Uniform Controllability the Semi-discrete 1-D Wave Equation

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Exact controllability problem:

Given $T \geq 2$ and $(u^0, u^1) \in L^2(0,1) \times H^{-1}(0,1)$ there exists a control function $v \in L^2(0,T)$ such that the solution of the wave equation

$$(1) \qquad \begin{cases} u'' - u_{xx} = 0 & \text{for } x \in (0,1), \;\; t > 0 \\ u(t,0) = 0 & \text{for } t > 0 \\ u(t,1) = v(t) & \text{for } t > 0 \\ u(0,x) = u^0(x) & \text{for } x \in (0,1) \\ u'(0,x) = u^1(x) & \text{for } x \in (0,1) \end{cases}$$

satisfies

$$(2) \qquad u(T,\cdot) = u'(T,\cdot) = 0.$$

- (u, u') is the state
- v is the control
- The state is driven from (u^0, u^1) to $(0,0)$ in time T by acting on the boundary with the control v.

- Fattorini H. O. and Russell D. L.: *Exact controllability theorems for linear parabolic equations in one space dimension*, Arch. Rat. Mech. Anal., 4 (1971), 272-292.

- Russell D. L.: *A unified boundary controllability theory for hyperbolic and parabolic partial differential equations*, Studies in Appl. Math., 52 (1973), 189-211.

MOMENTS THEORY + NOHARMONIC FOURIER ANALYSIS

- Lions J.-L.: Contrôlabilité exacte perturbations et stabilisation de systèmes distribués, Tome 1, Masson, Paris, 1988.

HILBERT UNIQUENESS METHOD (HUM)

- Glowinski R., Li C. H. and Lions J.-L.: *A numerical approach to the exact boundary controllability of the wave equation (I). Dirichlet controls: Description of the numerical methods*, Jap. J. Appl. Math. 7 (1990), 1-76.

NUMERICAL METHODS FOR APPROXIMATION OF THE HUM CONTROLS

Finite differences method

$N \in \mathbb{N}^*$, $h = \frac{1}{N+1}$, $x_j = jh$, $0 \leq j \leq N+1$.

$$(3) \qquad \begin{cases} u_j''(t) = \frac{u_{j+1}(t)+u_{j-1}(t)-2u_j(t)}{h^2}, \ t > 0 \\ u_0(t) = 0, \ t > 0 \\ u_{N+1}(t) = v_h(t), \ t > 0 \\ u_j(0) = u_j^0, \ u_j'(0) = u_j^1, \ 1 \leq j \leq N. \end{cases}$$

Discrete controllability problem: given $T > 0$ and $(U_h^0, U_h^1) = (u_j^0, u_j^1)_{1 \leq j \leq N} \in \mathbb{R}^{2N}$, there exists a control function $v_h \in L^2(0,T)$ such that the solution u of (3) satisfies

$$(4) \qquad u_j(T) = u_j'(T) = 0, \ \forall j = 1, 2, ..., N.$$

System (3) consists of N linear differential equations with N unknowns $u_1, u_2, ..., u_N$.

$u_j(t) \approx u(t, x_j)$ if $(U_h^0, U_h^1) \approx (u^0, u^1)$.

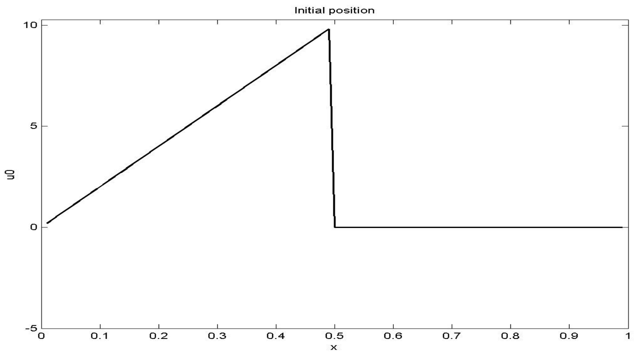
- Existence of the discrete control v_h.

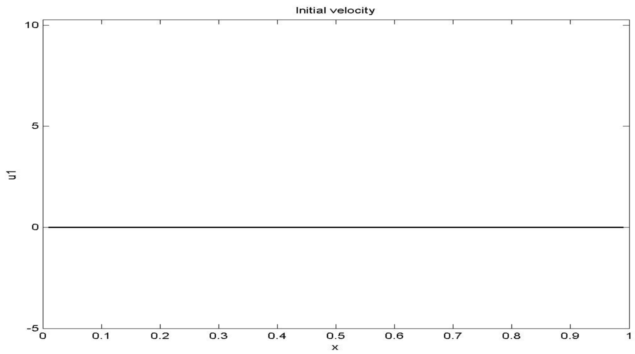
- Boundedness of the sequence $(v_h)_{h>0}$ in $L^2(0, T)$.

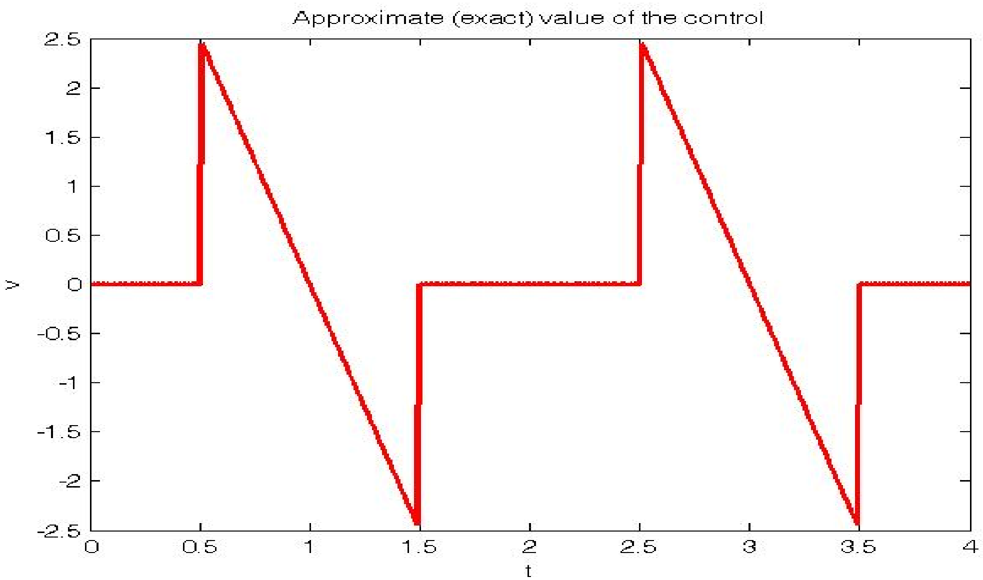
- Convergence of the sequence $(v_h)_{h>0}$ to a control v of the wave equation (1).

- The case of the HUM controls.

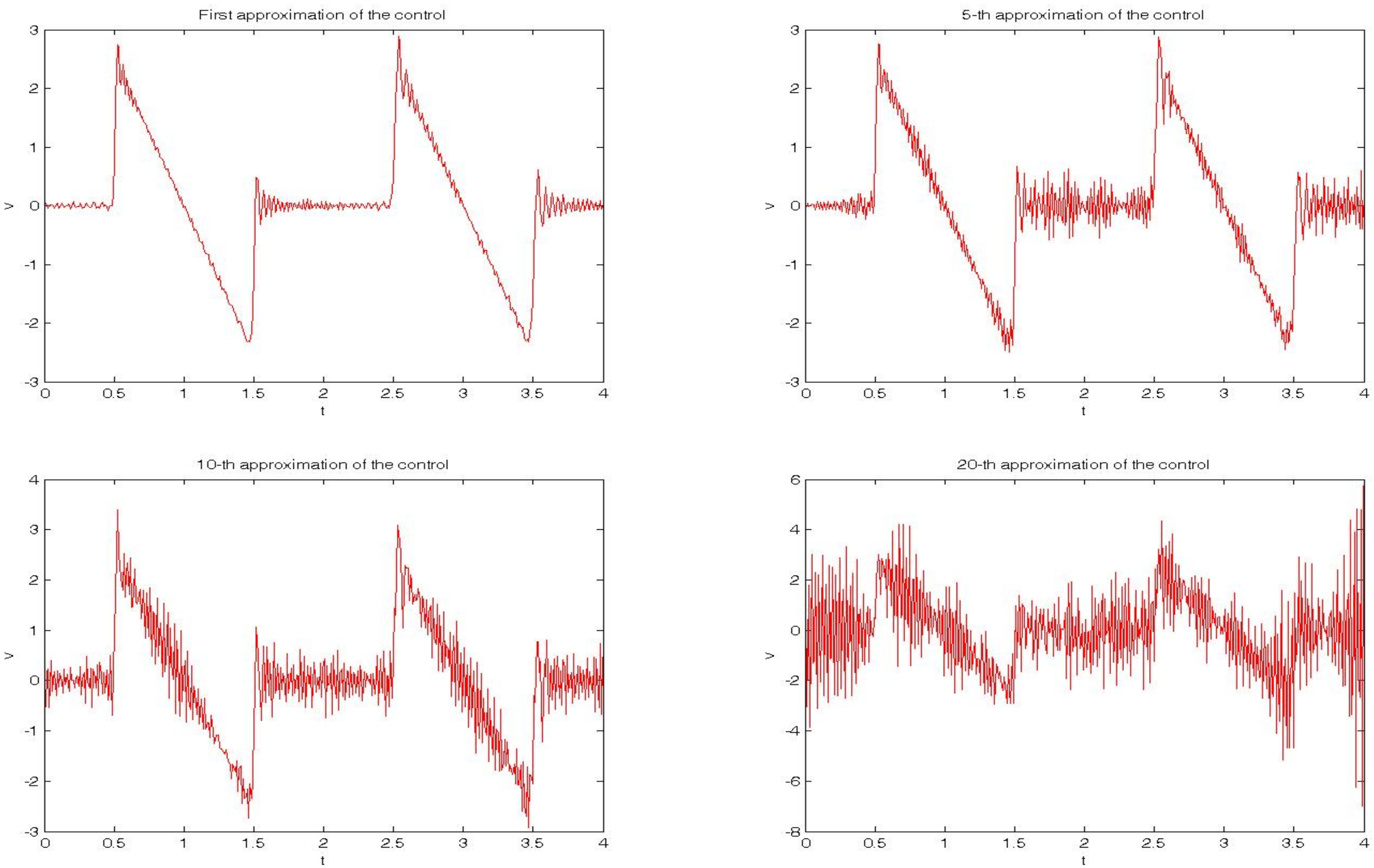
Numerical Experiments: $l = \dfrac{\Delta t}{h} = 1,\ h = 0.01$







Numerical Experiments: $l = \dfrac{\Delta t}{h} = 0.95$, $h = 0.01$



Spectral Analysis

The eigenvalues corresponding to this system are:

$$\nu_n(h) = \lambda_n(h)\, i, \quad 1 \leq |n| \leq N,$$

$$\lambda_n(h) = \frac{2}{h} \sin\left(\frac{n\pi h}{2}\right),\ 1 \leq |n| \leq N.$$

The eigenfunctions are:

$$\varphi_n(h) = \sqrt{2}(\sin(j\pi n h))_{1\leq j\leq N}.$$

- $\lambda_n(h) \approx n\pi$ for n small.

- $\lambda_{n+1}(h) - \lambda_n(h) = \frac{4}{h} \sin\left(\frac{\pi h}{4}\right) \cos\left(\frac{(2n+1)\pi h}{4}\right) \approx$

  $\approx \pi \cos\left(\frac{(2n+1)\pi h}{4}\right) \sim \pi h$ for $n \sim N$.

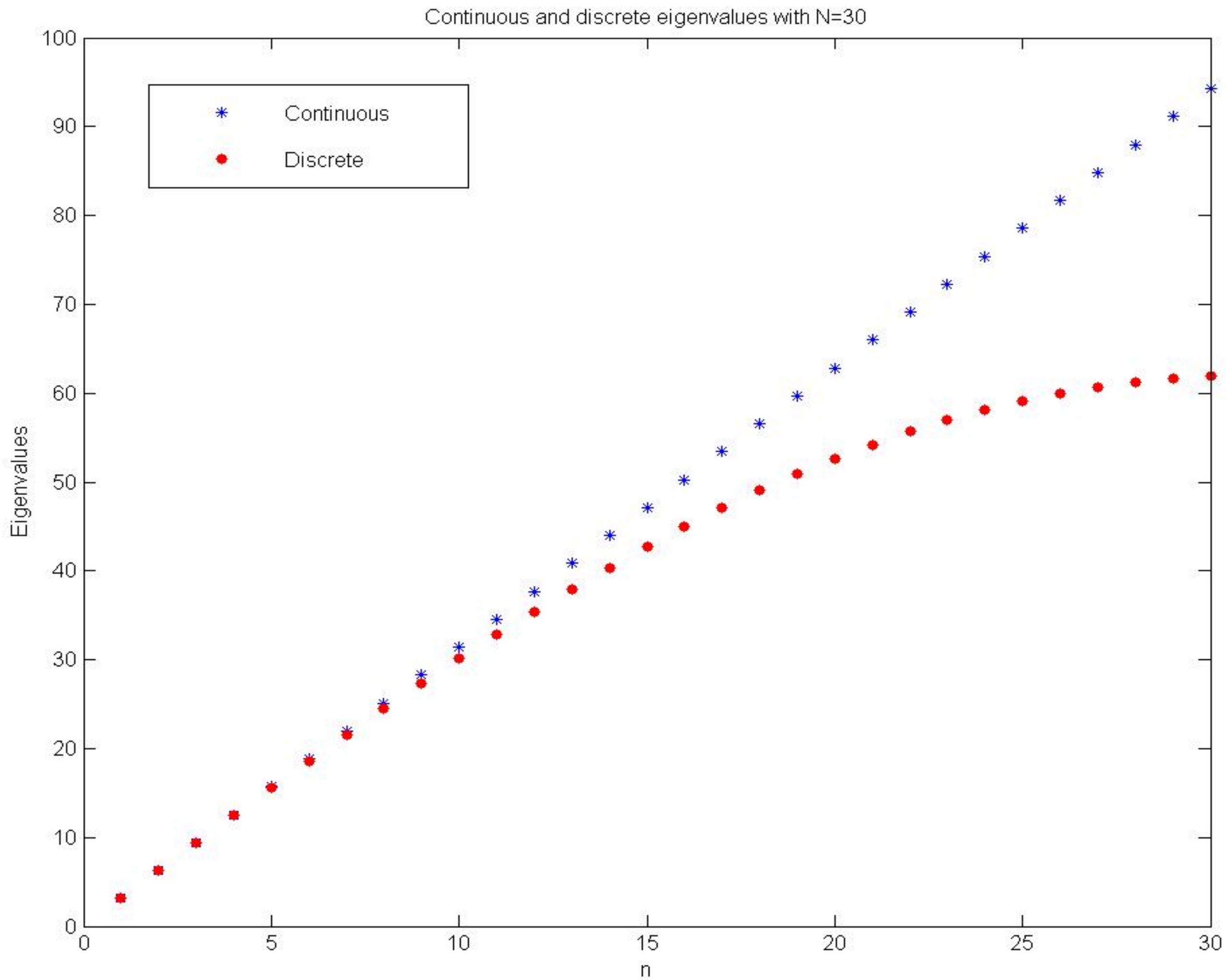


Fig 1. *Eigenvalues of the continuous and finite differences discrete equations.*

Problem of moments

Property. *System (3) is controllable if and only if for any initial data* $(U_h^0, U_h^1) = \sum_{n=1}^{N}(a_n^0, a_n^1)\varphi_n(h)$ *there exists* $v_h \in L^2(0,T)$ *such that*

$$(5) \qquad \int_0^T v_h(t)e^{-i\lambda_n(h)t}dt = \frac{(-1)^n h}{\sqrt{2}\sin(|n|\pi h)}\left(i\lambda_n(h)a^0_{|n|} + a^1_{|n|}\right), \; 1 \le |n| \le N.$$

(PROBLEM OF MOMENTS)

- $(U_h^0, U_h^1) = (\varphi_m(h), 0) \Rightarrow \int_0^T v_h^{0,m}(t)e^{-i\lambda_n(h)t}dt = \frac{(-1)^m h i\lambda_{|m|}(h)}{\sqrt{2}\sin(|m|\pi h)}\delta_{mn}, \; 1 \le |n| \le N.$
- $(U_h^0, U_h^1) = (0, \varphi_m(h)) \Rightarrow \int_0^T v_h^{1,m}(t)e^{-i\lambda_n(h)t}dt = \frac{(-1)^m h}{\sqrt{2}\sin(|m|\pi h)}\delta_{mn}, \; 1 \le |n| \le N.$
- $(U_h^0, U_h^1) = \sum_{n=1}^{N}(a_n^0, a_n^1)\varphi_n(h) \Rightarrow v_h = \sum_{1\le|m|\le N}\left(a_m^0 v_h^{0,m} + a_m^1 v_h^{1,m}\right).$

Definition. *$(\Theta_m)_{1\le|m|\le N}$ is a biorthogonal sequence to the family of complex exponentials $\left(e^{-i\lambda_j(h)t}\right)_{1\le|j|\le N}$ in $L^2\left(-\frac{T}{2},\frac{T}{2}\right)$ if*

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} \Theta_m(t)e^{-i\lambda_n(h)t}dt = \delta_{mn}, \quad 1 \le |n| \le N. \tag{6}$$

A control of the initial data $(U_h^0, U_h^1) = \sum_{n=1}^{N}(a_n^0, a_n^1)\varphi_n(h)$ is given by

$$v_h = \sum_{1\le|m|\le N} \frac{(-1)^m h}{\sqrt{2}\sin(|m|\pi h)} e^{i\lambda_m(h)\frac{T}{2}}\Theta_m\left(t-\frac{T}{2}\right)\left(i\lambda_m(h)a_{|m|}^0 + a_{|m|}^1\right).$$

Theorem. *(S. M., Numer. Math. 2002) If* $T > 0$ *is independent of* h *and* $(\psi_m)_{\substack{|m|\leq N\\ m\neq 0}}$ *is any biorthogonal to* $\left(e^{i\lambda_n t}\right)_{\substack{|n|\leq N\\ n\neq 0}}$ *in* $L^2\left(-\frac{T}{2},\frac{T}{2}\right)$ *there exists a positive constants* C *independent of* N, *such that*

$$\| \psi_N \|_{L^2} \geq Ce^{\sqrt{N}}. \tag{7}$$

- There are regular initial data (exponentially small coefficients $(a_n)_n$) that are not uniformly controllable.

- The problems come from trying to control the high, spurious, numerical frequencies.

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- Negreanu M. and Zuazua E.: *Uniform boundary controllability of a discrete 1-D wave equation*, System and Control Letters, 48 (2003), pp. 261-280.

- Castro C. and M. S.: *Boundary controllability of a linear semi-discrete 1-D wave equation derived from a mixed finite element method*, Numer. Math., 102 (2006), pp. 413-462.

- Münch A.: *A uniformly controllable and implicit scheme for the 1-D wave equation*, M2NA, 39 (2005), pp. 377-418.

Finite differences method with numerical viscosity

$N \in \mathbb{N}^*$, $h = \frac{1}{N+1}$, $x_j = jh$, $0 \leq j \leq N+1$.

$$(8)\quad \begin{cases} u_j''(t) = \frac{u_{j+1}(t)+u_{j-1}(t)-2u_j(t)}{h^2} + \varepsilon\frac{u_{j+1}'(t)+u_{j-1}'(t)-2u_j'(t)}{h^2},\ t>0 \\ u_0(t) = 0,\ t>0 \\ u_{N+1}(t) = v_h(t),\ t>0 \\ u_j(0) = u_j^0,\ u_j'(0) = u_j^1,\ 1 \leq j \leq N. \end{cases}$$

Discrete controllability problem: given $T > 0$ and $(U_h^0, U_h^1) = (u_j^0, u_j^1)_{1\leq j\leq N} \in \mathbb{R}^{2N}$, there exists a control function $v_h \in L^2(0,T)$ such that the solution u of (3) satisfies

$$(9)\qquad u_j(T) = u_j'(T) = 0,\ \forall j = 1, 2, ..., N.$$

The term $\varepsilon\dfrac{u_{j+1}'(t)+u_{j-1}'(t)-2u_j'(t)}{h^2}$ is a numerical viscosity which vanishes in the limit:

$$\lim_{h\to 0} \varepsilon = 0.$$

- Tcheugoué Tébou L. R. and Zuazua E.: *Uniform exponential long time decay for the space semi-discretization of a locally damped wave equation via an artificial numerical viscosity*, Numer. Math., 95 (2003), pp. 563-598.

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- DiPerna R. J.: *Convergence of approximate solutions to conservation laws*, Arch. Rational Mech. Anal., 82 (1983), pp. 27-70.

- Majda A. and Osher S.: *Numerical viscosity and the entropy condition*, Comm. Pure Appl. Math., 32 (1979), pp. 797-838.

Spectral Analysis We chose $\varepsilon = h$, but other choices are possible. The eigenvalues corresponding to this system are:

$$\mu_n(h) = i\frac{2}{h}\sin\left(\frac{n\pi h}{2}\right)\left(\cos\left(\frac{n\pi h}{2}\right) + i\,\sin\left(\frac{n\pi h}{2}\right)\right), \quad 1 \leq |n| \leq N.$$

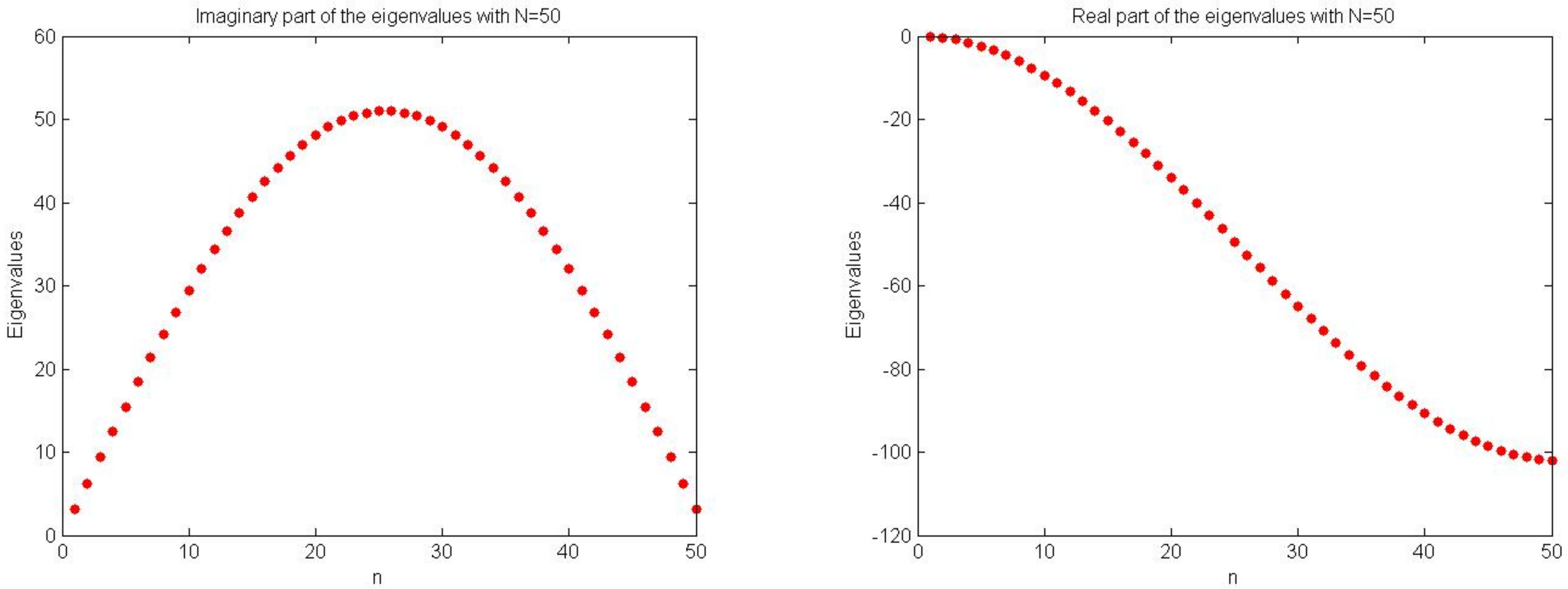


Fig 2. *Imaginary and real part of the eigenvalues of the finite differences discrete equation with viscosity.*

Problem of moments

Property. *System (8) is controllable if and only if for any initial data* $(U_h^0, U_h^1) = \sum_{n=1}^{N} (a_n^0, a_n^1)\varphi_n(h)$ *there exists* $v_h \in L^2(0,T)$ *such that*

$$(10) \qquad \int_0^T v_h(t) e^{-\mu_n(h)t} dt = \frac{(-1)^n h}{\sqrt{2}\sin(|n|\pi h)} \left(\frac{(\lambda_n(h))^2}{\overline{\mu}_n(h)} a_{|n|}^0 + a_{|n|}^1 \right), \ 1 \le |n| \le N.$$

(PROBLEM OF MOMENTS)

If $(\Theta_m)_{1\le|m|\le N}$ is a biorthogonal sequence to the family of complex exponentials $\left(e^{-\mu_j(h)t}\right)_{1\le|j|\le N}$ in $L^2\left(-\frac{T}{2}, \frac{T}{2}\right)$, then a control of the initial data $(U_h^0, U_h^1) = \sum_{n=1}^{N} (a_n^0, a_n^1)\varphi_n(h)$ is given by

$$v_h = \sum_{1\le|m|\le N} \frac{(-1)^m h}{\sqrt{2}\sin(|m|\pi h)} e^{\mu_m(h)\frac{T}{2}} \Theta_m\left(t - \frac{T}{2}\right) \left(\frac{(\lambda_m(h))^2}{\overline{\mu}_m(h)} a_{|m|}^0 + a_{|m|}^1 \right).$$

Theorem. *For any $T > 0$ sufficiently large but independent of h, there exists a sequence $(\Theta_m)_{\substack{|m|\leq N \\ m\neq 0}}$, biorthogonal in $L^2\left(-\frac{T}{2}, \frac{T}{2}\right)$ to the family $\left(e^{-\mu_j(h)t}\right)_{\substack{|j|\leq N \\ j\neq 0}}$, such that*

$$(11) \qquad ||\Theta_m||_{L^2\left(-\frac{T}{2},\frac{T}{2}\right)} \leq C \cos\left(\frac{m\pi h}{2}\right) e^{\omega\,|\Re(\mu_m)|}, \quad 1 \leq |m| \leq N$$

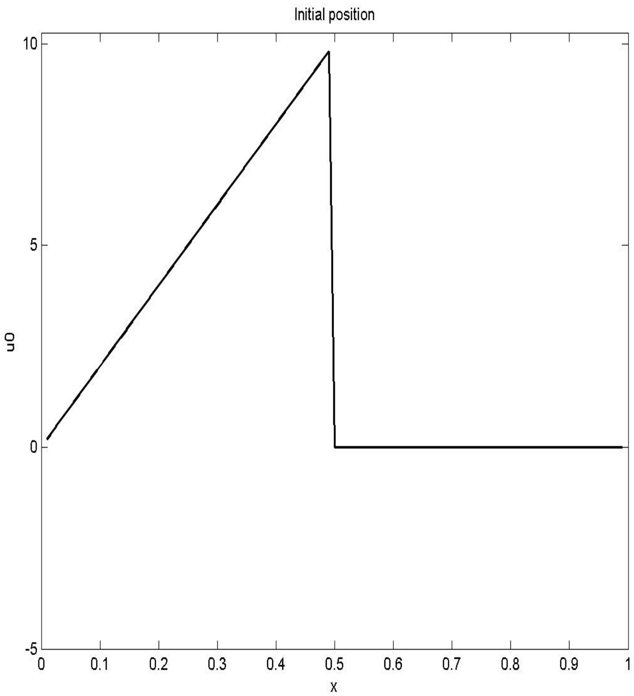
where C and ω are positive constants, independent of m and N.

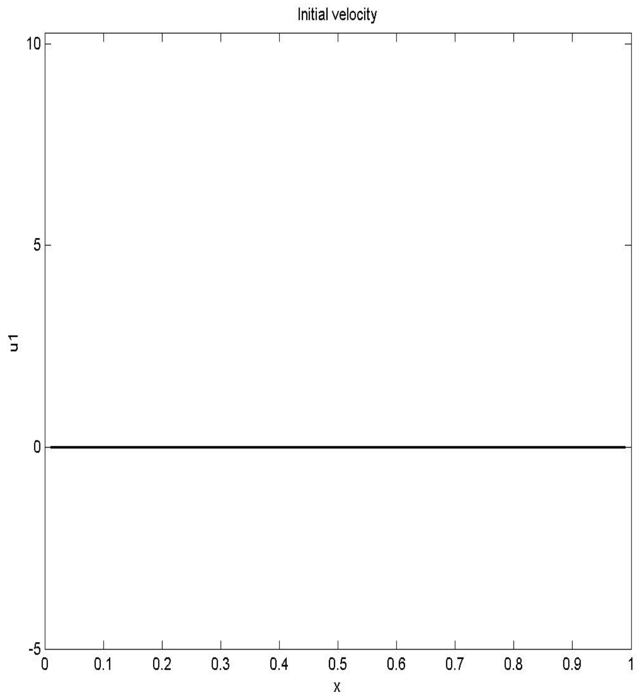
- Any initial data of (1) such that

$$(12) \qquad \sum_{n\geq 1} \left(|a_n^0| + \frac{1}{n\pi}|a_n^1|\right) < \infty$$

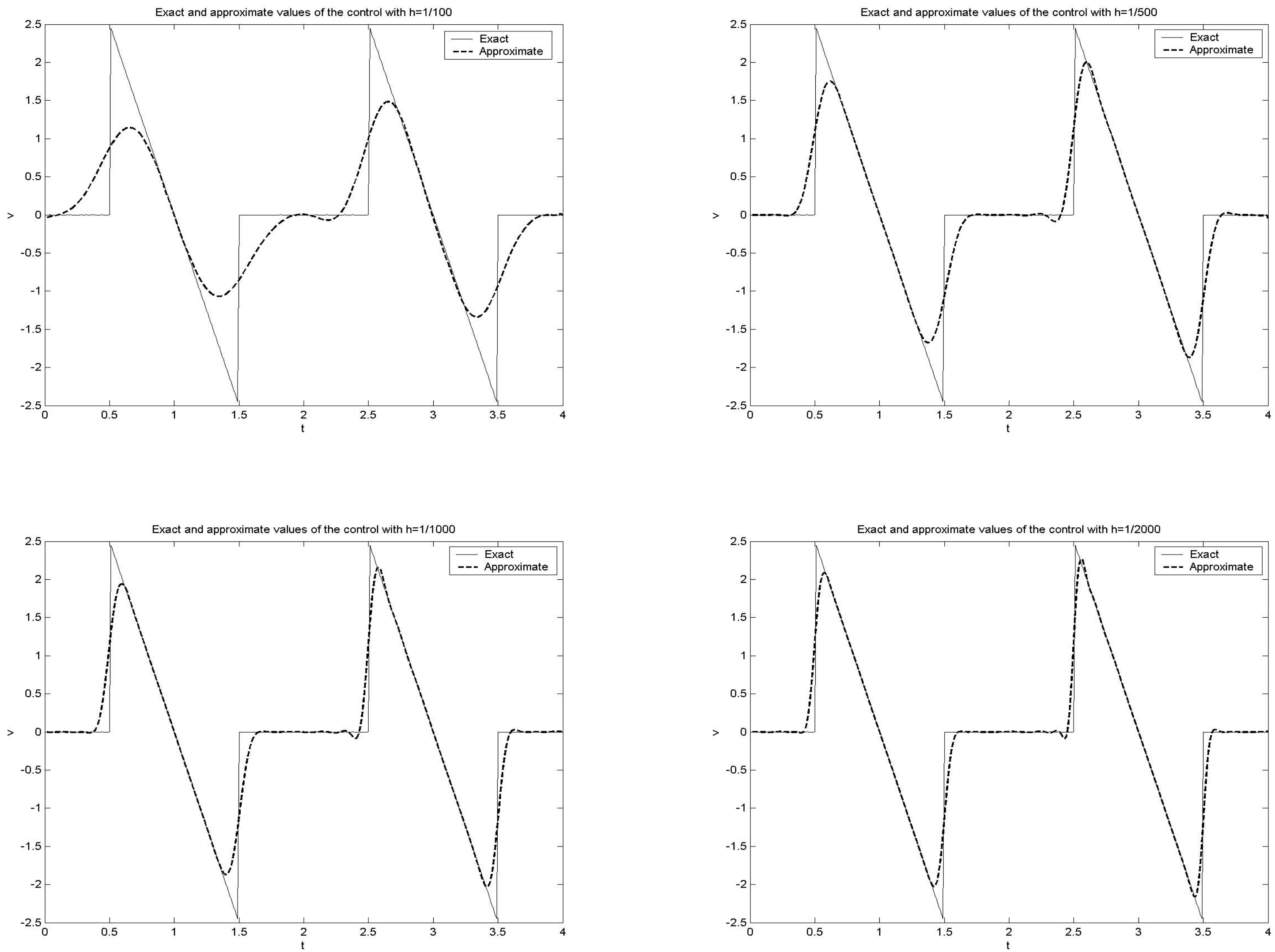
are uniformly controllable.

Numerical Experiments: Initial data (u^0, u^1) to be controlled.





Numerical Experiments: Approximations of the control with four different values of h and $\dfrac{\Delta t}{h} = 7/8$



Open problem: Improve the rate of convergence

Numerical results obtained with $\Delta t = 7/8h$ and $\epsilon = h$.

h	1/100	1/500	1/1000	1/2000
$\|v_h^1\|_{L^2}$	1.4739	1.8103	1.8845	1.9354
$\|v_h^1 - v\|_{L^2}/\|v\|_{L^2}$	0.4882	0.3209	0.2699	0.2264

Numerical results obtained with $\Delta t = 7/8h$ and $\epsilon = h^{1.5}$.

h	1/100	1/500	1/1000
$\|v_h^{1.5}\|_{L^2}$	1.8496	1.9877	2.0101
$\|v_h^{1.5} - v\|_{L^2}/\|v\|_{L^2}$	0.0801	0.0114	0.0005

$$\|v\|_{L^2} = 2.0106$$

Open problem: Changing the viscosity

$$u_j''(t) = (\Delta_h u)_j + \varepsilon(\Delta_h u)_j \tag{13}$$

$$u_j''(t) = (\Delta_h u)_j - \varepsilon(\Delta_h^2 u)_j. \tag{14}$$

Open problem: evaluation of the error

$$||v_h - v|| \leq Ch^r$$

Benasque, 2009 (???)