Conservative finite difference schemes and adaptive mesh refinement techniques for hyperbolic systems of conservation laws

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Outline

- Finite-difference Shu-Osher schemes
- Adaptive mesh refinement
- A look at the complete algorithm
- Some issues

Problem statement

Hyperbolic system of conservation laws (1D case):

$$\begin{cases} u_t + f(u)_x = 0 & \text{in } \mathbb{R} \times \mathbb{R}^+ \\ u(x,0) = u_0(x) & \text{in } \mathbb{R} \end{cases}$$

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Finally solve the ODE system

$$u_t + \frac{\hat{f}_{j+\frac{1}{2}} - \hat{f}_{j-\frac{1}{2}}}{\Delta x} = 0$$

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for unknown ϕ, whose average on cell $[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$ is $f(u(x_i,t))$ ($x_i = i\Delta x$).

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- TVD Runge-Kutta methods are used for time evolution

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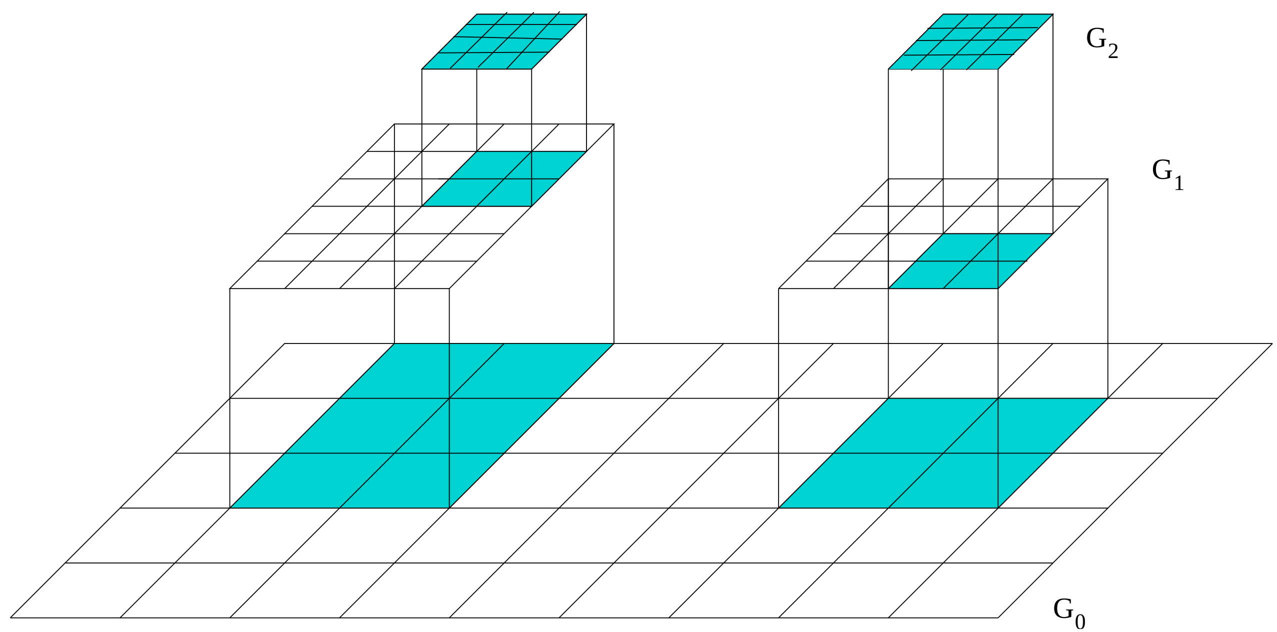

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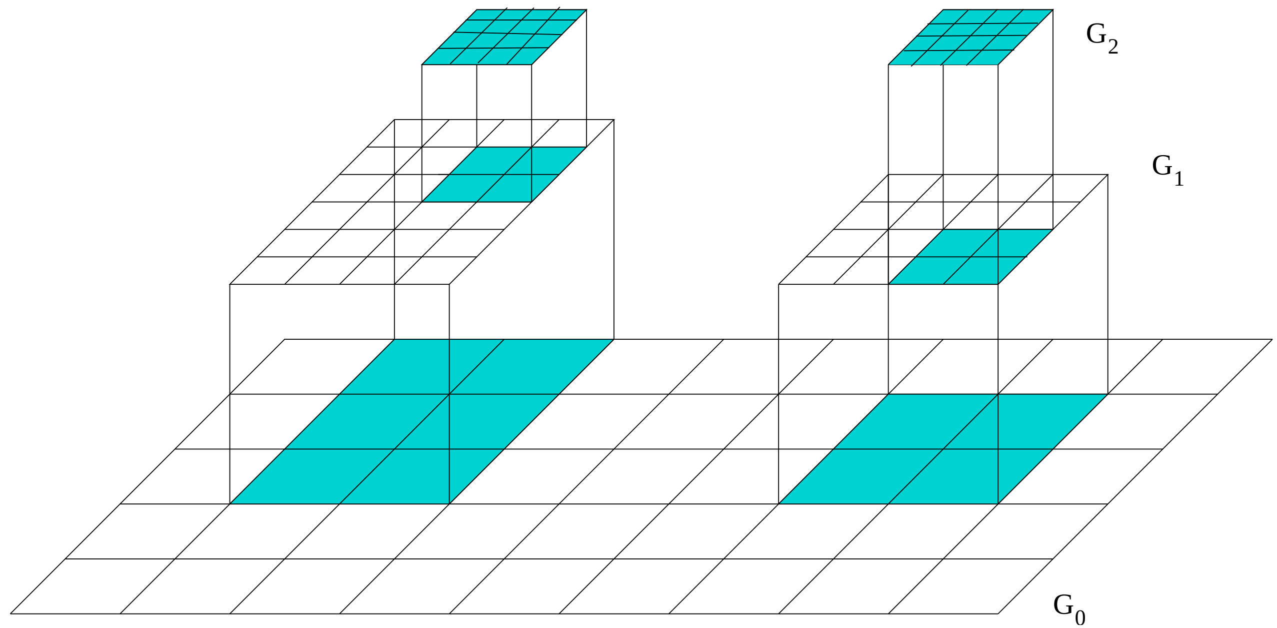

Key idea: To reduce the total number of cell updates (flux computations).

A look at the complete algorithm

We use a grid hierarchy $G_0, \ldots, G_L$:

- $G_l \equiv$ union of Cartesian patches of uniform mesh size
- G_l is finer than G_{l-1} and $G_l \subseteq G_{l-1}$ (nestedness)
- Singularities **never** cross a fine mesh boundary (moving grids)

$\Rightarrow$ **Adaptive mesh refinement(AMR) [Berger, Oliger]**.

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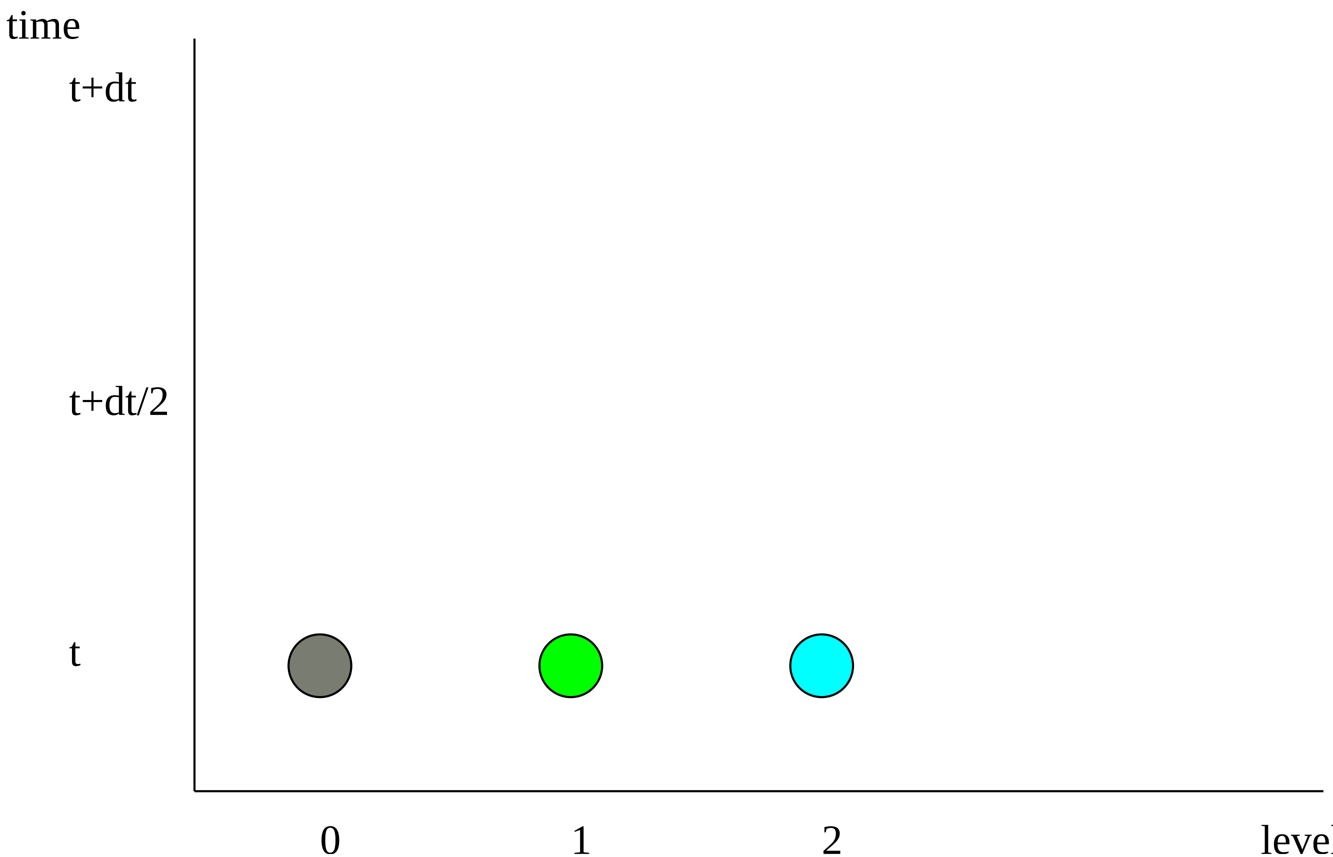

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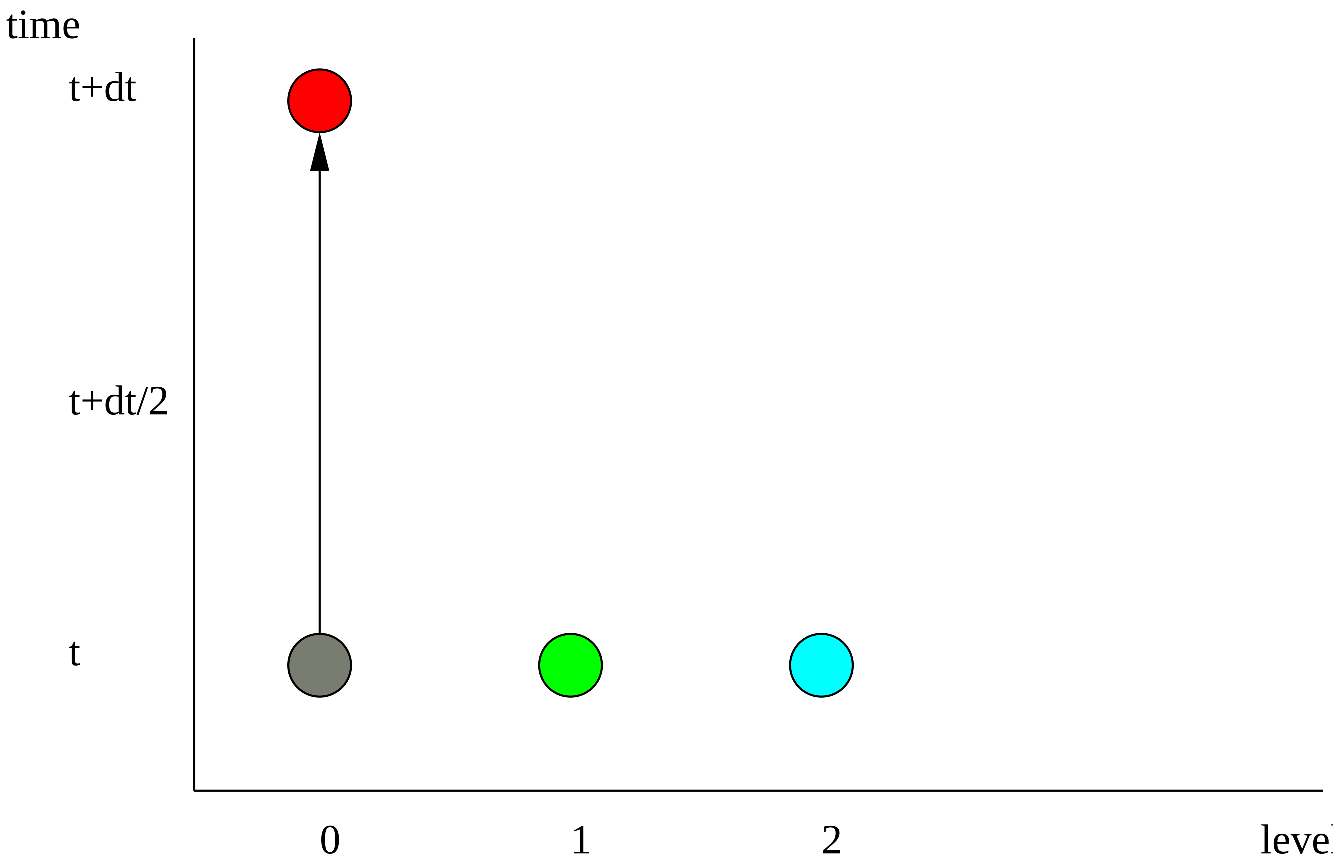

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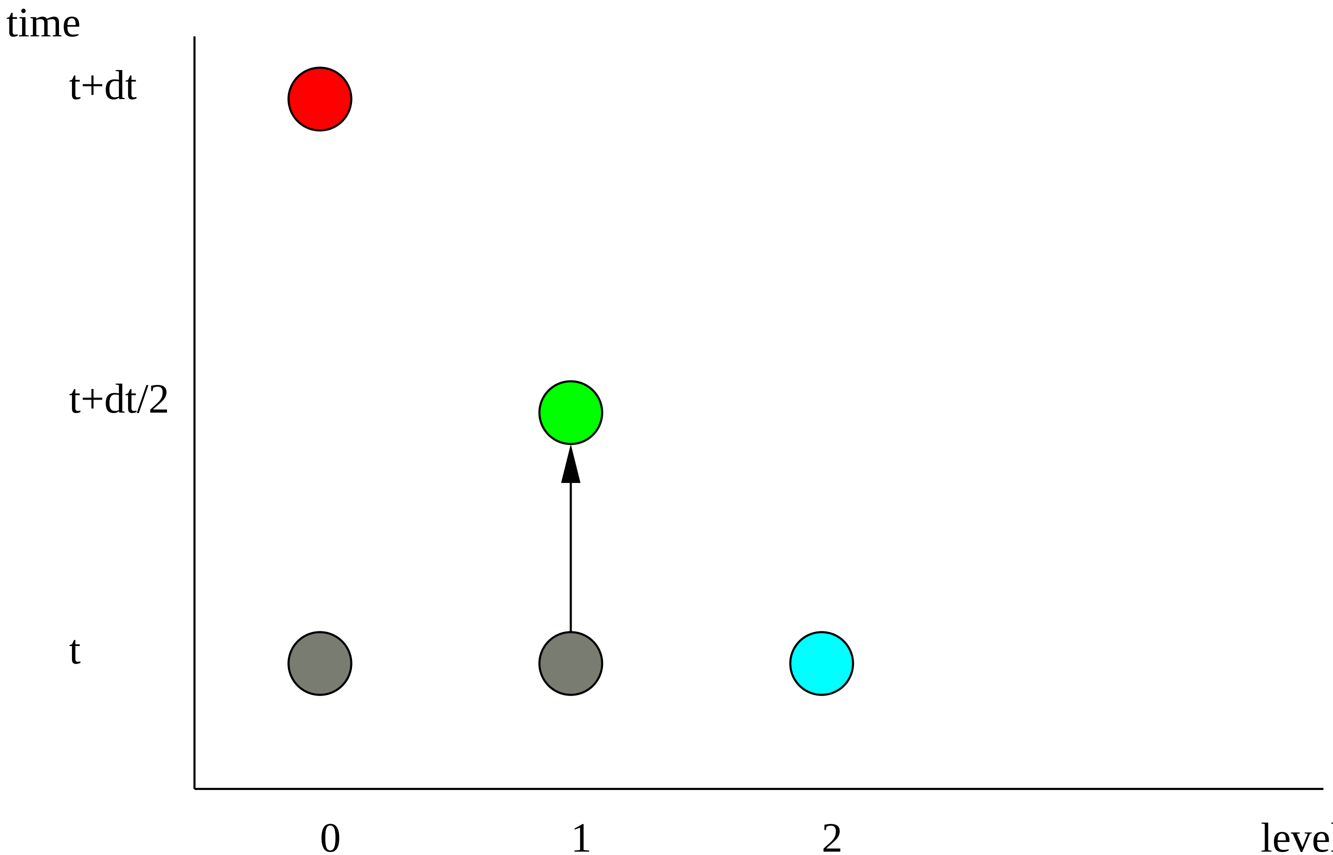

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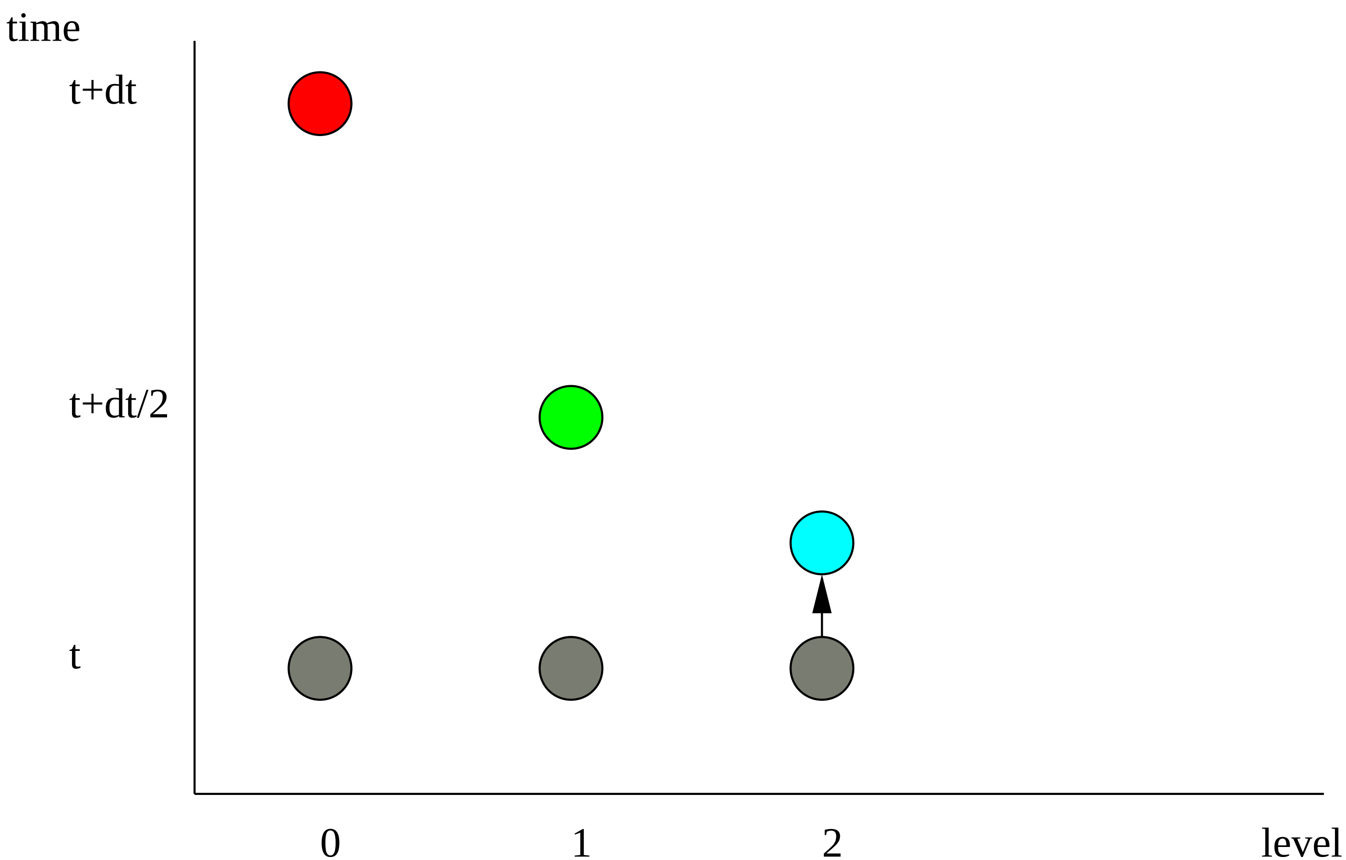

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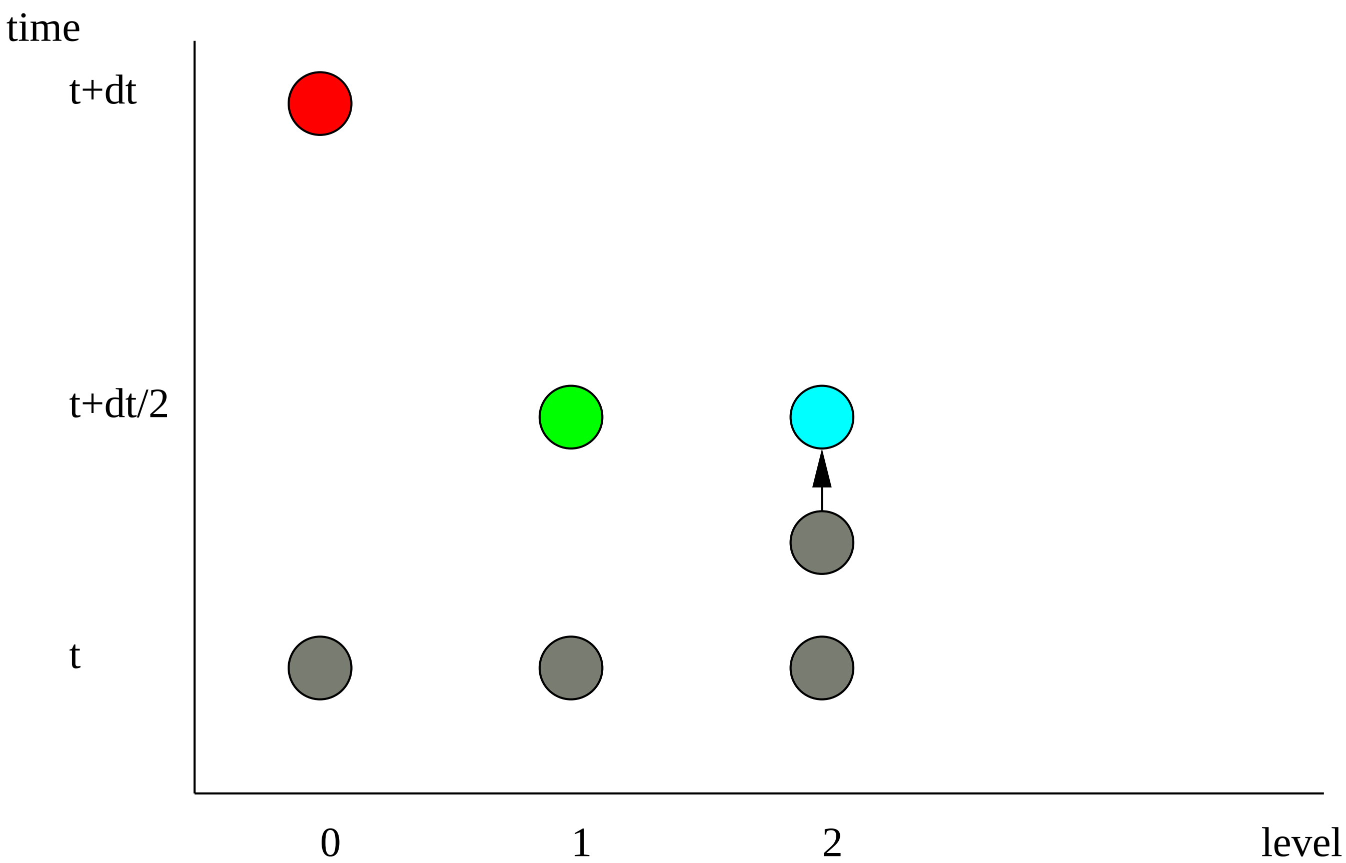

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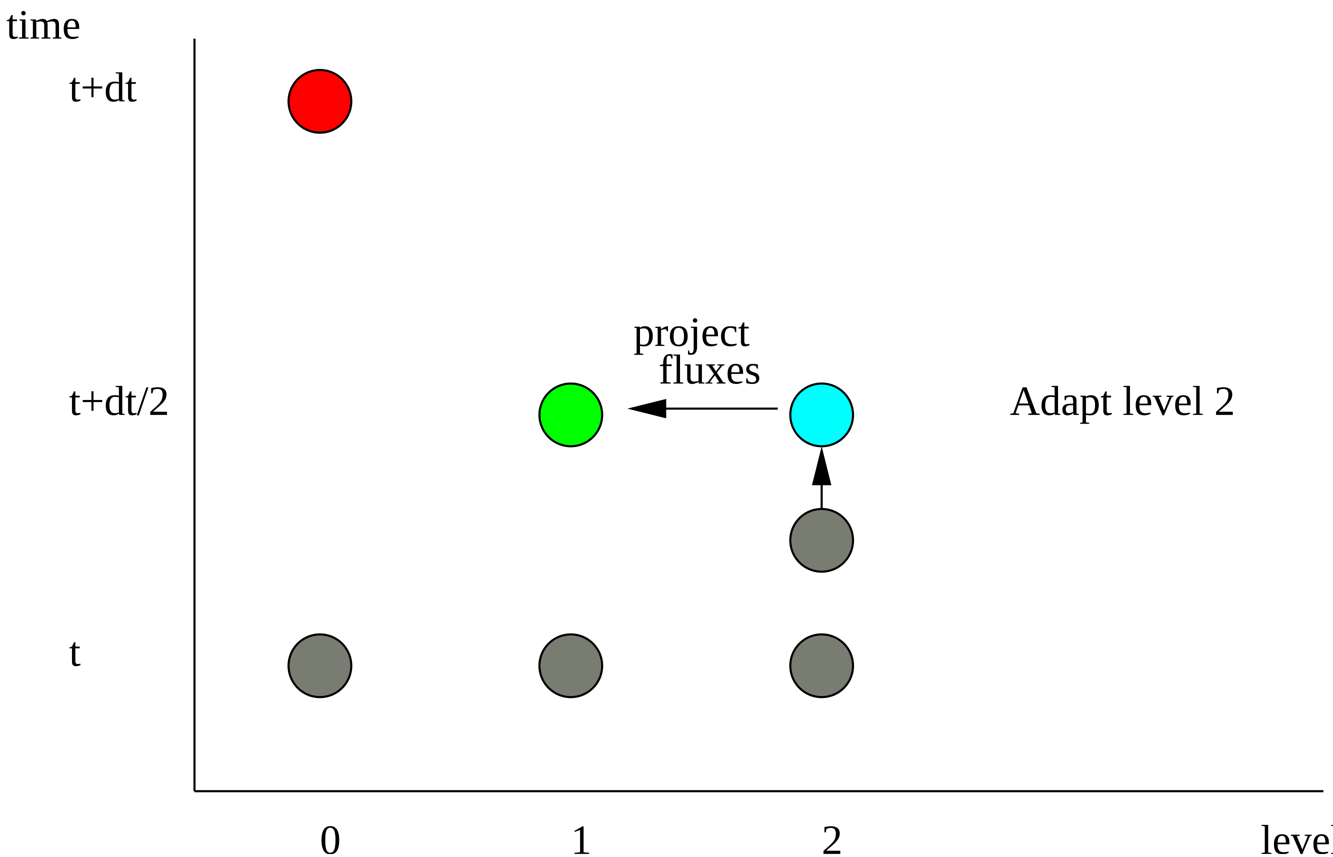

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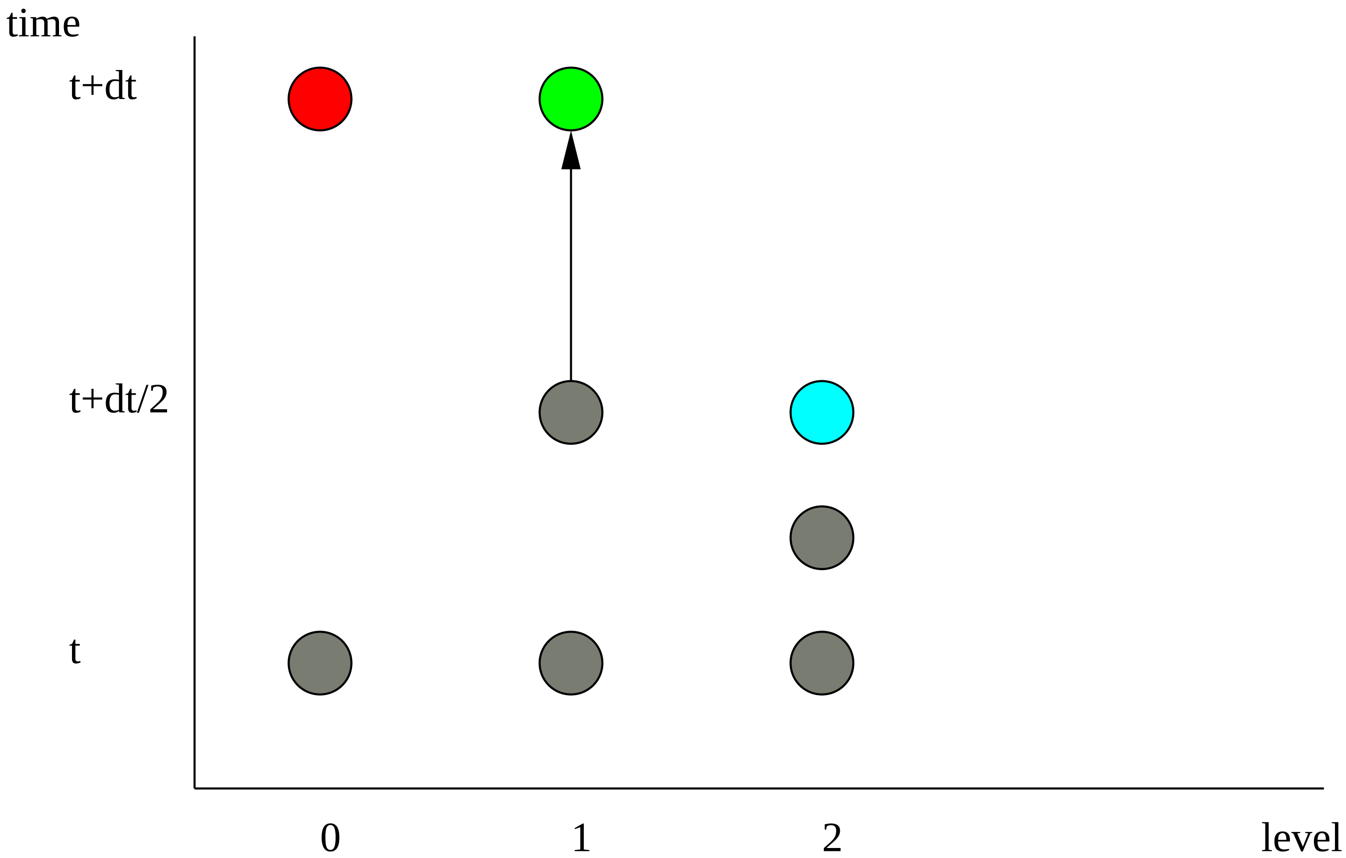

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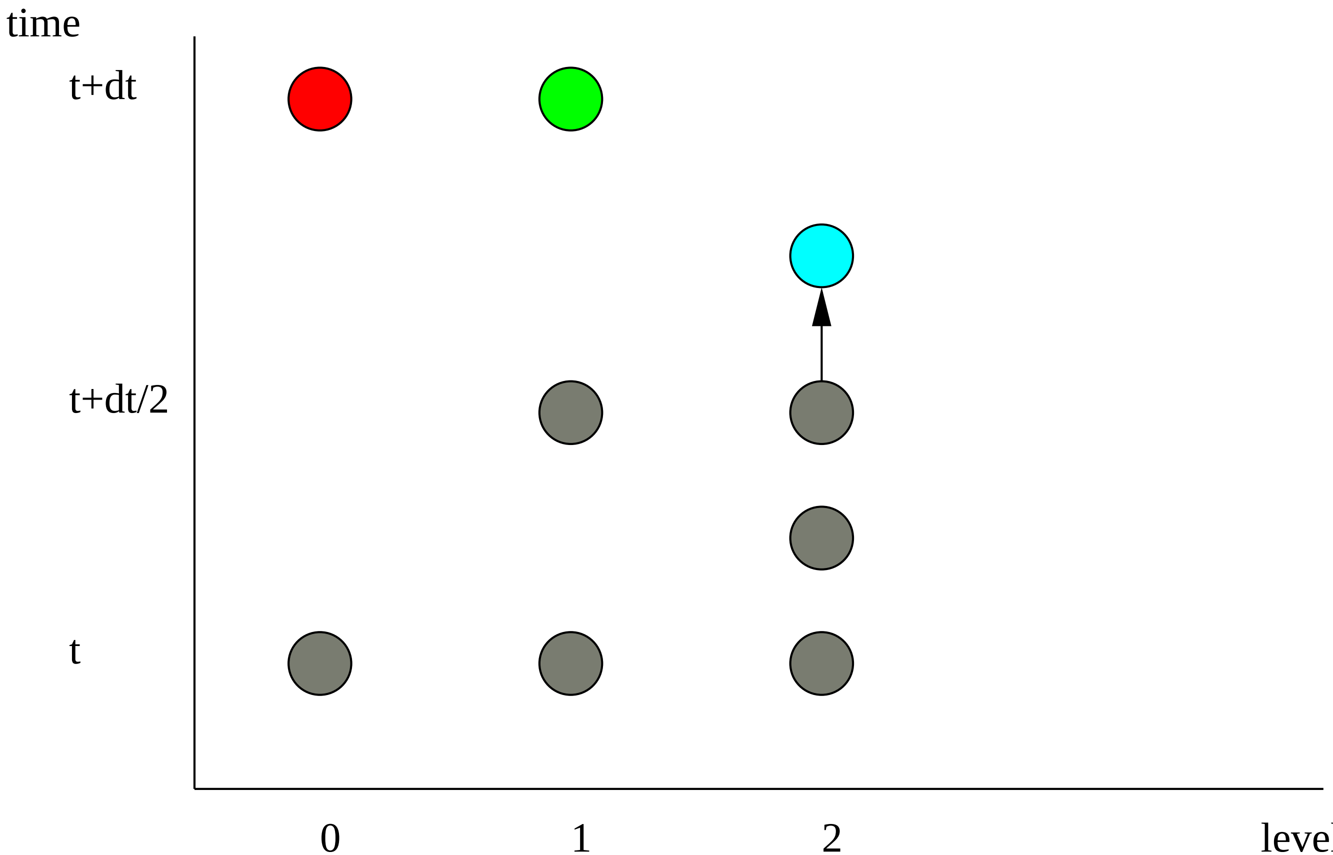

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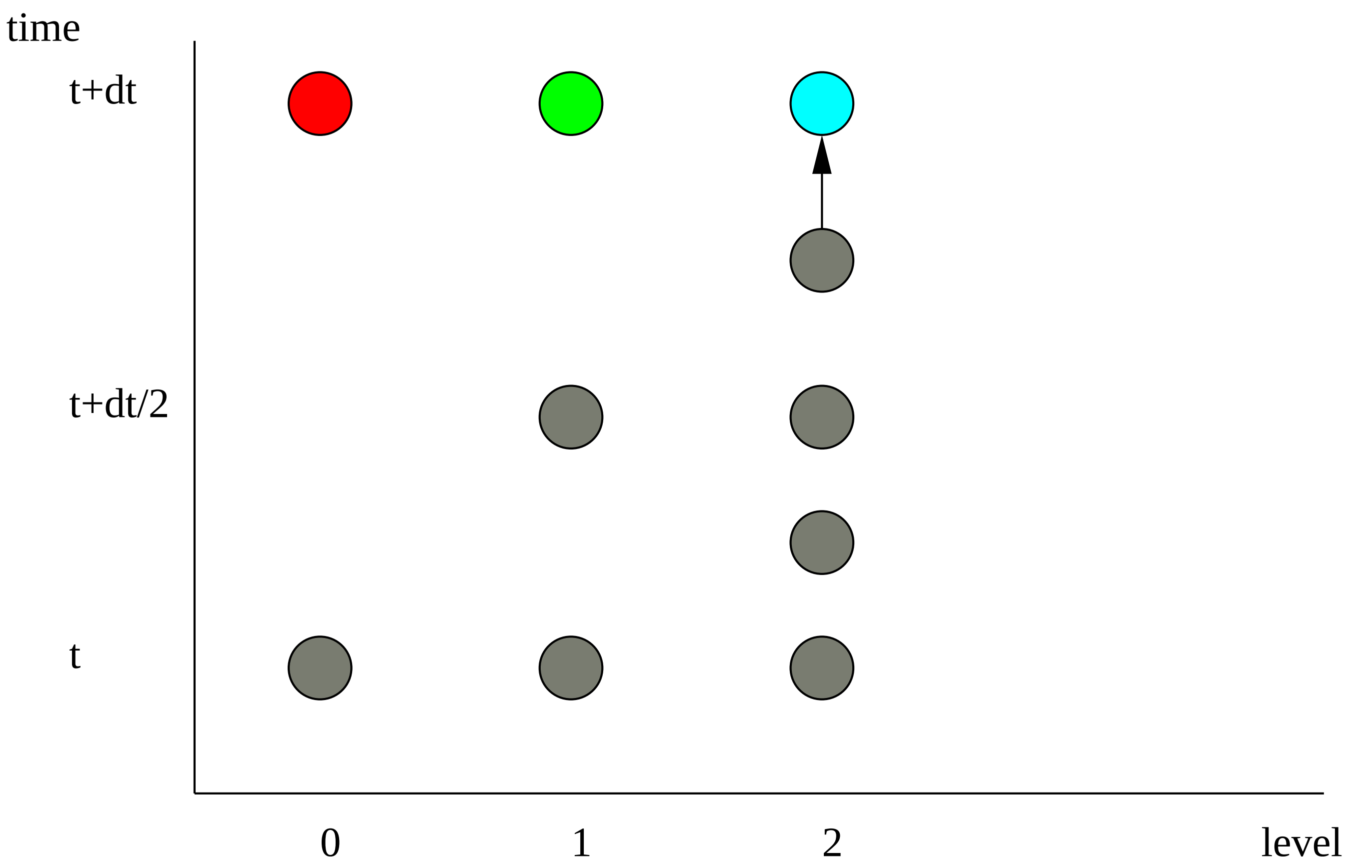

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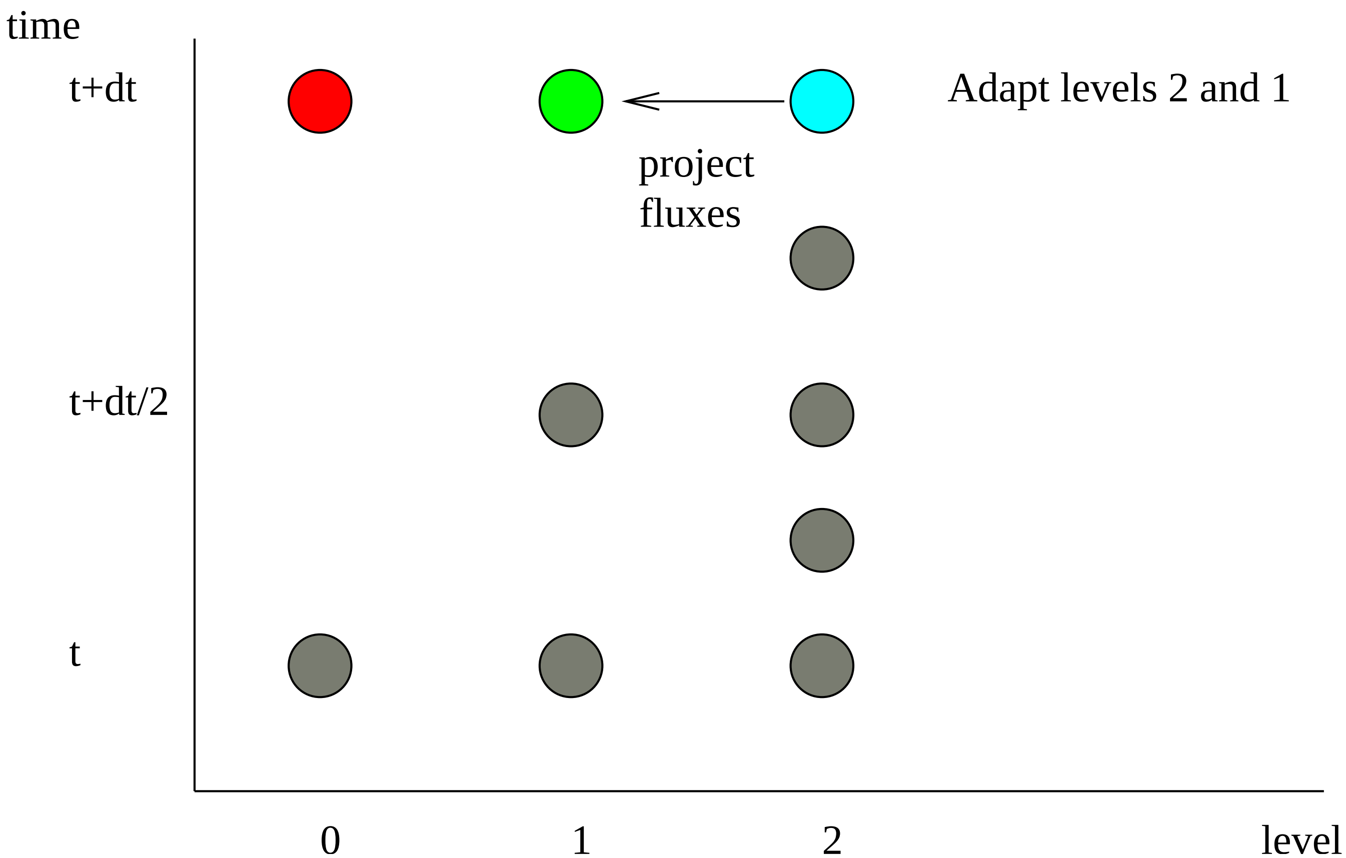

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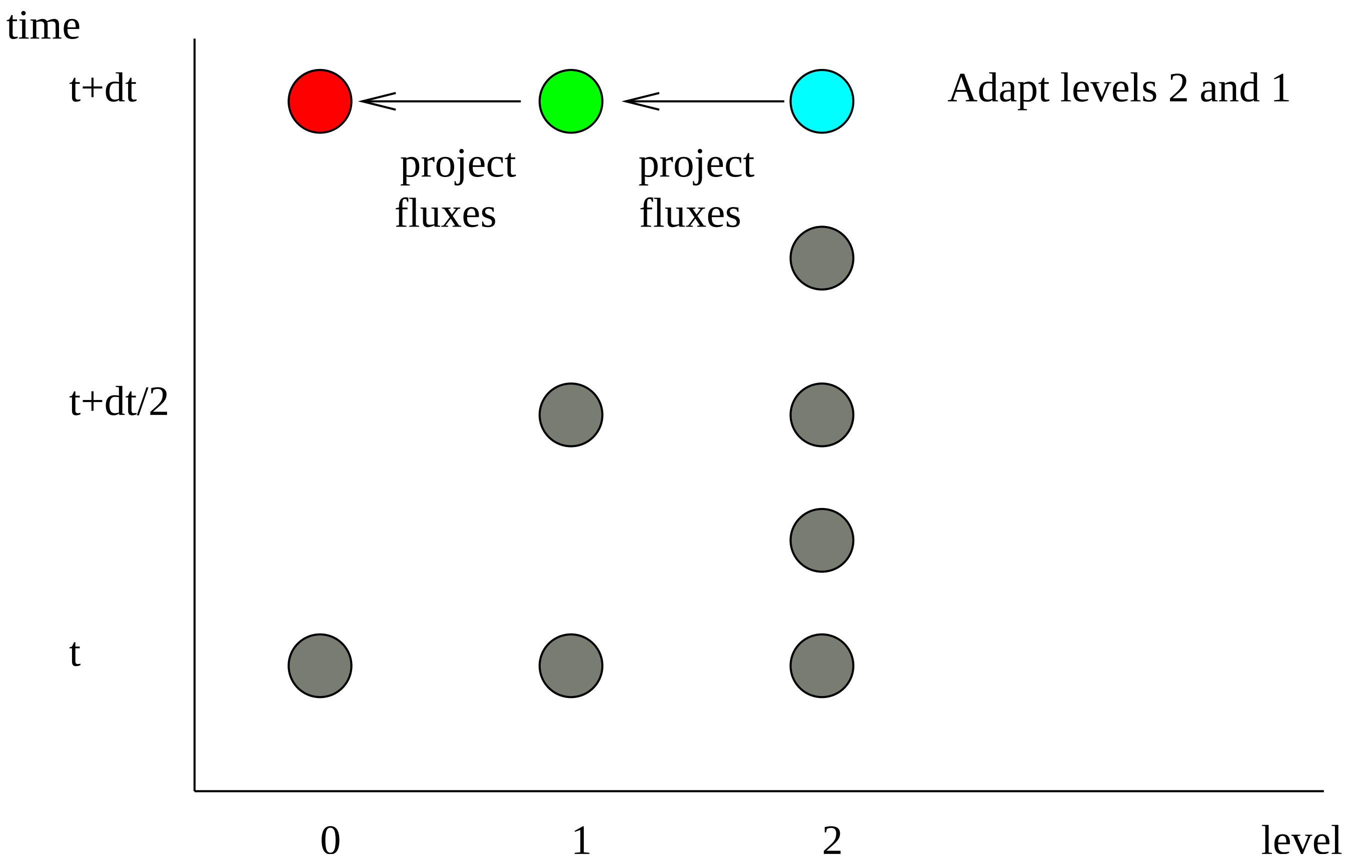

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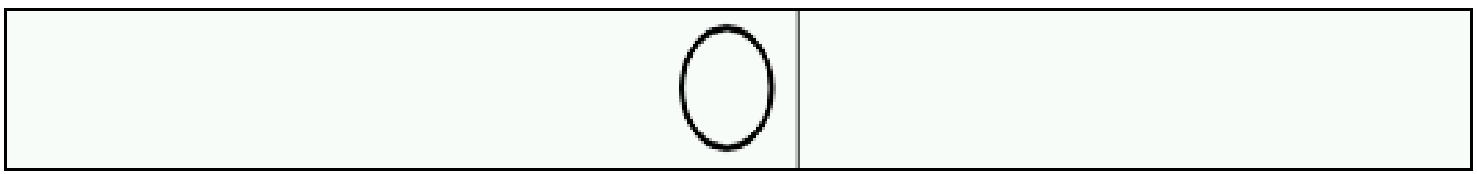
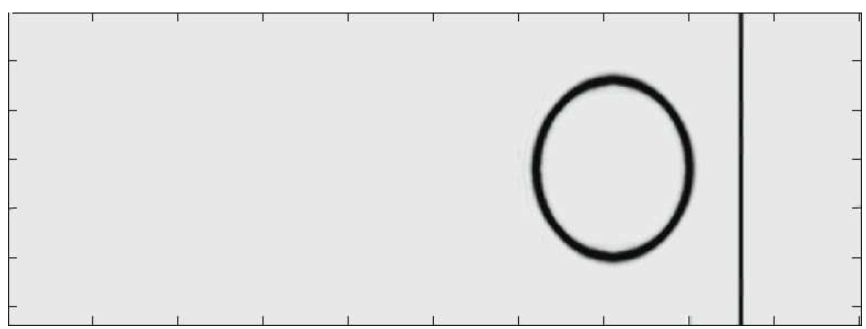
- Algorithm implementation, Parallelisation

Shock-Helium bubble

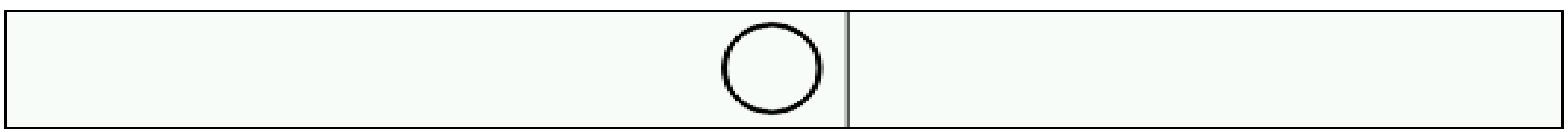
- Mach 1.22 shock interaction with Helium bubble **[Haas & Sturtevant]**, **[Karni & Quirk]**, **[Marquina & Mulet]**.
- Basic scheme: Shu-Osher+Donat-Marquina+WENO 5 reconstruction $\Rightarrow 5^{\text{th}}$ order space accuracy + 3^{rd} order time accuracy.





Parallel implementation

- Parallelization by domain decomposition: split G_0 and evenly assign each piece (along with overlying pieces of each G_l) to processors.



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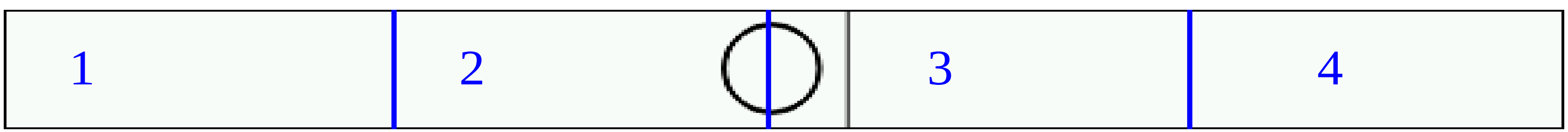

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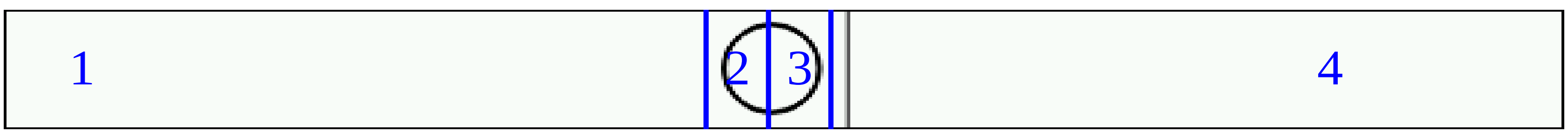

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  - but can not be static, since now processor 1 does almost all the work.

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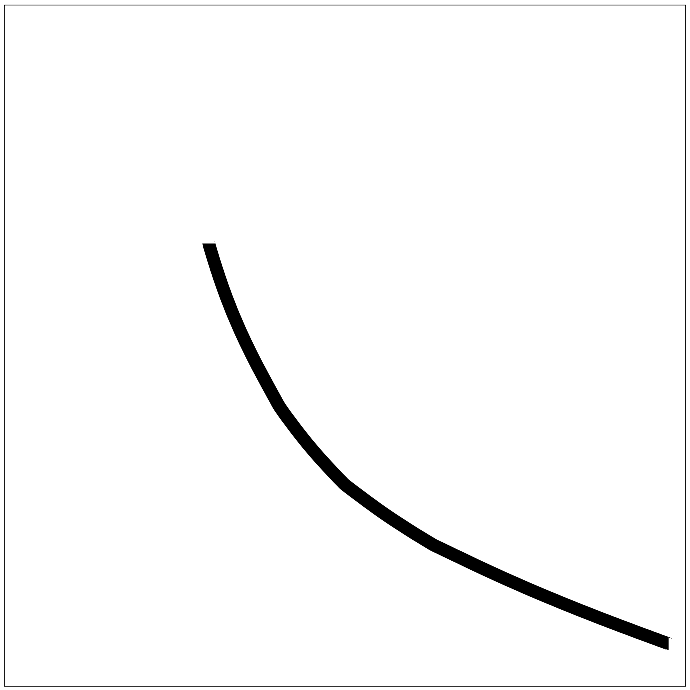

discontinuity

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- Need multidimensional partitioning.

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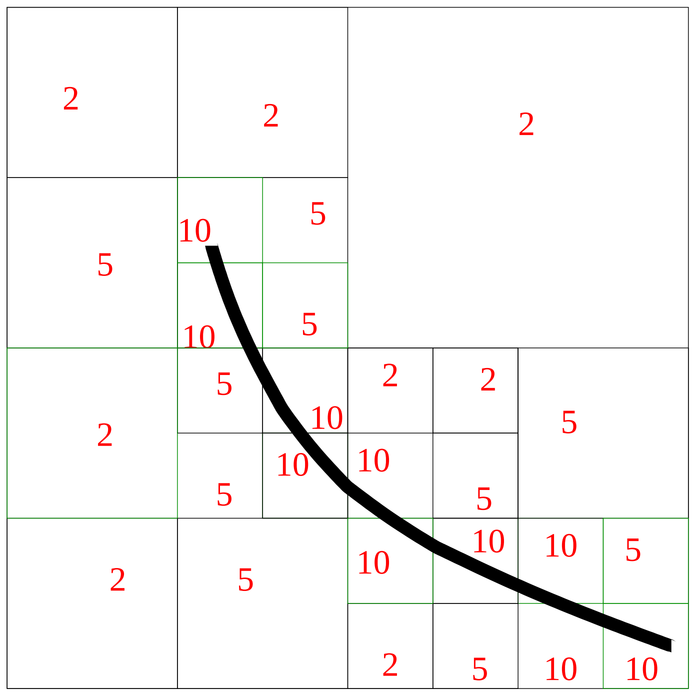

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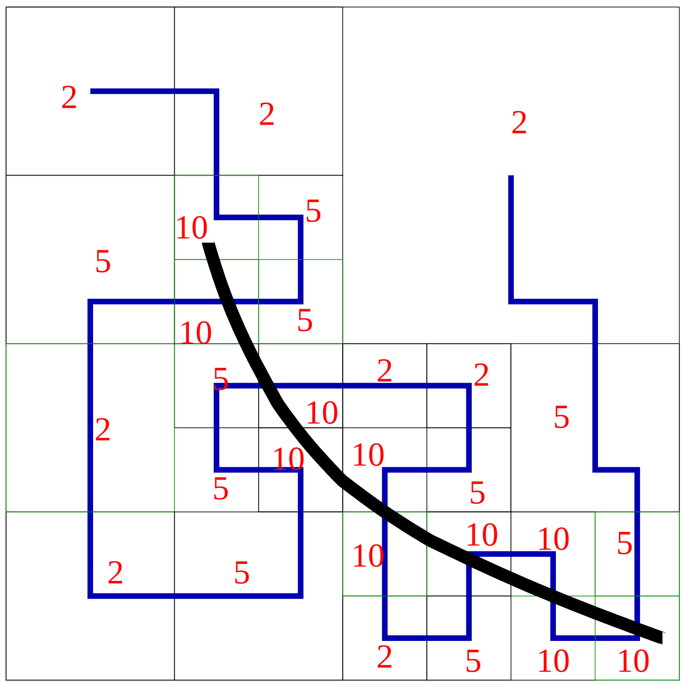

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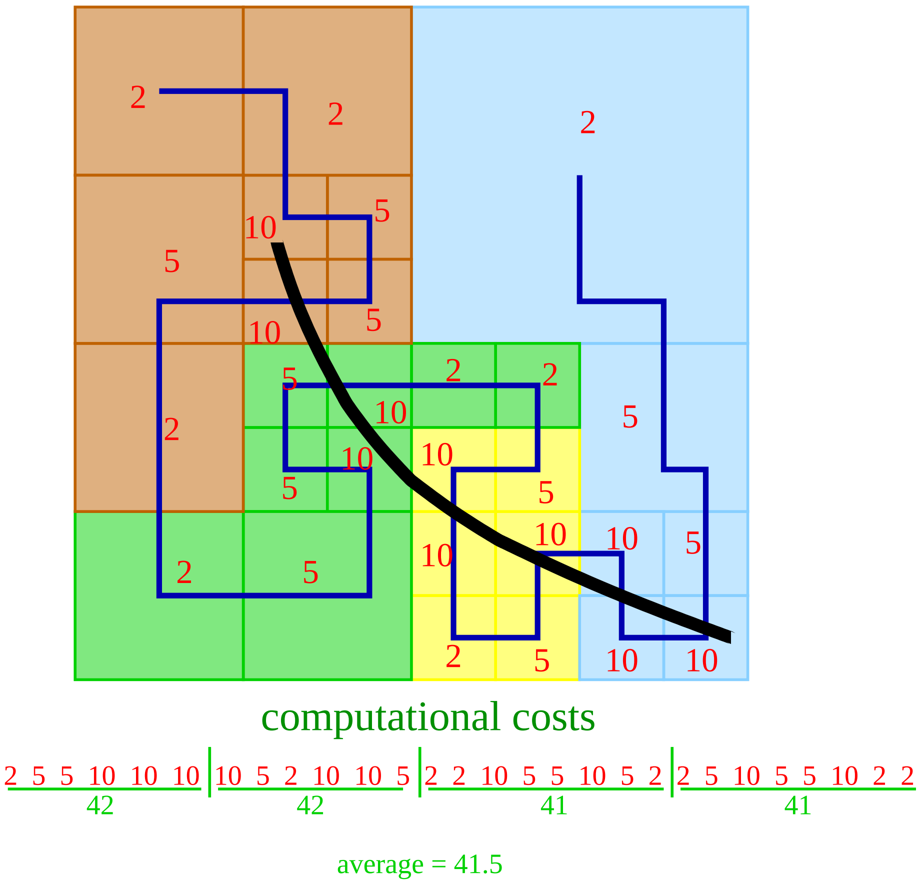

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