2)_t$ and integrate in space and next we multiply the second and the third equations respectively by $\ddot a_1-\ddot a_2$ and $\dot\omega_1-\dot\omega_2$. We obtain

$$\int_{\Omega_F(0)}\big|(\tilde u_1-\tilde u_2)_t\big|^2dx+\frac{\mu}{2}\frac{d}{dt}\int_{\Omega_F(0)}\big|\epsilon(\tilde u_1-\tilde u_2)\big|^2dx+\mu'\frac{d}{dt}\int_{\Omega_F(0)}\big|\nabla\cdot(\tilde u_1-\tilde u_2)\big|^2dx$$
$$+m|\ddot a_1-\ddot a_2|^2+\big(\hat J_1(\dot\omega_1-\dot\omega_2)\cdot(\dot\omega_1-\dot\omega_2)\big)$$
$$=\int_{\Omega_F(0)}\Big(f_1(\hat\rho_1,\hat u_1,\hat a_1,\hat\omega_1)-f_1(\hat\rho_2,\hat u_2,\hat a_2,\hat\omega_2)\Big)\cdot\big(\tilde u_1-\tilde u_2\big)_tdx-\int_{\Omega_F(0)}(p^0\nabla(\tilde\rho_1-\tilde\rho_2))\cdot(\tilde u_1-\tilde u_2)_t\,dx$$
$$+\big(f_2(\hat\rho_1,\hat u_1,\hat a_1,\hat\omega_1)-f_2(\hat\rho_2,\hat u_2,\hat a_2,\hat\omega_2)\big)\cdot\big(\ddot a_1-\ddot a_2\big)-\left(\int_{\partial\Omega_S(0)}p^0(\tilde\rho_1-\tilde\rho_2)n\,d\sigma\right)\cdot(\ddot a_1-\ddot a_2)$$
$$+\Big((\hat J_2-\hat J_1)\dot\omega_2+\big(\hat J_1\hat\omega_1\big)\wedge(\omega_1-\omega_2)+\hat J_1\big(\hat\omega_1-\hat\omega_2\big)\wedge\omega_2+(\hat J_1-\hat J_2)\hat\omega_2\wedge\omega_2\Big)\cdot\big(\dot\omega_1-\dot\omega_2\big)$$
$$+\left(f_3(\hat\rho_1,\hat u_1,\hat a_1,\hat\omega_1)-f_3(\hat\rho_2,\hat u_2,\hat a_2,\hat\omega_2)-\int_{\partial\Omega_S(0)}\big(\widehat Q_1x\big)\wedge(p^0(\tilde\rho_1-\tilde\rho_2)n)\,d\sigma\right)\cdot\big(\dot\omega_1-\dot\omega_2\big)$$
$$+\left(\int_{\partial\Omega_S(0)}\big((\widehat Q_1-\widehat Q_2)x\big)\wedge\Big(\big(2\mu\epsilon(\tilde u_2)+\mu'(\nabla\cdot\tilde u_2)Id-p^0\tilde\rho_2Id\big)n\Big)\,d\sigma\right)\cdot\big(\dot\omega_1-\dot\omega_2\big)$$
$$+\int_{\partial\Omega_S(0)}\Big(\big(2\mu\epsilon(\tilde u_2-\tilde u_1)+\mu'(\nabla\cdot(\tilde u_2-\tilde u_1))Id\big)n\Big)\cdot\big(\dot\omega_2\wedge\big((\widehat Q_1-\widehat Q_2)x\big)\big)\,d\sigma$$
$$+\int_{\partial\Omega_S(0)}\Big(\big(2\mu\epsilon(\tilde u_2-\tilde u_1)+\mu'(\nabla\cdot(\tilde u_2-\tilde u_1))Id\big)n\Big)\cdot\big(\omega_2\wedge\big((\dot{\widehat Q}_1-\dot{\widehat Q}_2)x\big)\big)\,d\sigma$$
$$+\int_{\partial\Omega_S(0)}\Big(\big(2\mu\epsilon(\tilde u_2-\tilde u_1)+\mu'(\nabla\cdot(\tilde u_2-\tilde u_1))Id\big)n\Big)\cdot\big((\omega_1-\omega_2)\wedge\big(\dot{\widehat Q}_1x\big)\big)\,d\sigma.$$

Observe that, here, we just need estimates of $f_i(\hat\rho_1,\hat u_1,\hat a_1,\hat\omega_1)-f_i(\hat\rho_2,\hat u_2,\hat a_2,\hat\omega_2)$ $(i=1,2,3)$ in L^2 norms. This was already done in (68) and (69).

Then, arguing as before, we find the existence of $C>0$ such that

$$\|\tilde u_1-\tilde u_2\|_{H^1_s(L^2)}+\|\tilde u_1-\tilde u_2\|_{L^\infty_s(H^1)}+\|a_1-a_2\|_{H^2_s}+\|\omega_1-\omega_2\|_{H^1_s}$$
$$\leqslant C(R(\|\hat\rho_1-\hat\rho_2\|_{L^\infty_s(H^1)}+\|\hat u_1-\hat u_2\|_{L^2_s(H^2)}+\|\hat a_1-\hat a_2\|_{H^1_s})+\sqrt{R}(\|\hat\omega_1-\hat\omega_2\|_{L^2_s}+\|\omega_1-\omega_2\|_{L^2_s})) \tag{71}$$
$$+C\|\tilde\rho_1-\tilde\rho_2\|_{L^\infty_s(H^1)}+\sqrt{R}\|\tilde u_1-\tilde u_2\|_{L^2_s(H^2)}.$$

Moreover, according to the equation and the boundary conditions satisfied by $\tilde u_1-\tilde u_2$, we also obtain that $\tilde u_1-\tilde u_2$ is bounded in $L^2_s(H^2)$ and is bounded by the right-hand side of (71). If we reassemble this result with (66) and (70), we get

$$\|\tilde\rho_1-\tilde\rho_2\|_{L^\infty_s(H^1)}+\|\tilde u_1-\tilde u_2\|_{L^2_s(H^2)}+\|\tilde u_1-\tilde u_2\|_{H^1_s(L^2)}+\|a_1-a_2\|_{H^2_s}+\|\omega_1-\omega_2\|_{H^1_s}$$
$$\leqslant C\sqrt{R}e^{Cs/R}\Big(\|\hat\rho_1-\hat\rho_2\|_{L^\infty_s(H^1)}+\|\hat u_1-\hat u_2\|_{L^2_s(H^2)}+\|\hat a_1-\hat a_2\|_{H^1_s}+\|\hat\omega_1-\hat\omega_2\|_{L^2_s}\Big).$$

Next, we recall that our final time was given by $s = CR$ (see (55)). Thus we obtain that

$$\|\tilde{\rho}_1 - \tilde{\rho}_2\|_{L^\infty_s(H^1)} + \|\tilde{u}_1 - \tilde{u}_2\|_{L^2_s(H^2)} + \|\tilde{u}_1 - \tilde{u}_2\|_{H^1_s(L^2)} + \|a_1 - a_2\|_{H^2_s} + \|\omega_1 - \omega_2\|_{H^1_s}$$
$$\leqslant C\sqrt{R}\Big(\|\hat{\rho}_1 - \hat{\rho}_2\|_{L^\infty_s(H^1)} + \|\hat{u}_1 - \hat{u}_2\|_{L^2_s(H^2)} + \|\hat{a}_1 - \hat{a}_2\|_{H^1_s} + \|\hat{\omega}_1 - \hat{\omega}_2\|_{L^2_s}\Big).$$

¿From the definition of $\tilde{u}_{F,i}$ (see (58)) and taking R small enough, we directly get the existence of a positive constant $\alpha < 1$ such that

$$\|\tilde{\rho}_1 - \tilde{\rho}_2\|_{L^\infty_s(H^1)} + \|\tilde{u}_{F,1} - \tilde{u}_{F,2}\|_{L^2_s(H^2)} + \|a_1 - a_2\|_{H^1_s} + \|\omega_1 - \omega_2\|_{L^2_s}$$
$$\leqslant \alpha\Big(\|\hat{\rho}_1 - \hat{\rho}_2\|_{L^\infty_s(H^1)} + \|\hat{u}_{F,1} - \hat{u}_{F,2}\|_{L^2_s(H^2)} + \|\hat{a}_1 - \hat{a}_2\|_{H^1_s} + \|\hat{\omega}_1 - \hat{\omega}_2\|_{L^2_s}\Big).$$

Thus, Λ is a contraction and we obtain the existence and uniqueness of a fixed point on $(0, s)$. Finally, we easily check that this fixed point is solution of (30).

4 A priori estimates: proof of Proposition 8

The proof of this result is inspired by the works [17] and [18], where the authors dealt with the compressible Navier-Stokes equations.

For the sake of simplicity, in this paragraph we will denote our solution by (ρ, u, a, ω) instead of $(\tilde{\rho}, \tilde{u}, a, \omega)$ and the velocity associated to the structure will be denoted by v instead of $\tilde{v}$. The equations are

$$\left\{\begin{array}{ll}
\rho_t + ((\nabla\chi)^{-1}(u - v)) \cdot \nabla\rho + \overline{\rho}(\nabla \cdot u) = f_0 & \text{in } (0,T) \times \Omega_F(0),\\
u_t - 2\mu\nabla \cdot (\epsilon(u)) - \mu'\nabla \cdot ((\nabla \cdot u)Id) + p^0\nabla\rho = f_1 & \text{in } (0,T) \times \Omega_F(0),\\
m\ddot{a} = \displaystyle\int_{\partial\Omega_S(0)} \big(2\mu\epsilon(u) + \mu'(\nabla \cdot u)Id - p^0\rho Id\big)\, n\, d\sigma + f_2 & \text{in } (0,T),\\
J\dot{\omega} = \displaystyle\int_{\partial\Omega_S(0)} Qx \wedge \big(2\mu\epsilon(u) + \mu'(\nabla \cdot u)Id - p^0\rho Id\big)\, n\, d\sigma & \\
\qquad + (J\omega) \wedge \omega + f_3 & \text{in } (0,T),\\
u = \dot{a} + \omega \wedge Qx & \text{in } (0,T) \times \partial\Omega_S(0),\\
u = 0 & \text{on } (0,T) \times \partial\Omega,\\
\rho(0,\cdot) = \rho_0 - \overline{\rho},\ u(0,\cdot) = u_0 & \text{in } \Omega_F(0),\\
a(0) = 0,\ \dot{a}(0) = a_0,\ \omega(0) = \omega_0. &
\end{array}\right. \tag{72}$$

where f_0, f_1, f_2 and f_3 are given by (32)-(35) (with $(\tilde{\rho}, \tilde{u})$ replaced by (ρ, u)).

In the following lines, we will give several lemmas where we will present a priori estimates of different nature:

• Global estimates associated to an elliptic operator (Lemma 9) and energy-type estimates associated to the compressible Stokes system (Lemmas 10 and 11).

• Interior estimates for the compressible Stokes system (Lemma 12).

• Estimates close to the boundary $\partial\Omega_F(0)$. First, we will estimate the tangential derivatives (Lemma 13) and then the normal ones (Lemma 14).

• Global estimates for the Stokes operator.

In the above lines, the term 'Global' refers to estimates on the whole domain $\Omega_F(0)$.

All this will be proved under the hypothesis $(\rho, u, a, \omega) \in X(0, T)$. The conclusion of all these Lemmas will be the following inequality:

$$\begin{aligned}
N_{0,T}(\rho, u, a, \omega) \leqslant C\Big(&\|f_0\|_{H^1_T(L^2)} + \|f_0\|_{L^2_T(H^3)} + \|f_1\|_{L^2_T(H^2)} + \|f_1\|_{H^1_T(L^2)} + \|f_1\|_{L^\infty_T(H^1)}\\
&+ \|f_2\|_{W^{1,1}_T} + \|f_2\|_{H^1_T} + \|f_3\|_{W^{1,1}_T} + \|f_3\|_{H^1_T} + \|\partial_t u(0,\cdot)\|_{H^1} + \|u_0\|_{H^3} + \|\partial_t\rho(0,\cdot)\|_{L^2}\\
&+ \|\rho_0\|_{H^3} + |\ddot{a}(0)| + |a_0| + |\dot{\omega}(0)| + |\omega_0| + N^{3/2}_{0,T}(\rho, u, a, \omega) + N^2_{0,T}(\rho, u, a, \omega)\Big).
\end{aligned} \tag{73}$$

Next, using the equations of u, ρ, a and ω, we see that

$$\|\partial_t u(0,\cdot)\|_{H^1} \leqslant C(\|u_0\|_{H^3} + \|\rho_0\|_{H^2} + \|f_1\|_{L^\infty_T(H^1)}),$$

$$\|\partial_t \rho(0,\cdot)\|_{L^2} \leqslant C(\|u_0\|_{H^2} + \|f_0\|_{L^\infty_T(L^2)} + N^2_{0,T}(\rho,u,a,\omega)),$$
$$|\ddot{a}(0)| \leqslant C(\|u_0\|_{H^2} + \|\rho_0\|_{H^1} + \|f_2\|_{L^\infty_T})$$

and
$$|\dot{\omega}(0)| \leqslant C(\|u_0\|_{H^2} + \|\rho_0\|_{H^1} + \|f_3\|_{L^\infty_T} + N^2_{0,T}(\rho,u,a,\omega)).$$

Then, from the expression of f_i ($0 \leqslant i \leqslant 3$), we have

$$\|f_0\|_{H^1_T(L^2)} + \|f_0\|_{L^2_T(H^3)} + \|f_1\|_{L^2_T(H^2)} + \|f_1\|_{H^1_T(L^2)} + \|f_2\|_{W^{1,1}_T}$$
$$+\|f_1\|_{L^\infty_T(H^1)} + \|f_2\|_{H^1_T} + \|f_3\|_{W^{1,1}_T} + \|f_3\|_{H^1_T} \leqslant N^2_{0,T}(\rho,u,a,\omega).$$

Finally, using the assumption $N_{0,T}(\rho,u,a,\omega) \leqslant \delta_1$ with δ_1 small enough, we conclude the proof of Proposition 8. Consequently, from now on we concentrate in the proof of inequality (73).

4.1 Technical results

All along the proof of these Lemmas and for the sake of simplicity we will adopt the notation $N(0,T)$ instead of $N_{0,T}(\rho,u,a,\omega)$.

Furthermore, C will stand for generic positive constants which do not depend on t but which may depend on $\Omega_S(0)$, $\Omega_F(0)$ and Ω.

The first result concerns a classical elliptic estimate for u:

Lemma 9 *Let $k = 2, 3$. Then, we have*

$$\|u\|_{L^\infty_T(H^k)} \leqslant C(\|u\|_{W^{1,\infty}_T(H^{k-2})} + \|\rho\|_{L^\infty_T(H^{k-1})} + \|a\|_{W^{1,\infty}_T} + \|\omega\|_{L^\infty_T} + \|f_1\|_{L^\infty_T(H^{k-2})}). \tag{74}$$

Lemma 10 *Let $k = 0, 1$. Then, there exist a small constant $\delta > 0$ and a constant $C > 0$ such that*

$$\begin{aligned}&\|u\|_{W^{k,\infty}_T(L^2)} + \|\rho\|_{W^{k,\infty}_T(L^2)} + \|u\|_{H^k_T(H^1)} + \|a\|_{W^{k+1,\infty}_T} + \|\omega\|_{W^{k,\infty}_T}\\ &\leqslant \delta\|\rho\|_{H^k_T(L^2)} + C(\|f_0\|_{H^k_T(L^2)} + \|f_1\|_{H^k_T(L^2)} + \|f_2\|_{W^{k,1}_T} + \|f_3\|_{W^{k,1}_T}\\ &+\|\partial^k_t u(0,\cdot)\|_{L^2} + \|\partial^k_t \rho(0,\cdot)\|_{L^2} + |\partial^{k+1}_t a(0)| + |\partial^k_t \omega(0)| + N^{3/2}(0,T) + N^2(0,T)).\end{aligned} \tag{75}$$

Proof:

1) $k = 0$. We multiply the equation of ρ by $p^0\rho/\overline{\rho}$ and the equation of u by u. Integrating in Q_s for $s \in (0,T)$ and adding up both expressions, we obtain:

$$\begin{aligned}&\frac{1}{2}\sup_{s\in(0,T)}\left(|\dot{a}|^2(s) + |\omega|^2(s) + \int_{\Omega_F(0)}\left(\frac{p^0}{\overline{\rho}}|\rho|^2 + |u|^2\right)(s)\,dx\right) + \iint_{Q_T}(2\mu|\epsilon(u)|^2 + \mu'|\nabla\cdot u|^2)\,dx\,d\tau\\ &\leqslant C\left(\iint_{Q_T}(|u| + |v| + |\nabla v| + |\nabla u|)|\rho|^2(|(\nabla\chi)^{-1}| + |\nabla((\nabla\chi)^{-1})|)\,dx\,d\tau\right.\\ &+\iint_{Q_T}(|f_0|^2 + |f_1|^2 + \delta(|\rho|^2 + |u|^2))\,dx\,d\tau + \int_{\Omega_F(0)}(|u_0|^2 + |\rho_0|^2)\,dx + |a_0|^2 + |\omega_0|^2\\ &\left.+\sup_{s\in(0,T)}|\dot{a}|\int_0^T|f_2|d\tau + \sup_{s\in(0,T)}|\omega|\int_0^T|f_3|d\tau\right),\end{aligned} \tag{76}$$

for $\delta > 0$ small enough. Here, we have put together the integrals coming from the third term in the equation of ρ and the fourth term in the equation of u and we have integrated by parts. We have also used the fact that according to the definition (9) of J, we have $(\dot{J}\omega)\cdot\omega = 0$.

Thanks to (43), we get the $L^2_T(H^1)$ norm of u in the left hand side of (76), which allows to absorb $\delta\|u\|^2_{L^2_T(L^2)}$. Moreover, we can estimate the nonlinear term in (76) in the following way:

$$\iint_{Q_T} (|u| + |v| + |\nabla u| + |\nabla v|)|\rho|^2(|(\nabla\chi)^{-1}| + |\nabla((\nabla\chi)^{-1})|)\, dx\, d\tau$$
$$\leqslant C(\|u\|_{L^2_T(W^{1,\infty})} + \|v\|_{L^2_T(W^{1,\infty})})\|\rho\|^2_{L^4_T(L^\infty)} \leqslant CN^3(0,T).$$

The definition of $N(0,T)$ was given in (36). The estimate of $\|\rho\|^2_{L^4_T(L^\infty)}$ comes from an interpolation argument, while the estimate on $\|v\|_{L^2_T(W^{1,\infty})}$ comes from (20). Thus, we deduce (75) for $k = 0$

2) $k = 1$**.** First, we differentiate (72) with respect to t:

$$\begin{cases} (\rho_t)_t + ((\nabla\chi)^{-1}(u-v))\cdot\nabla(\rho_t) + \overline{\rho}(\nabla\cdot u_t) = f_{0,t} - ((\nabla\chi)^{-1}(u-v))_t\cdot\nabla\rho & \text{in } (0,T)\times\Omega_F(0),\\ (u_t)_t - 2\mu\nabla\cdot(\epsilon(u_t)) - \mu'\nabla\cdot((\nabla\cdot u_t)Id) + p^0\nabla\rho_t = f_{1,t} & \text{in } (0,T)\times\Omega_F(0),\\ m\dddot{a} = \displaystyle\int_{\partial\Omega_S(0)} \big(2\mu\epsilon(u_t) + \mu'(\nabla\cdot u_t)Id - p^0\rho_t Id\big)\, n\, d\sigma + f_{2,t} & \text{in } (0,T),\\ J\ddot{\omega} = f_{3,t} - \dot{J}\dot{\omega} + (\dot{J}\omega)\wedge\omega + (J\dot{\omega})\wedge\omega + (J\omega)\wedge\dot{\omega} & \\ \qquad + \displaystyle\int_{\partial\Omega_S(0)} Qx\wedge\big(2\mu\epsilon(u_t) + \mu'(\nabla\cdot u_t)Id - p^0\rho_t Id\big)\, n\, d\sigma & \\ \qquad + \displaystyle\int_{\partial\Omega_S(0)} \dot{Q}x\wedge\big(2\mu\epsilon(u) + \mu'(\nabla\cdot u)Id - p^0\rho Id\big)\, n\, d\sigma & \text{in } (0,T),\\ u_t = \ddot{a} + \dot{\omega}\wedge Qx + \omega\wedge\dot{Q}x & \text{in } (0,T)\times\partial\Omega_S(0),\\ u_t = 0 & \text{in } (0,T)\times\partial\Omega. \end{cases} \tag{77}$$

Now, we multiply the equation of ρ_t by $(p^0/\overline{\rho})\rho_t$ and the equation of u_t by u_t. Let us see that the boundary terms provide estimates for $\dot{\omega}$ and $\ddot{a}$ and the remaining terms are bounded by $C(N^3(0,T) + N^4(0,T))$ and the data:

$$\begin{aligned} &\iint_{\Sigma_s} (2\mu\epsilon(u_t)n + \mu'\nabla\cdot u_t n - p^0\rho_t n)\cdot u_t\, d\sigma\, d\tau = \frac{m}{2}(|\ddot{a}(t)|^2 - |\ddot{a}(0)|^2) - \int_0^s \ddot{a} f_{2,t}\, d\tau\\ &+ \iint_{\Sigma_s} \dot{\omega}\cdot(Qx\wedge(2\mu\epsilon(u_t)n + \mu'\nabla\cdot u_t n - p^0\rho_t n))\, d\sigma\, d\tau\\ &+ \iint_{\Sigma_s} \omega\wedge(\dot{Q}x)\cdot(2\mu\epsilon(u_t)n + \mu'\nabla\cdot u_t n - p^0\rho_t n)\, d\sigma\, d\tau. \end{aligned} \tag{78}$$

In order to develop the third term in the right hand side of (78) we use that

$$\begin{aligned} &\dot{\omega}\cdot\int_{\partial\Omega_S(0)} Qx\wedge\big(2\mu\epsilon(u_t) + \mu'(\nabla\cdot u_t)Id - p^0\rho_t Id\big)\, n\, d\sigma\\ &\qquad = (J\ddot{\omega})\cdot\dot{\omega} - f_{3,t}\cdot\dot{\omega} + (\dot{J}\dot{\omega})\cdot\dot{\omega} - ((\dot{J}\omega)\wedge\omega)\cdot\dot{\omega} - ((J\dot{\omega})\wedge\omega)\cdot\dot{\omega}\\ &\qquad -\dot{\omega}\cdot\int_{\partial\Omega_S(0)} \dot{Q}x\wedge\big(2\mu\epsilon(u) + \mu'(\nabla\cdot u)Id - p^0\rho Id\big)\, n\, d\sigma. \end{aligned} \tag{79}$$

Observe that we have

$$(J\ddot{\omega})\cdot\dot{\omega} = \frac{1}{2}\frac{d}{dt}((J\dot{\omega})\cdot\dot{\omega}) - \frac{1}{2}(\dot{J}\dot{\omega})\cdot\dot{\omega}.$$

Now, we integrate between $\tau = 0$ and $\tau = s$ in (79) and we use that J is a definite positive matrix (see (10) . We find

$$\begin{aligned} &\iint_{\Sigma_s} \dot{\omega}\cdot(Qx\wedge(2\mu\epsilon(u_t)n + \mu'\nabla\cdot u_t n - p^0\rho_t n))\, d\sigma\, d\tau\\ &\geq (C_J/2)|\dot{\omega}(s)|^2 - C|\dot{\omega}(0)|^2 - C\sup_{s\in(0,T)}|\dot{\omega}|\int_0^s |f_{3,t}|\, d\tau - C(N^3(0,T) + N^4(0,T)). \end{aligned} \tag{80}$$

We also observe that the last term in (78) is estimated by $N^3(0,T)$.

Similarly as before, we find

$$\frac{1}{2}\sup_{s\in(0,T)}\left(|\ddot{a}|^2(s)+|\dot{\omega}|^2(s)+\int_{\Omega_F(0)}\left(\frac{p^0}{\overline{\rho}}|\rho_t|^2+|u_t|^2\right)(s)\,dx\right)+\iint_{Q_T}(2\mu|\epsilon(u_t)|^2+\mu'|\nabla\cdot u_t|^2)\,dx\,d\tau$$
$$\leqslant C\left(\iint_{Q_T}(|u|+|v|+|\nabla v|+|\nabla u|)|\rho_t|^2(|(\nabla\chi)^{-1}|+|\nabla((\nabla\chi)^{-1})|)\,dx\,d\tau\right.$$
$$+\int_{\Omega_F(0)}(|\partial_t u(0,\cdot)|^2+|\partial_t\rho(0,\cdot)|^2)\,dx+|\ddot{a}(0)|^2+|\dot{\omega}(0)|^2+\iint_{Q_T}(|f_{0,t}|^2+|f_{1,t}|^2+\delta(|\rho_t|^2+|u_t|^2))\,dx\,d\tau$$
$$\left.+\sup_{s\in(0,T)}|\ddot{a}|\int_0^T|f_{2,t}|\,d\tau+\sup_{s\in(0,T)}|\dot{\omega}|\int_0^T|f_{3,t}|\,d\tau+N^3(0,T)+N^4(0,T)\right),$$

for $\delta>0$ small enough. Thus, we deduce (75) for $k=1$.

Lemma 11 *Let $k=0,1$. Then, there exists a constant $C>0$ such that*

$$\begin{aligned}&\|u\|_{H_T^{k+1}(L^2)}+\|\rho\|_{H_T^{k+1}(L^2)}+\|u\|_{W_T^{k,\infty}(H^1)}+\|a\|_{H_T^{k+2}}+\|\omega\|_{H_T^{k+1}}\\&\leqslant C(\|u\|_{H_T^k(H^1)}+\|\rho\|_{W_T^{k,\infty}(L^2)}+\|f_0\|_{H_T^k(L^2)}+\|f_1\|_{H_T^k(L^2)}+\|f_2\|_{H_T^k}+\|f_3\|_{H_T^k}\\&+\|\partial_t^k u(0,\cdot)\|_{H^1}+N^{3/2}(0,T)+N^2(0,T)).\end{aligned}\tag{81}$$

Proof:

1) $k=0$**.** We multiply the equation of ρ by ρ_t and the equation of u by u_t. Integrating in Q_s for $s\in(0,T)$ and adding up both expressions, we obtain:

$$\begin{aligned}&\frac{1}{2}\int_{\Omega_F(0)}(2\mu|\epsilon(u)(s)|^2+\mu'|\nabla\cdot u(s)|^2)\,dx+\iint_{Q_s}(|\rho_t|^2+|u_t|^2)\,dx\,d\tau\\&+\iint_{\Sigma_s}u_t\cdot(2\mu\epsilon(u)+\mu'(\nabla\cdot u)Id-p^0\rho Id)nd\sigma\,d\tau+\iint_{Q_s}(\overline{\rho}(\nabla\cdot u)\rho_t-p^0(\nabla\cdot u_t)\rho)\,dx\,d\tau\\&\qquad\leqslant C\left(\iint_{Q_s}(|v|+|u|)|\rho_t||\nabla\rho||(\nabla\chi)^{-1}|\,dx\,d\tau+\int_{\Omega_F(0)}|\nabla u_0|^2dx\right.\\&\qquad\left.+\iint_{Q_s}(|f_0|^2+|f_1|^2+\delta(|\rho_t|^2+|u_t|^2))\,dx\,d\tau\right),\end{aligned}\tag{82}$$

for $\delta>0$ small enough. The boundary term yields

$$\iint_{\Sigma_s}u_t\cdot(2\mu\epsilon(u)+\mu'(\nabla\cdot u)Id-p^0\rho Id)nd\sigma\,d\tau=m\int_0^s|\ddot{a}|^2\,d\tau-\int_0^s\ddot{a}\,f_2\,d\tau$$
$$+\int_0^s((J\dot{\omega})\cdot\dot{\omega}-((J\omega)\wedge\omega)\cdot\dot{\omega}-f_3\cdot\dot{\omega})d\tau+\omega\cdot\iint_{\Sigma_s}\dot{Q}x\wedge(2\mu\epsilon(u)+\mu'(\nabla\cdot u)Id-p^0\rho Id)n\,d\sigma\,d\tau$$
$$\geq\int_0^s((m/2)|\ddot{a}|^2+(C_J/2)|\dot{\omega}|^2)d\tau-C\left(\int_0^s(|f_2|^2+|f_3|^2)d\tau+N^3(0,T)\right).$$

The last integral in the left hand side of (82) can be estimated as follows:

$$\begin{aligned}&\iint_{Q_s}(\overline{\rho}(\nabla\cdot u)\rho_t-p^0(\nabla\cdot u_t)\rho)\,dx\,d\tau\leqslant\frac{1}{2}\iint_{Q_s}|\rho_t|^2dx\,d\tau+C\iint_{Q_s}|\nabla u|^2dx\,d\tau\\&+\iint_{Q_s}p^0(\nabla\cdot u)\rho_t\,dx\,d\tau-p^0\int_{\Omega_F(0)}((\nabla\cdot u)\rho)(\tau)\big|_{\tau=0}^{\tau=s}dx.\end{aligned}\tag{83}$$

Taking the supremum in s in (82), one can readily deduce (81) for $k = 0$.

2) $k = 1$**.** As in the proof of Lemma 10, we consider system (77). Let us multiply the equation of ρ_t by ρ_{tt} and the equation of u_t by u_{tt}. This provides:

$$\begin{aligned}
&\frac{1}{2}\int_{\Omega_F(0)} (2\mu|\epsilon(u_t)(s)|^2 + \mu'|\nabla\cdot u_t(s)|^2)\,dx + \iint_{Q_s} (|\rho_{tt}|^2 + |u_{tt}|^2)\,dx\,d\tau \\
&+ \iint_{\Sigma_s} u_{tt}\cdot(2\mu\epsilon(u_t) + \mu'(\nabla\cdot u_t)Id - p^0\rho_t Id)n d\sigma\,d\tau + \iint_{Q_s} (\overline{\rho}(\nabla\cdot u_t)\rho_{tt} - p^0(\nabla\cdot u_{tt})\rho_t)\,dx\,d\tau \\
&\leqslant C\left(\iint_{Q_s} (|\rho_{tt}||(\nabla\chi)^{-1}|((|v|+|u|)|\nabla\rho_t| + (|u_t|+|v_t|)|\nabla\rho|)\,dx\,d\tau + \int_{\Omega_F(0)} |\partial_t\nabla u(0,\cdot)|^2 dx\right. \\
&\left.+ \iint_{Q_s} (|f_{0,t}|^2 + |f_{1,t}|^2 + \delta(|\rho_{tt}|^2 + |u_{tt}|^2))\,dx\,d\tau\right),
\end{aligned} \tag{84}$$

for $\delta > 0$ small enough. Let us now compute the boundary terms. On $(0,T)\times\partial\Omega_S(0)$, we have:

$$u_{tt} = \dddot{a} + \ddot{\omega}\wedge Qx + 2\dot{\omega}\wedge\dot{Q}x + \omega\wedge\ddot{Q}x.$$

Thus,

$$\begin{aligned}
&\iint_{\Sigma_s} u_{tt}\cdot(2\mu\epsilon(u_t) + \mu'(\nabla\cdot u_t)Id - p^0\rho_t Id)n\,d\sigma\,d\tau = m\int_0^s |\dddot{a}|^2 d\tau - \int_0^s \dddot{a}\cdot f_{2,t}\,d\tau \\
&+ \iint_{\Sigma_s} \ddot{\omega}\cdot(Qx\wedge(2\mu\epsilon(u_t) + \mu'(\nabla\cdot u_t)Id - p^0\rho_t Id)n)\,d\sigma\,d\tau \\
&+2\iint_{\Sigma_s} (\dot{\omega}\wedge\dot{Q}x)\cdot(2\mu\epsilon(u_t) + \mu'(\nabla\cdot u_t)Id - p^0\rho_t Id)n\,d\sigma\,d\tau \\
&+ \iint_{\Sigma_s} (\omega\wedge\ddot{Q}x)\cdot(2\mu\epsilon(u_t) + \mu'(\nabla\cdot u_t)Id - p^0\rho_t Id)n\,d\sigma\,d\tau.
\end{aligned} \tag{85}$$

In order to develop the third term in the right hand side of (78) we use that

$$\begin{aligned}
&\ddot{\omega}\cdot\int_{\partial\Omega_S(0)} Qx\wedge\big(2\mu\epsilon(u_t) + \mu'(\nabla\cdot u_t)Id - p^0\rho_t Id\big)\,n\,d\sigma \\
&\qquad = (J\ddot{\omega})\cdot\ddot{\omega} - f_{3,t}\cdot\ddot{\omega} + (\dot{J}\dot{\omega})\cdot\ddot{\omega} - ((\dot{J}\omega)\wedge\omega)\cdot\ddot{\omega} - ((J\dot{\omega})\wedge\omega)\cdot\ddot{\omega} - ((J\omega)\wedge\dot{\omega})\cdot\ddot{\omega} \\
&\qquad -\ddot{\omega}\cdot\int_{\partial\Omega_S(0)} \dot{Q}x\wedge\big(2\mu\epsilon(u) + \mu'(\nabla\cdot u)Id - p^0\rho Id\big)\,n\,d\sigma.
\end{aligned} \tag{86}$$

Now, we integrate between $\tau = 0$ and $\tau = s$ in (86); we find

$$\begin{aligned}
&\iint_{\Sigma_s} \ddot{\omega}\cdot(Qx\wedge(2\mu\epsilon(u_t)n + \mu'\nabla\cdot u_t n - p^0\rho_t n))\,d\sigma\,d\tau \\
&\geq (C_J/2)\int_0^s |\ddot{\omega}|^2 d\tau - C\left(\int_0^s |f_{3,t}|^2\,d\tau + N^3(0,T) + N^4(0,T)\right),
\end{aligned} \tag{87}$$

where C_J is such that (10) is satisfied. The two last terms in the right hand side of (85) are also bounded by $N^3(0,T)$.

Finally, the last term in the left hand side of (84) is estimated as the corresponding term in the previous case (see (83)). Taking the supremum in s in (84), we conclude inequality () for $k = 1$.

As a conclusion, we deduce inequality (81) for $k = 1$.

Let us do some interior estimates now. Before doing this, we introduce the total time derivative of ρ:

$$\frac{d\rho}{dt} := \rho_t + ((\nabla\chi)^{-1}(u - v))\cdot\nabla\rho \quad \text{in } (0,T)\times\Omega_F(0). \tag{88}$$

Let $\zeta_0 \in C_c^\infty(\Omega_F(0))$.

Lemma 12 *For $1 \leqslant k \leqslant 3$, we have*

$$\begin{aligned} &\|\zeta_0\rho\|_{L^\infty_T(H^k)} + \|\zeta_0\rho\|_{L^2_T(H^k)} + \|\zeta_0 u\|_{L^\infty_T(H^k)} + \|\zeta_0 u\|_{L^2_T(H^{k+1})} + \|\zeta_0 \frac{d\rho}{dt}\|_{L^2_T(H^k)} \\ &\leqslant C(\|u\|_{L^\infty_T(H^{k-1})} + \|u\|_{L^2_T(H^k)} + \|\rho_0\|_{H^k} + \|u_0\|_{H^k} + \|f_0\|_{L^2_T(H^k)} + \|f_1\|_{L^2_T(H^{k-1})} + N^{3/2}(0,T)). \end{aligned} \tag{89}$$

Proof: In this proof, we denote by D^ℓ all possible derivatives in x of order ℓ. We divide the proof in two steps:

• First, we apply the operator $\zeta_0^2 \partial_{x_j} D^{k-1}$ to the equation of ρ, we multiply it by $\partial_{x_j} D^{k-1}\rho$, we sum up in $j = 1, 2, 3$ and we integrate in Q_s:

$$\sum_{j=1}^{3} \iint_{Q_s} \zeta_0^2 \partial_{x_j} D^{k-1}(\rho_t + ((\nabla\chi)^{-1}(u - v)) \cdot \nabla\rho + \overline{\rho}(\nabla \cdot u) - f_0)\partial_{x_j} D^{k-1}\rho \, dx \, d\tau = 0.$$

Integrating by parts with respect to x in the nonlinear term, this gives

$$\begin{aligned} &\|\zeta_0\rho\|^2_{L^\infty_T(H^k)} + \overline{\rho} \iint_{Q_s} \zeta_0^2 \nabla(D^{k-1}(\nabla \cdot u)) \cdot \nabla(D^{k-1}\rho) \, dx \, d\tau \\ &\leqslant \delta\|\zeta_0 D^k\rho\|^2_{L^2_T(L^2)} + C(\|\rho_0\|^2_{H^k} + \|f_0\|^2_{L^2_T(H^k)} + N^3(0,T)), \end{aligned} \tag{90}$$

with $\delta > 0$ small enough.

Next, we apply the operator $\frac{\overline{\rho}}{2\mu+\mu'}\zeta_0^2 D^{k-1}$ to the j-th equation of u, we multiply it by $\partial_{x_j} D^{k-1}\rho$, we sum up in $j = 1, 2, 3$ and we integrate in Q_s:

$$\frac{\overline{\rho}}{2\mu + \mu'} \sum_{j=1}^{3} \iint_{Q_s} \zeta_0^2 D^{k-1}(u_t - 2\mu\nabla \cdot (\epsilon(u)) - \mu'\nabla \cdot ((\nabla \cdot u)Id) + p^0\nabla\rho - f_1)_j \partial_{x_j} D^{k-1}\rho \, dx \, d\tau = 0. \tag{91}$$

We integrate by parts in t and then in x in the term concerning u_t and we use the equation of ρ. We deduce that

$$\iint_{Q_s} \zeta_0^2 D^{k-1} u_{j,t} \partial_{x_j} D^{k-1}\rho \, dx \, d\tau \leqslant \delta\|\zeta_0\rho\|^2_{L^\infty_T(H^k)} + C(\|f_0\|^2_{L^2_T(H^k)} + \|u\|^2_{L^2_T(H^k)} + \|u\|^2_{L^\infty_T(H^{k-1})} + N^3(0,T)).$$

As long as the elliptic term $-2\mu\nabla \cdot (\epsilon(u)) - \mu'\nabla \cdot ((\nabla \cdot u)Id)$ is concerned, we rewrite it as

$$-2\mu\nabla \cdot (\epsilon(u)) - \mu'\nabla \cdot ((\nabla \cdot u)Id) = -\mu\Delta u - (\mu + \mu')\nabla(\nabla \cdot u)$$

and we integrate by parts twice (with respect to x) in the laplacian term. Observe that there is no boundary term when we integrate in x since ζ_0 has a compact support. Then, from (91) we deduce that

$$\begin{aligned} &\frac{\overline{\rho}p^0}{2\mu + \mu'}\|\zeta_0 D^k\rho\|^2_{L^2_T(L^2)} - \overline{\rho} \iint_{Q_s} \zeta_0^2 \nabla(D^{k-1}(\nabla \cdot u)) \cdot \nabla(D^{k-1}\rho) \, dx \, d\tau \leqslant \delta(\|\zeta_0 D^k\rho\|^2_{L^2_T(L^2)} + \|\zeta_0\rho\|^2_{L^\infty_T(H^k)}) \\ &+C(\|u\|^2_{L^\infty_T(H^{k-1})} + \|u\|^2_{L^2_T(H^k)} + \|\rho_0\|^2_{H^k} + \|f_0\|^2_{L^2_T(H^k)} + \|f_1\|^2_{L^2_T(H^{k-1})} + N^3(0,T)). \end{aligned}$$

This, together with (90) and taking the supremum in s, yields

$$\begin{aligned} &\|\zeta_0\rho\|_{L^\infty_T(H^k)} + \|\zeta_0\rho\|_{L^2_T(H^k)} \leqslant C(\|u\|_{L^\infty_T(H^{k-1})} + \|u\|_{L^2_T(H^k)} \\ &+\|\rho_0\|_{H^k} + \|f_0\|_{L^2_T(H^k)} + \|f_1\|_{L^2_T(H^{k-1})} + N^{3/2}(0,T)). \end{aligned} \tag{92}$$

• Let us now apply the operator $(p^0/\overline{\rho})\zeta_0^2 D^k$ to the equation of ρ, we multiply it by $D^k\rho$ and we integrate in Q_s:

$$\frac{p^0}{\overline{\rho}} \iint_{Q_s} \zeta_0^2 D^k(\rho_t + ((\nabla\chi)^{-1}(u - v)) \cdot \nabla\rho + \overline{\rho}(\nabla \cdot u) - f_0) D^k\rho \, dx \, d\tau = 0.$$

This gives

$$\begin{aligned}&\|\zeta_0\rho\|^2_{L^\infty_s(H^k)}+p^0\iint_{Q_s}\zeta_0^2 D^k(\nabla\cdot u)D^k\rho\,dx\,d\tau\\&\leqslant \delta\|\zeta_0 D^k\rho\|^2_{L^2_T(L^2)}+C(\|\rho_0\|^2_{H^k}+\|f_0\|^2_{L^2_T(H^k)}+N^3(0,T)).\end{aligned}\tag{93}$$

Next, we apply the operator $\zeta_0^2 D^k$ to the j-th equation of u, we multiply it by $D^k u_j$, we sum up in $j=1,2,3$ and we integrate in Q_s:

$$\sum_{j=1}^{3}\iint_{Q_s}\zeta_0^2 D^k(u_t-2\mu\nabla\cdot(\epsilon(u))-\mu'\nabla\cdot((\nabla\cdot u)Id)+p^0\nabla\rho-f_1)_j D^k u_j\,dx\,d\tau=0.$$

We integrate by parts in x in the second, third, fourth and fifth terms of the last integral. We deduce that

$$\begin{aligned}&\|\zeta_0 u\|^2_{L^\infty_T(H^k)}+\|\zeta_0 u\|^2_{L^2_T(H^{k+1})}-p^0\iint_{Q_s}\zeta_0^2 D^k(\nabla\cdot u)D^k\rho\,dx\,d\tau\\&\leqslant \delta\|\zeta_0\rho\|^2_{L^2_T(H^k)}+C(\|u\|^2_{L^2_T(H^k)}+\|u_0\|^2_{H^k}+\|f_1\|^2_{L^2_T(H^{k-1})}),\end{aligned}$$

for $\delta>0$ small enough. This, together with (93), yields

$$\begin{aligned}&\|\zeta_0\rho\|_{L^\infty_T(H^k)}+\|\zeta_0 u\|_{L^\infty_T(H^k)}+\|\zeta_0 u\|_{L^2_T(H^{k+1})}\\&\leqslant \delta\|\zeta_0\rho\|_{L^2_T(H^k)}+C(\|u\|_{L^2_T(H^k)}+\|u_0\|_{H^k}+\|\rho_0\|_{H^k}+\|f_0\|_{L^2_T(H^k)}+\|f_1\|_{L^2_T(H^{k-1})}+N^{3/2}(0,T)).\end{aligned}$$

Combining this with (92) we obtain the conclusion (89).

Let us now do some estimates close to the boundary. For this, we consider a finite covering $\{\mathcal{O}_i\}_{i=1}^K$ of $\partial\Omega_F(0)$. For each $1\leqslant i\leqslant K$, we consider a function

$$\begin{array}{rcl}\theta^i:(-M_1^i,M_1^i)\times(-M_2^i,M_2^i)&\to&\partial\Omega_F(0)\cap\mathcal{O}_i\\(\phi_1,\phi_2)&\to&\theta^i(\phi_1,\phi_2)\end{array}\tag{94}$$

satisfying

$$\partial_{\phi_1}\theta^i\cdot\partial_{\phi_2}\theta^i=0,\ |\partial_{\phi_1}\theta^i|=1,\ |\partial_{\phi_2}\theta^i|\geq C>0.$$

Then, we perform the change of variables

$$x(\phi_1,\phi_2,r)=rn(\phi_1,\phi_2)+\theta^i(\phi_1,\phi_2)\in\mathcal{O}_i\cap\Omega_F(0)\quad r\in(0,r_1^i),\ (\phi_1,\phi_2)\in(-M_1^i,M_1^i)\times(-M_2^i,M_2^i),$$

where n denotes the inward unit normal vector to $\partial\Omega_F(0)$.

In what follows, we will establish some estimates of u and ρ in $\Omega_F(0)\cap\mathcal{O}_i$ for each $1\leqslant i\leqslant K$ (see Lemmas 13, 14 and 15 below). For the sake of simplicity, we drop the index i from now on. In the new variables, we define

$$e_1=\partial_{\phi_1}\theta\quad\text{and}\quad e_2=\frac{\partial_{\phi_2}\theta}{|\partial_{\phi_2}\theta|}.$$

Observe that $e_1\cdot e_2=0$, $e_1\cdot n=0$ and $e_2\cdot n=0$ and so we can write

$$\partial_{\phi_1}n=\alpha e_1+\beta e_2\ \text{ and }\ \partial_{\phi_2}n=\alpha' e_1+\beta' e_2.$$

Let us compute the Jacobian of this change of variables:

$$\partial_{\phi_1}x=(1+\alpha r)e_1+\beta r e_2,\quad \partial_{\phi_2}x=\alpha' r e_1+(\beta' r+|\partial_{\phi_2}\theta|)e_2\quad\text{and}\quad\partial_r x=n,$$

that is to say,

$$\begin{pmatrix}\partial_{\phi_1}x\\\partial_{\phi_2}x\\\partial_r x\end{pmatrix}=\begin{pmatrix}1+\alpha r&\beta r&0\\\alpha' r&\beta' r+|\partial_{\phi_2}\theta|&0\\0&0&1\end{pmatrix}\begin{pmatrix}e_1\\e_2\\n\end{pmatrix}.$$

Inversely, we have

$$\begin{pmatrix} \partial_x\phi_1 \\ \partial_x\phi_2 \\ \partial_x r \end{pmatrix} = \frac{1}{\bar J}\begin{pmatrix} \beta' r + |\partial_{\phi_2}\theta| & -\beta r & 0 \\ -\alpha' r & 1+\alpha r & 0 \\ 0 & 0 & \bar J \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \\ n \end{pmatrix},$$

where $\bar J := (1+\alpha r)(\beta' r + |\partial_{\phi_2}\theta|) - \alpha'\beta r^2$ is the Jacobian.

Consequently, we have

$$\nabla_x = \partial_x = \frac{1}{\bar J}(A_1e_1 + A_2e_2)\partial_{\phi_1} + \frac{1}{\bar J}(A_3e_1 + A_4e_2)\partial_{\phi_2} + n\partial_r \tag{95}$$

where

$$A_1 = \beta' r + |\partial_{\phi_2}\theta|,\ A_2 = -\beta r,\ A_3 = -\alpha' r \ \text{ and } \ A_4 = 1+\alpha r.$$

Let us now write the equations of (72) in the new variables (ϕ_1, ϕ_2, r):

$$\begin{cases} \rho_t + ((\nabla_x\chi)^{-1}(u-v))\cdot\left(\frac{1}{\bar J}(A_1e_1 + A_2e_2)\partial_{\phi_1}\rho + \frac{1}{\bar J}(A_3e_1 + A_4e_2)\partial_{\phi_2}\rho + n\partial_r\rho\right) \\ \quad +\bar\rho\left(\frac{1}{\bar J}(A_1e_1 + A_2e_2)\cdot\partial_{\phi_1}u + \frac{1}{\bar J}(A_3e_1 + A_4e_2)\cdot\partial_{\phi_2}u + n\cdot\partial_r u\right) = f_0 \\ (t,\phi_1,\phi_2,r)\in(0,T)\times(-M_1,M_1)\times(-M_2,M_2)\times(0,r_1) \end{cases} \tag{96}$$

and

$$\begin{cases} u_t - \frac{\mu}{\bar J^2}((A_1^2 + A_2^2)\partial^2_{\phi_1\phi_1}u + 2(A_1A_3 + A_2A_4)\partial^2_{\phi_1\phi_2}u + (A_3^2 + A_4^2)\partial^2_{\phi_2\phi_2}u + \bar J^2\partial^2_{rr}u) + P_1(D)u \\ \quad +\frac{1}{\bar J}\partial_{\phi_1}\left(\frac{\mu+\mu'}{\bar\rho}\frac{d\rho}{dt} + p^0\rho\right)(A_1e_1 + A_2e_2) + \frac{1}{\bar J}\partial_{\phi_2}\left(\frac{\mu+\mu'}{\bar\rho}\frac{d\rho}{dt} + p^0\rho\right)(A_3e_1 + A_4e_2) \\ \quad +\partial_r\left(\frac{\mu+\mu'}{\bar\rho}\frac{d\rho}{dt} + p^0\rho\right)n = f_1 + \frac{\mu+\mu'}{\bar\rho}\nabla_x f_0 \\ (t,\phi_1,\phi_2,r)\in(0,T)\times(-M_1,M_1)\times(-M_2,M_2)\times(0,r_1), \end{cases} \tag{97}$$

where $P_1(D)$ is a first order differential operator in the x variables. In order to obtain this last equation, observe that we have used that

$$u_t - \mu\Delta u + \frac{\mu+\mu'}{\bar\rho}\nabla\left(\frac{d\rho}{dt} - f_0\right) + p^0\nabla\rho = f_1$$

and then we have rewritten this in the new variables. In the new variables $\frac{d\rho}{dt}$ is given by the first line in (96).

Let $\zeta_1 \in C_c^\infty(\mathcal{O}_i)$ for $1 \leqslant i \leqslant K$ (we also drop here the index i).

We first estimate the tangential derivatives:

Lemma 13 *Let $k = 1, 2, 3$. For any $\delta > 0$, there exists a positive constant C such that*

$$\begin{aligned} &\|\zeta_1 D^k_\phi\rho\|_{L^\infty_T(L^2)} + \|\zeta_1 D^k_\phi u\|_{L^\infty_T(L^2)} + \|\zeta_1 D^k_\phi u\|_{L^2_T(H^1)} + \|\zeta_1 D^k_\phi\frac{d\rho}{dt}\|_{L^2_T(L^2)} \\ &\leqslant \delta(\|\zeta_1 D^k_\phi\rho\|_{L^2_T(L^2)} + \|u\|_{L^2_T(H^2)} + \|\rho\|_{L^2_T(H^1)}) + C(\|u\|_{L^2_T(H^k)} + \|\omega\|_{L^2_T} + \|\rho_0\|_{H^k} \\ &+\|u_0\|_{H^k} + \|f_0\|_{L^2_T(H^k)} + \|f_1\|_{L^2_T(H^{k-1})} + N^{3/2}(0,T)). \end{aligned} \tag{98}$$

Proof: The proof of this lemma is very similar to that of the second part of Lemma 12. In fact, we first apply the operator $\frac{p^0}{\overline{\rho}}\zeta_1^2 D_\phi^k$ to the equation of ρ in (72) and multiply it by $D_\phi^k\rho$:

$$\frac{p^0}{\overline{\rho}}\iint_{Q_s}\zeta_1^2 D_\phi^k(\rho_t+((\nabla_x\chi)^{-1}(u-v))\cdot\nabla_x\rho+\overline{\rho}(\nabla_x\cdot u)-f_0)D_\phi^k\rho\,dx\,d\tau=0.$$

For the nonlinear term, since the expression of ∇_x is given by (95), we have

$$\iint_{Q_s}\zeta_1^2 D_\phi^k(((\nabla_x\chi)^{-1}(u-v))\cdot\nabla_x\rho)D_\phi^k\rho\,dx\,d\tau\geq\iint_{Q_s}\zeta_1^2(((\nabla_x\chi)^{-1}(u-v))\cdot\nabla_x D_\phi^k\rho)D_\phi^k\rho\,dx\,d\tau-CN^3(0,T).$$

Thus, this implies that

$$\begin{aligned}\|\zeta_1 D_\phi^k\rho\|_{L^2}^2(s)&+p^0\iint_{Q_s}\zeta_1^2 D_\phi^k(\nabla_x\cdot u)D_\phi^k\rho\,dx\,d\tau\\&\leqslant\delta\|\zeta_1 D_\phi^k\rho\|_{L_T^2(L^2)}^2+C(\|\rho_0\|_{H^k}^2+\|f_0\|_{L_T^2(H^k)}^2+N^3(0,T)).\end{aligned}\tag{99}$$

Then, we apply the operator $\zeta_1^2 D_\phi^k$ to the j-th equation of u, multiply it by $D_\phi^k u_j$ and sum up in j:

$$\sum_{j=1}^3\iint_{Q_s}\zeta_1^2 D_\phi^k(u_t-2\mu\nabla_x\cdot\epsilon(u)-\mu'\nabla_x(\nabla\cdot u)+p^0\nabla_x\rho-f_1)_j D_\phi^k u_j\,dx\,d\tau=0.\tag{100}$$

Observe that, since ζ_1 has compact support, we have

$$-\iint_{Q_s}\zeta_1^2 D_\phi^k\nabla_x\cdot\epsilon(u)_j\,D_\phi^k u_j\,dx\,d\tau=(-1)^{k+1}\iint_{Q_s}\nabla_x\cdot\epsilon(u)_j\,D_\phi^k(\zeta_1^2 D_\phi^k u_j)\,dx\,d\tau.\tag{101}$$

Integrating by parts on the x variable, we deduce:

$$\begin{aligned}&-\iint_{Q_s}\zeta_1^2 D_\phi^k\nabla_x\cdot\epsilon(u)_j\,D_\phi^k u_j\,dx\,d\tau=(-1)^k\iint_{Q_s}\epsilon(u)_j\cdot\nabla_x D_\phi^k(\zeta_1^2 D_\phi^k u_j)\,dx\,d\tau\\&+(-1)^k\iint_{\Sigma_s}(\epsilon(u)_j\cdot n)D_\phi^k(\zeta_1^2 D_\phi^k u_j)\,d\sigma\,d\tau\\&\geq(-1)^k\iint_{Q_s}\epsilon(u)_j\cdot\nabla_x D_\phi^k(\zeta_1^2 D_\phi^k u_j)\,dx\,d\tau-C(\|u\|_{L_T^2(H^2)}^2+\|\omega\|_{L_T^2}^2),\end{aligned}\tag{102}$$

where we have used that $|D_\phi^\ell u|\leqslant C|\omega|$ for all $|\ell|\geq 1$. We recall now that (see (95))

$$\nabla_x=B_1\partial_{\phi_1}+B_2\partial_{\phi_2}+B_3\partial_r,$$

where B_j $(j=1,2,3)$ are smooth functions of ϕ. Then, we notice that

$$D_\phi^k\nabla_x h\;=\;\nabla_x D_\phi^k h+\sum_{\ell=1}^k\begin{pmatrix}k\\ \ell\end{pmatrix}\big(D_\phi^\ell B_1 D_\phi^{k-\ell}\partial_{\phi_1}+D_\phi^\ell B_2 D_\phi^{k-\ell}\partial_{\phi_2}+D_\phi^\ell B_3 D_\phi^{k-\ell}\partial_r\big)h.$$

Let us take $h=\zeta_1^2 D_\phi^k u_j$. We have, for $1\leqslant\ell\leqslant k$,

$$\begin{aligned}&\left|\iint_{Q_s}\epsilon(u)_j\cdot\big(D_\phi^\ell B_1 D_\phi^{k-\ell}\partial_{\phi_1}(\zeta_1^2 D_\phi^k u_j)+D_\phi^\ell B_2 D_\phi^{k-\ell}\partial_{\phi_2}(\zeta_1^2 D_\phi^k u_j)+D_\phi^\ell B_3 D_\phi^{k-\ell}\partial_r(\zeta_1^2 D_\phi^k u_j)\big)\,dx\,d\tau\right|\\&\leqslant\left|\iint_{Q_s}D_\phi^{k-\ell}(D_\phi^\ell B_1\epsilon(u)_j)\partial_{\phi_1}(\zeta_1^2 D_\phi^k u_j)+D_\phi^{k-\ell}(D_\phi^\ell B_2\epsilon(u)_j)\partial_{\phi_2}(\zeta_1^2 D_\phi^k u_j)\,dx\,d\tau\right|\\&+\left|\iint_{Q_s}D_\phi^{k-\ell}(D_\phi^\ell B_3\epsilon(u)_j)\partial_r(\zeta_1^2 D_\phi^k u_j)\,dx\,d\tau\right|\leqslant C(\|u\|_{L_T^2(H^k)}^2+\delta\|\zeta_1 D_\phi^k u\|_{L_T^2(H^1)}^2).\end{aligned}\tag{103}$$

This implies that

$$
\begin{aligned}
&(-1)^k \iint_{Q_s} \epsilon(u)_j \cdot \nabla_x D_\phi^k(\zeta_1^2 D_\phi^k u_j)\, dx\, d\tau \\
&\geq (-1)^k \iint_{Q_s} \epsilon(u)_j \cdot D_\phi^k \nabla_x(\zeta_1^2 D_\phi^k u_j)\, dx\, d\tau - C(\|u\|^2_{L^2_T(H^k)} + \delta\|\zeta_1 D_\phi^k u\|^2_{L^2_T(H^1)}) \\
&\geq \iint_{Q_s} D_\phi^k \epsilon(u)_j \cdot \nabla_x(\zeta_1^2 D_\phi^k u_j)\, dx\, d\tau - C(\|u\|^2_{L^2_T(H^k)} + \delta\|\zeta_1 D_\phi^k u\|^2_{L^2_T(H^1)}).
\end{aligned} \tag{104}
$$

Analogously, we can prove that

$$
\begin{aligned}
&\iint_{Q_s} D_\phi^k \epsilon(u)_j \cdot \nabla_x(\zeta_1^2 D_\phi^k u_j)\, dx\, d\tau \\
&\geq \iint_{Q_s} \epsilon(\zeta_1 D_\phi^k u)_j \cdot \nabla_x(\zeta_1 D_\phi^k u_j)\, dx\, d\tau - C(\|u\|^2_{L^2_T(H^k)} + \delta\|\zeta_1 D_\phi^k u\|^2_{L^2_T(H^1)}).
\end{aligned} \tag{105}
$$

So, from (101)-(105), we have for the second term in (100):

$$
\begin{aligned}
-\iint_{Q_s} \zeta_1^2 D_\phi^k \nabla_x \cdot \epsilon(u)_j\, D_\phi^k u_j\, dx\, d\tau &\geq \iint_{Q_s} \epsilon(\zeta_1 D_\phi^k u)_j \cdot \nabla_x(\zeta_1 D_\phi^k u_j)\, dx\, d\tau \\
&\quad - C(\|u\|^2_{L^2_T(H^k)} + \delta\|\zeta_1 D_\phi^k u\|^2_{L^2_T(H^1)} + \|u\|^2_{L^2_T(H^2)} + \|\omega\|^2_{L^2_T}).
\end{aligned} \tag{106}
$$

The same can also be done for the third term in (100):

$$
\begin{aligned}
-\iint_{Q_s} \zeta_1^2 D_\phi^k \partial_{x_j}(\nabla_x \cdot u)\, D_\phi^k u_j\, dx\, d\tau &\geq \iint_{Q_s} |\nabla_x \cdot (\zeta_1 D_\phi^k u)|^2\, dx\, d\tau \\
&\quad - C(\|u\|^2_{L^2_T(H^k)} + \delta\|\zeta_1 D_\phi^k u\|^2_{L^2_T(H^1)} + \|u\|^2_{L^2_T(H^2)} + \|\omega\|^2_{L^2_T}).
\end{aligned} \tag{107}
$$

Equations (106) and (107) provide the estimate of $\|\zeta_1 D_\phi^k u\|^2_{L^2_T(H^1)}$ in (98).

Then, using the same arguments, one can prove that

$$
\begin{aligned}
p^0 \sum_{j=1}^{3} \iint_{Q_s} \zeta_1^2 D_\phi^k \partial_{x_j} \rho D_\phi^k u_j\, dx\, d\tau &\geq -p^0 \iint_{Q_s} \zeta_1^2 D_\phi^k(\nabla_x \cdot u) D_\phi^k \rho\, dx\, d\tau \\
&\quad - \delta(\|\rho\|^2_{L^2_T(H^1)} + \|\zeta_1 D_\phi^k \rho\|^2_{L^2_T(L^2)}) + C(\|\omega\|^2_{L^2_T} + \|u\|^2_{L^2_T(H^k)}).
\end{aligned} \tag{108}
$$

Then, inequality (98) is obtained by reassembling (99) (for $\|\zeta_1 D_\phi^k \rho\|_{L^2_T(L^2)}$), (100) (for $\|\zeta_1 D_\phi^k u\|_{L^\infty(L^2)}$), (106)-(107) (for $\|\zeta_1 D_\phi^k u\|^2_{L^2_T(H^1)}$ and, consequently, $\|\zeta_1 D_\phi^k \frac{d\rho}{dt}\|_{L^2_T(L^2)}$) and (108).

Let us now take the derivative with respect to r of equation (96), multiply it by $\mu/\overline{\rho}$ and sum it with equation (97) multiplied by n (which we can symbolically write as $(\mu/\overline{\rho})(96)_r + n \cdot (97)$):

$$
\begin{cases}
\dfrac{2\mu+\mu'}{\overline{\rho}} \left(\dfrac{d\rho}{dt}\right)_r + p^0 \rho_r = -n \cdot u_t - \dfrac{\mu}{\bar{J}^2}\left[(A_1^2 + A_2^2) n \cdot u_{\phi_1\phi_1} + 2(A_1A_3 + A_2A_4) n \cdot u_{\phi_1\phi_2}\right. \\
\left.+(A_3^2 + A_4^2) n \cdot u_{\phi_2\phi_2} - \bar{J}(A_1e_1 + A_2e_2) \cdot u_{\phi_1 r} - \bar{J}(A_3e_1 + A_4e_2) \cdot u_{\phi_2 r}\right] \\
-P_1(D)u \cdot n + f_1 \cdot n + \dfrac{2\mu+\mu'}{\overline{\rho}} f_{0,r}, \qquad \phi \in (-M_1, M_1) \times (-M_2, M_2),\, r \in (0, r_1).
\end{cases} \tag{109}
$$

Lemma 14 *For $0 \leqslant k+\ell \leqslant 2$, we have*

$$\|\zeta_1 D_\phi^k D_r^{\ell+1}\rho\|_{L^\infty_T(L^2)} + \|\zeta_1 D_\phi^k D_r^{\ell+1}\rho\|_{L^2_T(L^2)} + \|\zeta_1 D_\phi^k D_r^{\ell+1}\frac{d\rho}{dt}\|_{L^2_T(L^2)} \leqslant C(\|\zeta_1 D_\phi^{k+1} D_r^{\ell} u\|_{L^2_T(H^1)} + \|u_t\|_{L^2_T(H^{k+\ell})} + \|u\|_{L^2_T(H^{k+\ell+1})} + \|\rho_0\|_{H^{k+\ell+1}} + \|f_0\|_{L^2_T(H^{k+\ell+1})} + \|f_1\|_{L^2_T(H^{k+\ell})} + N^{3/2}(0,T)). \tag{110}$$

Proof: We take the operator $D_\phi^k D_r^\ell$ in equation (109), multiply this by ζ_1 and square the whole expression. The left hand side term is

$$\iint_{Q_s} \zeta_1^2 \left(\left| \frac{2\mu+\mu'}{\overline{\rho}} \right|^2 \left| D_\phi^k D_r^{\ell+1}\frac{d\rho}{dt}\right|^2 + (p^0)^2 |D_\phi^k D_r^{\ell+1}\rho|^2 + \frac{4\mu+2\mu'}{\overline{\rho}} p^0 D_\phi^k D_r^{\ell+1}\frac{d\rho}{dt} D_\phi^k D_r^{\ell+1}\rho \right) dx\, d\tau.$$

$$= \left\| \zeta_1 \frac{2\mu+\mu'}{\overline{\rho}} D_\phi^k D_r^{\ell+1}\frac{d\rho}{dt}\right\|^2_{L^2_s(L^2)} + \left\|\zeta_1 p^0 D_\phi^k D_r^{\ell+1}\rho\right\|^2_{L^2_s(L^2)} + \frac{4\mu+2\mu'}{\overline{\rho}} p^0 \left(\zeta_1 D_\phi^k D_r^{\ell+1}\frac{d\rho}{dt}, \zeta_1 D_\phi^k D_r^{\ell+1}\rho \right)_{L^2_T(L^2)}.$$

Using the definition of $\frac{d\rho}{dt}$ (see (88)), the last term in this expression readily provides the $L^\infty_T(L^2)$ norm of $\zeta_1 D_\phi^k D_r^{\ell+1}\rho$ in terms of $\|\rho_0\|_{H^{k+\ell+1}}$ and $N^{3/2}(0,T)$.

Lemma 15 *For $0 \leqslant k+\ell \leqslant 2$, we have*

$$\|\zeta_1 D_\phi^k u\|_{L^2_T(H^{2+\ell})} + \|\zeta_1 D_\phi^k \rho\|_{L^2_T(H^{1+\ell})} \leqslant C \left(\left\| \zeta_1 D_\phi^k \frac{d\rho}{dt} \right\|_{L^2_T(H^{1+\ell})} + \|u_t\|_{L^2_T(H^{k+\ell})} + \|u\|_{L^2_T(H^{k+\ell+1})} + \|\rho\|_{L^2_T(H^{k+\ell})} + \|a\|_{H^1_T} + \|\omega\|_{L^2_T} + \|f_0\|_{L^2_T(H^{1+k+\ell})} + \|f_1\|_{L^2_T(H^{k+\ell})} \right). \tag{111}$$

Proof: We regard the equation of the fluid in (72) as a stationary Stokes system for each $s \in (0,T)$:

$$\begin{cases} -2\mu\nabla\cdot(\epsilon(u)) - \mu'\nabla\cdot((\nabla\cdot u)Id) + p^0\nabla\rho = f_1 - u_t & \text{in } \mathcal{O}_i \cap \Omega_F(0), \\ \nabla\cdot u = \dfrac{1}{\overline{\rho}}\left(f_0 - \dfrac{d\rho}{dt}\right) & \text{in } \mathcal{O}_i \cap \Omega_F(0), \\ u = (\dot{a} + \omega\wedge Qx)\mathbf{1}_{\partial\Omega_S(0)} & \text{on } \mathcal{O}_i \cap \partial\Omega_F(0). \end{cases}$$

Recall that $\dfrac{d\rho}{dt}$ was given in (88). Now, we differentiate the equation of u with respect to $\phi = (\phi_1,\phi_2)$ k times and we multiply by ζ_1. Denoting $(u_k,\rho_k) := \zeta_1 D_\phi^k(u,\rho)$, we have

$$\begin{cases} -2\mu\nabla\cdot(\epsilon(u_k)) - \mu'\nabla\cdot((\nabla\cdot u_k)Id) + p^0\nabla(\rho_k) = f_{1,k} & \text{in } \mathcal{O}_i \cap \Omega_F(0), \\ \nabla\cdot u_k = \dfrac{\zeta_1}{\overline{\rho}}\left(D_\phi^k f_0 - D_\phi^k\dfrac{d\rho}{dt}\right) + P_k(D)u & \text{in } \mathcal{O}_i \cap \Omega_F(0), \\ u_k = (\zeta_1 D_\phi^k(\dot{a} + \omega\wedge Qx))\mathbf{1}_{\partial\Omega_S(0)} & \text{on } \mathcal{O}_i \cap \partial\Omega_F(0), \end{cases}$$

where

$$f_{1,k} = \zeta_1 D_\phi^k f_1 - \zeta_1 D_\phi^k u_t + P_{k+1}(D)u + P_k(D)\rho,$$

with $P_j(D)$ a differential operator in the x variable of order $j = k, k+1$. Observe that the boundary condition can be lifted by a $C^\infty(\overline{\mathcal{O}_i \cap \Omega_F(0)})$ function. Finally, we use the following classical regularity result for the stationary Stokes problem (see, for instance, [**?**]):

Let U be a regular domain, $m \geq 0$, $\alpha > 0$, $f \in H^m(U)$ and $g \in H^{m+1}(U)$. Then, the solution (h,π) of

$$\begin{cases} -\alpha\Delta h + \nabla\pi = f & \textit{in } U, \\ \nabla\cdot h = g & \textit{in } U, \\ h = 0 & \textit{on } \partial U \end{cases}$$

belongs to $H^{m+2}(U) \times H^{m+1}(U)$ *and there exists a constant* $C > 0$ *such that*

$$\|(h,\pi)\|_{H^{m+2}(U)\times H^{m+1}(U)} \leqslant C(\|f\|_{H^m(U)} + \|g\|_{H^{m+1}(U)}).$$

4.2 Combination of Lemmas and conclusion

In this paragraph we will gather all the technical results previously stated and conclude the proof of Proposition 8.

1) We apply Lemma 10 for $k = 0$:

$$\begin{aligned}&\|u\|_{L^\infty_T(L^2)} + \|\rho\|_{L^\infty_T(L^2)} + \|u\|_{L^2_T(H^1)} + \|a\|_{W^{1,\infty}_T} + \|\omega\|_{L^\infty_T}\\ &\leqslant \delta\|\rho\|_{L^2_T(L^2)} + C(\|f_0\|_{L^2_T(L^2)} + \|f_1\|_{L^2_T(L^2)} + \|f_2\|_{L^1_T} + \|f_3\|_{L^1_T}\\ &+\|u_0\|_{L^2} + \|\rho_0\|_{L^2} + |a_0| + |\omega_0| + N^{3/2}(0,T) + N^2(0,T)).\end{aligned} \tag{112}$$

2) We apply Lemma 11 for $k = 0$:

$$\begin{aligned}&\|u\|_{H^1_T(L^2)} + \|\rho\|_{H^1_T(L^2)} + \|u\|_{L^\infty_T(H^1)} + \|a\|_{H^2_T} + \|\omega\|_{H^1_T}\\ &\leqslant C(\|u\|_{L^2_T(H^1)} + \|\rho\|_{L^\infty_T(L^2)} + \|f_0\|_{L^2_T(L^2)} + \|f_1\|_{L^2_T(L^2)} + \|f_2\|_{L^2_T} + \|f_3\|_{L^2_T}\\ &+\|u_0\|_{H^1} + N^{3/2}(0,T) + N^2(0,T)).\end{aligned} \tag{113}$$

3) We apply Lemma 10 for $k = 1$:

$$\begin{aligned}&\|u\|_{W^{1,\infty}_T(L^2)} + \|\rho\|_{W^{1,\infty}_T(L^2)} + \|u\|_{H^1_T(H^1)} + \|a\|_{W^{2,\infty}_T} + \|\omega\|_{W^{1,\infty}_T}\\ &\leqslant \delta\|\rho\|_{H^1_T(L^2)} + C(\|f_0\|_{H^1_T(L^2)} + \|f_1\|_{H^1_T(L^2)} + \|f_2\|_{W^{1,1}_T} + \|f_3\|_{W^{1,1}_T}\\ &+\|\partial_t u(0,\cdot)\|_{L^2} + \|\partial_t\rho(0,\cdot)\|_{L^2} + |\partial_t^2 a(0)| + |\partial_t\omega(0)| + N^{3/2}(0,T) + N^2(0,T)).\end{aligned} \tag{114}$$

Putting together estimates (112), (113) and (114) and taking $\delta > 0$ sufficiently small, we have:

$$\begin{aligned}&\|u\|_{W^{1,\infty}_T(L^2)} + \|u\|_{L^\infty_T(H^1)} + \|u\|_{H^1_T(H^1)} + \|\rho\|_{W^{1,\infty}_T(L^2)} + \|\rho\|_{H^1_T(L^2)}\\ &+\|a\|_{W^{2,\infty}_T} + \|a\|_{H^2_T} + \|\omega\|_{W^{1,\infty}_T} + \|\omega\|_{H^1_T} \leqslant C(\|f_0\|_{H^1_T(L^2)} + \|f_1\|_{H^1_T(L^2)} + \|f_2\|_{W^{1,1}_T} + \|f_2\|_{L^2_T}\\ &+\|f_3\|_{W^{1,1}_T} + \|f_3\|_{L^2_T} + \|\partial_t u(0,\cdot)\|_{L^2} + \|u_0\|_{H^1} + \|\partial_t\rho(0,\cdot)\|_{L^2} + \|\rho_0\|_{L^2} + |\partial_t^2 a(0)| + |a_0|\\ &+|\partial_t\omega(0)| + |\omega_0| + N^{3/2}(0,T) + N^2(0,T)).\end{aligned} \tag{115}$$

4) We apply Lemma 12 for $k = 1$:

$$\begin{aligned}&\|\zeta_0\rho\|_{L^\infty_T(H^1)} + \|\zeta_0\rho\|_{L^2_T(H^1)} + \|\zeta_0 u\|_{L^\infty_T(H^1)} + \|\zeta_0 u\|_{L^2_T(H^2)} + \|\zeta_0\frac{d\rho}{dt}\|_{L^2_T(H^1)}\\ &\leqslant C(\|u\|_{L^\infty_T(L^2)} + \|u\|_{L^2_T(H^1)} + \|\rho_0\|_{H^1} + \|u_0\|_{H^1} + \|f_0\|_{L^2_T(H^1)} + \|f_1\|_{L^2_T(L^2)} + N^{3/2}(0,T)).\end{aligned} \tag{116}$$

5) We apply Lemma 13 for $k = 1$:

$$\|\zeta_1 D_\phi \rho\|_{L^\infty_T(L^2)} + \|\zeta_1 D_\phi u\|_{L^\infty_T(L^2)} + \|\zeta_1 D_\phi u\|_{L^2_T(H^1)} + \|\zeta_1 D_\phi \frac{d\rho}{dt}\|_{L^2_T(L^2)}$$
$$\leqslant \delta(\|\zeta_1 D_\phi \rho\|_{L^2_T(L^2)} + \|u\|_{L^2_T(H^2)} + \|\rho\|_{L^2_T(H^1)}) + C(\|u\|_{L^2_T(H^1)} + \|\omega\|_{L^2_T} + \|\rho_0\|_{H^1} + \|u_0\|_{H^1} \tag{117}$$
$$+\|f_0\|_{L^2_T(H^1)} + \|f_1\|_{L^2_T(L^2)} + N^{3/2}(0,T)).$$

6) We apply Lemma 14 for $k = \ell = 0$:

$$\|\zeta_1 D_r \rho\|_{L^\infty_T(L^2)} + \|\zeta_1 D_r \rho\|_{L^2_T(L^2)} + \|\zeta_1 D_r \frac{d\rho}{dt}\|_{L^2_T(L^2)} \leqslant C(\|\zeta_1 D_\phi u\|_{L^2_T(H^1)}$$
$$+\|u_t\|_{L^2_T(L^2)} + \|u\|_{L^2_T(H^1)} + \|\rho_0\|_{H^1} + \|f_0\|_{L^2_T(H^1)} + \|f_1\|_{L^2_T(L^2)} + N^{3/2}(0,T)). \tag{118}$$

¿From (116), (117) and (118), we deduce in particular that we can estimate u in $L^2_T(H^2)$ in the interior and ρ in $L^\infty_T(H^1)$. Precisely, we have

$$\|\zeta_0 u\|_{L^2_T(H^2)} + \|\rho\|_{L^\infty_T(H^1)} + \|\zeta_0 \rho\|_{L^2_T(H^1)} + \|\frac{d\rho}{dt}\|_{L^2_T(H^1)}$$
$$\leqslant \delta(\|\zeta_1 D_\phi \rho\|_{L^2_T(L^2)} + \|u\|_{L^2_T(H^2)} + \|\rho\|_{L^2_T(H^1)}) + C(\|u\|_{L^\infty_T(L^2)} + \|u\|_{L^2_T(H^1)} + \|u\|_{H^1_T(L^2)} \tag{119}$$
$$+\|\omega\|_{L^2_T} + \|u_0\|_{H^1} + \|\rho_0\|_{H^1} + \|f_0\|_{L^2_T(H^1)} + \|f_1\|_{L^2_T(L^2)} + N^{3/2}(0,T)).$$

7) We apply Lemma 9 for $k = 2$:

$$\|u\|_{L^\infty_T(H^2)} \leqslant C(\|u\|_{W^{1,\infty}_T(L^2)} + \|\rho\|_{L^\infty_T(H^1)} + \|a\|_{W^{1,\infty}_T} + \|\omega\|_{L^\infty_T} + \|f_1\|_{L^\infty_T(L^2)}). \tag{120}$$

8) We apply Lemma 15 for $k = \ell = 0$

$$\|\zeta_1 u\|_{L^2_T(H^2)} + \|\zeta_1 \rho\|_{L^2_T(H^1)} \leqslant C\left(\left\|\zeta_1 \frac{d\rho}{dt}\right\|_{L^2_T(H^1)} + \|u_t\|_{L^2_T(L^2)} + \|u\|_{L^2_T(H^1)} + \|\rho\|_{L^2_T(L^2)}\right.$$
$$\left. + \|a\|_{H^1_T} + \|\omega\|_{L^2_T} + \|f_0\|_{L^2_T(H^1)} + \|f_1\|_{L^2_T(L^2)}\right). \tag{121}$$

Using (119) in (120) and (121), we obtain estimates for u in $L^\infty_T(H^2) \cap L^2_T(H^2)$ and for ρ in $L^2_T(H^1)$:

$$\|u\|_{L^\infty_T(H^2)} + \|u\|_{L^2_T(H^2)} + \|\rho\|_{L^\infty_T(H^1)} + \|\rho\|_{H^1_T(H^1)} + \|\frac{d\rho}{dt}\|_{L^2_T(H^1)}$$
$$\leqslant C(\|u\|_{W^{1,\infty}_T(L^2)} + \|u\|_{L^2_T(H^1)} + \|u\|_{H^1_T(L^2)} + \|\rho\|_{H^1_T(L^2)} + \|a\|_{W^{1,\infty}_T} + \|a\|_{H^1_T} \tag{122}$$
$$+\|\omega\|_{L^\infty_T} + \|\omega\|_{L^2_T} + \|u_0\|_{H^1} + \|\rho_0\|_{H^1} + \|f_0\|_{L^2_T(H^1)}$$
$$+\|f_1\|_{L^\infty_T(L^2)} + \|f_1\|_{L^2_T(L^2)} + N^{3/2}(0,T) + N^2(0,T)).$$

Observe that we have added the term $\|\rho_t\|_{L^2_T(H^1)}$ in the left hand side of this inequality. This term comes from the estimate of $\dfrac{d\rho}{dt}$ in $L^2_T(H^1)$ since the nonlinear term goes to $N^2(0,T)$.

Therefore, putting this together with (115), we get

$$\|u\|_{W^{1,\infty}_T(L^2)} + \|u\|_{L^\infty_T(H^2)} + \|u\|_{L^2_T(H^2)} + \|u\|_{H^1_T(H^1)} + \|\rho\|_{W^{1,\infty}_T(L^2)} + \|\rho\|_{L^\infty_T(H^1)}$$
$$+\|\rho\|_{H^1_T(H^1)} + \|a\|_{W^{2,\infty}_T} + \|a\|_{H^2_T} + \|\omega\|_{W^{1,\infty}_T} + \|\omega\|_{H^1_T}$$
$$\leqslant C(\|f_0\|_{H^1_T(L^2)} + \|f_0\|_{L^2_T(H^1)} + \|f_1\|_{H^1_T(L^2)} + \|f_1\|_{L^\infty_T(L^2)} + \|f_2\|_{W^{1,1}_T} + \|f_2\|_{L^2_T} \tag{123}$$
$$+\|f_3\|_{W^{1,1}_T} + \|f_3\|_{L^2_T} + \|\partial_t u(0,\cdot)\|_{L^2} + \|u_0\|_{H^1} + \|\partial_t \rho(0,\cdot)\|_{L^2} + \|\rho_0\|_{H^1}$$
$$+|\partial_t^2 a(0)| + |a_0| + |\partial_t \omega(0)| + |\omega_0| + N^{3/2}(0,T) + N^2(0,T)).$$

9) We apply Lemma 12 for $k = 2$

$$\begin{aligned}&\|\zeta_0\rho\|_{L^\infty_T(H^2)} + \|\zeta_0\rho\|_{L^2_T(H^2)} + \|\zeta_0 u\|_{L^\infty_T(H^2)} + \|\zeta_0 u\|_{L^2_T(H^3)} + \|\zeta_0\frac{d\rho}{dt}\|_{L^2_T(H^2)}\\ &\leqslant C(\|u\|_{L^\infty_T(H^1)} + \|u\|_{L^2_T(H^2)} + \|\rho_0\|_{H^2} + \|u_0\|_{H^2} + \|f_0\|_{L^2_T(H^2)} + \|f_1\|_{L^2_T(H^1)} + N^{3/2}(0,T)).\end{aligned} \tag{124}$$

10) We apply Lemma 13 for $k = 2$:

$$\begin{aligned}&\|\zeta_1 D^2_\phi\rho\|_{L^\infty_T(L^2)} + \|\zeta_1 D^2_\phi u\|_{L^\infty_T(L^2)} + \|\zeta_1 D^2_\phi u\|_{L^2_T(H^1)} + \|\zeta_1 D^2_\phi\frac{d\rho}{dt}\|_{L^2_T(L^2)}\\ &\leqslant \delta(\|\zeta_1 D^2_\phi\rho\|_{L^2_T(L^2)} + \|\rho\|_{L^2_T(H^1)}) + C(\|u\|_{L^2_T(H^2)} + \|\omega\|_{L^2_T} + \|\rho_0\|_{H^2} + \|u_0\|_{H^2}\\ &+\|f_0\|_{L^2_T(H^2)} + \|f_1\|_{L^2_T(H^1)} + N^{3/2}(0,T)).\end{aligned} \tag{125}$$

11) We apply Lemma 14 for $k = 1$, $\ell = 0$:

$$\begin{aligned}&\|\zeta_1 D_\phi D_r\rho\|_{L^\infty_T(L^2)} + \|\zeta_1 D_\phi D_r\rho\|_{L^2_T(L^2)} + \|\zeta_1 D_\phi D_r\frac{d\rho}{dt}\|_{L^2_T(L^2)} \leqslant C(\|\zeta_1 D^2_\phi u\|_{L^2_T(H^1)}\\ &+\|u_t\|_{L^2_T(H^1)} + \|u\|_{L^2_T(H^2)} + \|\rho_0\|_{H^2} + \|f_0\|_{L^2_T(H^2)} + \|f_1\|_{L^2_T(H^1)} + N^{3/2}(0,T)).\end{aligned} \tag{126}$$

Observe that the first term in the right hand side in (126) can be estimated by the third term in the left hand side of (125).

12) We apply Lemma 15 for $k = 1$, $\ell = 0$:

$$\begin{aligned}\|\zeta_1 D_\phi u\|_{L^2_T(H^2)} + \|\zeta_1 D_\phi\rho\|_{L^2_T(H^1)} \leqslant C\Bigg(&\left\|\zeta_1 D_\phi\frac{d\rho}{dt}\right\|_{L^2_T(H^1)} + \|u_t\|_{L^2_T(H^1)} + \|u\|_{L^2_T(H^2)} + \|\rho\|_{L^2_T(H^1)}\\ &+ \|a\|_{H^1_T} + \|\omega\|_{L^2_T} + \|f_0\|_{L^2_T(H^2)} + \|f_1\|_{L^2_T(H^1)}\Bigg).\end{aligned} \tag{127}$$

The first term in the right hand side of (127) can be estimated with the help of the third term in the left hand side of (126) and the fourth term in the left hand side of (125).

13) We apply Lemma 14 for $k = 0$, $\ell = 1$:

$$\begin{aligned}&\|\zeta_1 D^2_r\rho\|_{L^\infty_T(L^2)} + \|\zeta_1 D^2_r\rho\|_{L^2_T(L^2)} + \|\zeta_1 D^2_r\frac{d\rho}{dt}\|_{L^2_T(L^2)} \leqslant C(\|\zeta_1 D_\phi D_r u\|_{L^2_T(H^1)}\\ &+\|u_t\|_{L^2_T(H^1)} + \|u\|_{L^2_T(H^2)} + \|\rho_0\|_{H^2} + \|f_0\|_{L^2_T(H^2)} + \|f_1\|_{L^2_T(H^1)} + N^{3/2}(0,T)).\end{aligned} \tag{128}$$

The norm $\|\zeta_1 D_\phi D_r u\|_{L^2_T(H^1)}$ is bounded thanks to the first term in the left hand side of (127).

14) We apply Lemma 15 for $k = 0$, $\ell = 1$:

$$\begin{aligned}\|\zeta_1 u\|_{L^2_T(H^3)} + \|\zeta_1\rho\|_{L^2_T(H^2)} \leqslant C\Bigg(&\left\|\zeta_1\frac{d\rho}{dt}\right\|_{L^2_T(H^2)} + \|u_t\|_{L^2_T(H^1)} + \|u\|_{L^2_T(H^2)} + \|\rho\|_{L^2_T(H^1)}\\ &+ \|a\|_{H^1_T} + \|\omega\|_{L^2_T} + \|f_0\|_{L^2_T(H^2)} + \|f_1\|_{L^2_T(H^1)}\Bigg).\end{aligned} \tag{129}$$

The first term in the right hand side of this inequality can be estimated with the help of third term in the right hand side of (128), the third in the left of (126) and the fourth in the left of (125).

Collecting expressions (124)-(129), we deduce

$$
\begin{aligned}
&\|u\|_{L^2_T(H^3)}+\|\rho\|_{L^\infty_T(H^2)}+\|\rho\|_{L^2_T(H^2)}+\|\frac{d\rho}{dt}\|_{L^2_T(H^2)}\\
&\leqslant C(\|u\|_{L^\infty_T(H^1)}+\|u\|_{L^2_T(H^2)}+\|u\|_{H^1_T(H^1)}+\|\rho\|_{L^2_T(H^1)}+\|a\|_{H^1_T}+\|\omega\|_{L^2_T}+\|u_0\|_{H^2}+\|\rho_0\|_{H^2}\\
&+\|f_0\|_{L^2_T(H^2)}+\|f_1\|_{L^2_T(H^1)}+N^{3/2}(0,T)).
\end{aligned}
\tag{130}
$$

Using (123), for the moment we have

$$
\begin{aligned}
&\|u\|_{W^{1,\infty}_T(L^2)}+\|u\|_{L^\infty_T(H^2)}+\|u\|_{L^2_T(H^3)}+\|u\|_{H^1_T(H^1)}+\|\rho\|_{W^{1,\infty}_T(H^1)}+\|\rho\|_{L^\infty_T(H^2)}\\
&+\|\rho\|_{H^1_T(H^2)}+\|\frac{d\rho}{dt}\|_{L^2_T(H^2)}+\|a\|_{W^{2,\infty}_T}+\|a\|_{H^2_T}+\|\omega\|_{W^{1,\infty}_T}+\|\omega\|_{H^1_T}\\
&\qquad\leqslant C(\|f_0\|_{H^1_T(L^2)}+\|f_0\|_{L^2_T(H^2)}+\|f_1\|_{L^2_T(H^1)}+\|f_1\|_{H^1_T(L^2)}+\|f_1\|_{L^\infty_T(L^2)}+\|f_2\|_{W^{1,1}_T}+\|f_2\|_{L^2_T}\\
&\qquad+\|f_3\|_{W^{1,1}_T}+\|f_3\|_{L^2_T}+\|\partial_t u(0,\cdot)\|_{L^2}+\|u_0\|_{H^2}+\|\partial_t\rho(0,\cdot)\|_{L^2}+\|\rho_0\|_{H^2}\\
&\qquad+|\partial_t^2 a(0)|+|a_0|+|\partial_t\omega(0)|+|\omega_0|+N^{3/2}(0,T)+N^2(0,T)).
\end{aligned}
\tag{131}
$$

The term $\|\rho\|_{W^{1,\infty}_T(H^1)}$ comes from $\|u\|_{L^\infty_T(H^2)}$ and the equation of ρ.

15) We apply Lemma 11 for $k=1$:

$$
\begin{aligned}
&\|u\|_{H^2_T(L^2)}+\|\rho\|_{H^2_T(L^2)}+\|u\|_{W^{1,\infty}_T(H^1)}+\|a\|_{H^3_T}+\|\omega\|_{H^2_T}\\
&\leqslant C(\|u\|_{H^1_T(H^1)}+\|\rho\|_{W^{1,\infty}_T(L^2)}+\|f_0\|_{H^1_T(L^2)}+\|f_1\|_{H^1_T(L^2)}+\|f_2\|_{H^1_T}+\|f_3\|_{H^1_T}\\
&+\|\partial_t u(0,\cdot)\|_{H^1}+N^{3/2}(0,T)+N^2(0,T)).
\end{aligned}
\tag{132}
$$

Now, we regard the equation satisfied by u_t as a stationary elliptic equation:

$$
\begin{cases}
-2\mu\nabla\cdot(\epsilon(u_t))-\mu'\nabla\cdot((\nabla\cdot u_t)Id)=-u_{tt}-p^0\nabla\rho_t+f_{1,t} & \text{in } (0,T)\times\Omega_F(0),\\
u_t=(\ddot a+\dot\omega\wedge Qx+\omega\wedge\dot Qx)\mathbf{1}_{\partial\Omega_S(0)} & \text{in } (0,T)\times\partial\Omega_F(0).
\end{cases}
$$

Then, we have

$$
\|u\|_{H^1_T(H^2)}\leqslant C(\|u\|_{H^2_T(L^2)}+\|\rho\|_{H^1_T(H^1)}+\|f_1\|_{H^1_T(L^2)}+\|a\|_{H^2_T}+\|\omega\|_{H^1_T}).
\tag{133}
$$

16) We apply Lemma 9 for $k=3$:

$$
\|u\|_{L^\infty_T(H^3)}\leqslant C(\|u\|_{W^{1,\infty}_T(H^1)}+\|\rho\|_{L^\infty_T(H^2)}+\|a\|_{W^{1,\infty}_T}+\|\omega\|_{L^\infty_T}+\|f_1\|_{L^\infty_T(H^1)}).
\tag{134}
$$

Combining the three estimates (132)-(134), we get

$$
\begin{aligned}
&\|u\|_{W^{1,\infty}_T(H^1)}+\|u\|_{L^\infty_T(H^3)}+\|u\|_{H^2_T(L^2)}+\|u\|_{H^1_T(H^2)}+\|\rho\|_{H^2_T(L^2)}+\|a\|_{H^3_T}+\|\omega\|_{H^2_T}\\
&\leqslant C(\|u\|_{H^1_T(H^1)}+\|\rho\|_{W^{1,\infty}_T(L^2)}+\|\rho\|_{L^\infty_T(H^2)}+\|\rho\|_{H^1_T(H^1)}+\|a\|_{W^{1,\infty}_T}+\|a\|_{H^2_T}\\
&+\|\omega\|_{H^1_T}+\|\omega\|_{L^\infty_T}+\|f_0\|_{H^1_T(L^2)}+\|f_1\|_{H^1_T(L^2)}+\|f_1\|_{L^\infty_T(H^1)}\\
&+\|f_2\|_{H^1_T}+\|f_3\|_{H^1_T}+\|\partial_t u(0,\cdot)\|_{H^1}+N^{3/2}(0,T)+N^2(0,T)).
\end{aligned}
\tag{135}
$$

17) We apply Lemma 12 for $k=3$:

$$
\begin{aligned}
&\|\zeta_0\rho\|_{L^\infty_T(H^3)}+\|\zeta_0\rho\|_{L^2_T(H^3)}+\|\zeta_0 u\|_{L^\infty_T(H^3)}+\|\zeta_0 u\|_{L^2_T(H^4)}+\|\zeta_0\frac{d\rho}{dt}\|_{L^2_T(H^3)}\\
&\leqslant C(\|u\|_{L^\infty_T(H^2)}+\|u\|_{L^2_T(H^3)}+\|\rho_0\|_{H^3}+\|u_0\|_{H^3}+\|f_0\|_{L^2_T(H^3)}+\|f_1\|_{L^2_T(H^2)}+N^{3/2}(0,T)).
\end{aligned}
\tag{136}
$$

18) We apply Lemma 13 for $k=3$:

$$\begin{aligned}&\|\zeta_1 D_\phi^3\rho\|_{L^\infty_T(L^2)}+\|\zeta_1 D_\phi^3 u\|_{L^\infty_T(L^2)}+\|\zeta_1 D_\phi^3 u\|_{L^2_T(H^1)}+\|\zeta_1 D_\phi^3\frac{d\rho}{dt}\|_{L^2_T(L^2)}\\&\leqslant \delta(\|\zeta_1 D_\phi^3\rho\|_{L^2_T(L^2)}+\|\rho\|_{L^2_T(H^1)})+C(\|u\|_{L^2_T(H^3)}+\|\omega\|_{L^2_T}+\|\rho_0\|_{H^3}+\|u_0\|_{H^3}\\&+\|f_0\|_{L^2_T(H^3)}+\|f_1\|_{L^2_T(H^2)}+N^{3/2}(0,T)).\end{aligned}\tag{137}$$

19) We apply Lemma 14 for $k=2$, $\ell=0$:

$$\begin{aligned}&\|\zeta_1 D_\phi^2 D_r\rho\|_{L^\infty_T(L^2)}+\|\zeta_1 D_\phi^2 D_r\rho\|_{L^2_T(L^2)}+\|\zeta_1 D_\phi^2 D_r\frac{d\rho}{dt}\|_{L^2_T(L^2)}\leqslant C(\|\zeta_1 D_\phi^3 u\|_{L^2_T(H^1)}\\&+\|u_t\|_{L^2_T(H^2)}+\|u\|_{L^2_T(H^3)}+\|\rho_0\|_{H^3}+\|f_0\|_{L^2_T(H^3)}+\|f_1\|_{L^2_T(H^2)}+N^{3/2}(0,T)).\end{aligned}\tag{138}$$

Observe that the norm $\|\zeta_1 D_\phi^3 u\|_{L^2_T(H^1)}$ is estimated with the third term in the left of (137).

20) We apply Lemma 15 for $k=2$, $\ell=0$:

$$\begin{aligned}\|\zeta_1 D_\phi^2 u\|_{L^2_T(H^2)}+\|\zeta_1 D_\phi^2\rho\|_{L^2_T(H^1)}\leqslant C\Bigg(\left\|\zeta_1 D_\phi^2\frac{d\rho}{dt}\right\|_{L^2_T(H^1)}+\|u_t\|_{L^2_T(H^2)}+\|u\|_{L^2_T(H^3)}+\|\rho\|_{L^2_T(H^2)}\\+\|a\|_{H^1_T}+\|\omega\|_{L^2_T}+\|f_0\|_{L^2_T(H^3)}+\|f_1\|_{L^2_T(H^2)}\Bigg).\end{aligned}\tag{139}$$

As for (129), the term on $\frac{d\rho}{dt}$ is bounded thanks to the third term in the left of (138) and the fourth term in the left of (137).

21) We apply Lemma 14 for $k=1$, $\ell=1$:

$$\begin{aligned}&\|\zeta_1 D_\phi D_r^2\rho\|_{L^\infty_T(L^2)}+\|\zeta_1 D_\phi D_r^2\rho\|_{L^2_T(L^2)}+\|\zeta_1 D_\phi D_r^2\frac{d\rho}{dt}\|_{L^2_T(L^2)}\leqslant C(\|\zeta_1 D_\phi^2 D_r u\|_{L^2_T(H^1)}\\&+\|u_t\|_{L^2_T(H^2)}+\|u\|_{L^2_T(H^3)}+\|\rho_0\|_{H^3}+\|f_0\|_{L^2_T(H^3)}+\|f_1\|_{L^2_T(H^2)}+N^{3/2}(0,T)).\end{aligned}\tag{140}$$

The first term in the left of (139) absorbs the first in the right of (140).

22) We apply Lemma 15 for $k=1$, $\ell=1$:

$$\begin{aligned}\|\zeta_1 D_\phi u\|_{L^2_T(H^3)}+\|\zeta_1 D_\phi\rho\|_{L^2_T(H^2)}\leqslant C\Bigg(\left\|\zeta_1 D_\phi\frac{d\rho}{dt}\right\|_{L^2_T(H^2)}+\|u_t\|_{L^2_T(H^2)}+\|u\|_{L^2_T(H^3)}+\|\rho\|_{L^2_T(H^2)}\\+\|a\|_{H^1_T}+\|\omega\|_{L^2_T}+\|f_0\|_{L^2_T(H^3)}+\|f_1\|_{L^2_T(H^2)}\Bigg).\end{aligned}\tag{141}$$

Observe that $\|\zeta_1 D_\phi\frac{d\rho}{dt}\|_{L^2_T(H^2)}$ is estimated with the help of the third term in the left of (140), the third in the left of (138) and the fourth in the left of (137).

23) We apply Lemma 14 for $k=0$, $\ell=2$:

$$\begin{aligned}&\|\zeta_1 D_r^3\rho\|_{L^\infty_T(L^2)}+\|\zeta_1 D_r^3\rho\|_{L^2_T(L^2)}+\|\zeta_1 D_r^3\frac{d\rho}{dt}\|_{L^2_T(L^2)}\leqslant C(\|\zeta_1 D_\phi D_r^2 u\|_{L^2_T(H^1)}\\&+\|u_t\|_{L^2_T(H^2)}+\|u\|_{L^2_T(H^3)}+\|\rho_0\|_{H^3}+\|f_0\|_{L^2_T(H^3)}+\|f_1\|_{L^2_T(H^2)}+N^{3/2}(0,T)).\end{aligned}\tag{142}$$

The term $\|\zeta_1 D_\phi D_r^2 u\|_{L^2_T(H^1)}$ is estimated with the first term in the left of (141).

24) We apply Lemma 15 for $k=0$, $\ell=2$:

$$\|\zeta_1 u\|_{L^2_T(H^4)} + \|\zeta_1\rho\|_{L^2_T(H^3)} \leqslant C\left(\left\|\zeta_1\frac{d\rho}{dt}\right\|_{L^2_T(H^3)} + \|u_t\|_{L^2_T(H^2)} + \|u\|_{L^2_T(H^3)} + \|\rho\|_{L^2_T(H^2)} + \|a\|_{H^1_T} + \|\omega\|_{L^2_T} + \|f_0\|_{L^2_T(H^3)} + \|f_1\|_{L^2_T(H^2)}\right). \tag{143}$$

The term concerning $\frac{d\rho}{dt}$ is estimated with the third terms in the left hand sides of (142), (140) and (138) and the fourth term in the left of (137).

Combining estimates (136)-(143), we get:

$$\begin{aligned} &\|u\|_{L^2_T(H^4)} + \|\rho\|_{L^2_T(H^3)} + \|\rho\|_{L^\infty_T(H^3)} + \|\frac{d\rho}{dt}\|_{L^2_T(H^3)} \\ &\leqslant C(\|u\|_{L^2_T(H^3)} + \|u\|_{L^\infty_T(H^3)} + \|u\|_{H^1_T(H^2)} + \|\rho\|_{L^2_T(H^2)} + \|a\|_{H^1_T} + \|\omega\|_{L^2_T} + \|u_0\|_{H^3} + \|\rho_0\|_{H^3} \\ &+\|f_0\|_{L^2_T(H^3)} + \|f_1\|_{L^2_T(H^2)} + N^{3/2}(0,T)). \end{aligned} \tag{144}$$

Together with (135), this estimate yields

$$\begin{aligned} &\|u\|_{W^{1,\infty}_T(H^1)} + \|u\|_{L^\infty_T(H^3)} + \|u\|_{L^2_T(H^4)} + \|u\|_{H^2_T(L^2)} + \|u\|_{H^1_T(H^2)} \\ &+\|\rho\|_{L^\infty_T(H^3)} + \|\rho\|_{H^1_T(H^3)} + \|\rho\|_{H^2_T(L^2)} + \|a\|_{H^3_T} + \|\omega\|_{H^2_T} \\ &\leqslant C(\|u\|_{L^2_T(H^3)} + \|u\|_{H^1_T(H^1)} + \|\rho\|_{H^1_T(H^1)} + \|\rho\|_{L^2_T(H^2)} + \|\rho\|_{W^{1,\infty}_T(L^2)} + \|\rho\|_{L^\infty_T(H^2)} + \|a\|_{W^{1,\infty}_T} \\ &+\|a\|_{H^2_T} + \|\omega\|_{H^1_T} + \|\omega\|_{L^\infty_T} + \|f_0\|_{L^2_T(H^3)} + \|f_0\|_{H^1_T(L^2)} + \|f_1\|_{L^2_T(H^2)} + \|f_1\|_{H^1_T(L^2)} + \|f_1\|_{L^\infty_T(H^1)} \\ &+\|f_2\|_{H^1_T} + \|f_3\|_{H^1_T} + \|u_0\|_{H^3} + \|\rho_0\|_{H^3} + \|\partial_t u(0,\cdot)\|_{H^1} + N^{3/2}(0,T) + N^2(0,T)). \end{aligned} \tag{145}$$

Observe that we have included the term $\|\rho_t\|_{L^2_T(H^3)}$ in the left hand side of this inequality. This comes from $\|\frac{d\rho}{dt}\|_{L^2_T(H^3)}$ (in the left of (144)).

Finally, we combine this estimate with (131) and we conclude inequality (73).

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