On Adaptive High Resolution Shock Capturing techniques for Multi-Class Traffic Flow problems

PDEs, Optimal Design and Numerics

Benasque Center for Sciences, 2007

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Outline

- Multi-Class **Lighthill-Whitham-Richards** traffic models
- HRSC numerical schemes for LWR Multi-Class Models: Characteristic-based schemes versus component-wise schemes.
  - Adaptive Mesh Refinement for Finite-Difference High Resolution Shock Capturing Schemes

Lighthill-Whitham-Richards (LWR) traffic model

Scalar hyperbolic conservation law for vehicle density $\rho(x,t)$:

- The total number of vehicles is conserved
- The *flow speed* v (average of speed of cars) is a function of $\rho(x,t)$.

$$\partial_t \rho + \partial_x(\rho v(\rho)) = 0,$$

where $v'(\rho) < 0$ and $(\rho v(\rho))'' < 0$ (concave flux).

Multi-Class LWR models

- Generalizations to multiple classes of drivers: e.g.
  - slow and fast cars (**Zhang & Jin 02**)
  - more general *Multiple classes* of drivers, depending on maximal speed attained under free flow **[Wong& Wong 02, Benzoni-Cavage & Colombo 03]**

  Class i, $1 \leq i \leq m$ with individual density $\rho_i(x,t)$ evolves by LWR equation

$$\partial_t \rho_i + \partial_x (\underbrace{\rho_i v_i(\rho_1, \dots \rho_m)}_{Q_i(\rho_1,\dots,\rho_m)}) = 0,$$

$$U_t + Q(U)_x = 0 \qquad U_i = \rho_i, \quad Q_i = \rho_i v_i.$$

- Hyperbolicity of system by studying the Jacobian matrix DQ:

$$DQ_{ij} = \frac{\partial Q_i}{\partial \rho_j} = \delta_{i,j} v_i + \rho_i \frac{\partial v_i}{\partial \rho_j}$$

MCLWR models

Working Assumption: Drivers belonging to different classes adjust their speed to the local traffic density **in the same way**.

- $v_i(\rho_1, \dots, \rho_m) = v_i(\rho)$, $\rho = \sum \rho_i$ the **total car density**
- **[Wong& Wong 02, Benzoni-Cavage & Colombo 03]** Assume $v_i(\rho) := \beta_i V(\rho)$, $\quad V(\rho)$ as in LWR model and $\beta_1 < \cdots < \beta_m$.
- $\dfrac{\partial v_i}{\partial \rho_j} = v_i'(\rho) = \beta_i V'(\rho)$, then :

$$DQ = \frac{\partial Q}{\partial U} = \text{diag}(v_i) + \begin{bmatrix} \rho_1 v_1' \\ \vdots \\ \rho_m v_m' \end{bmatrix} \begin{bmatrix} 1 & \dots & 1 \end{bmatrix}$$

- **Only numerical evidence** of hyperbolic character of MCLWR system until **Zhang et al 2006**!. By working out $det(DQ - \lambda I) = 0$ shows that, under appropriate conditions, eigenvalues of DQ are roots of

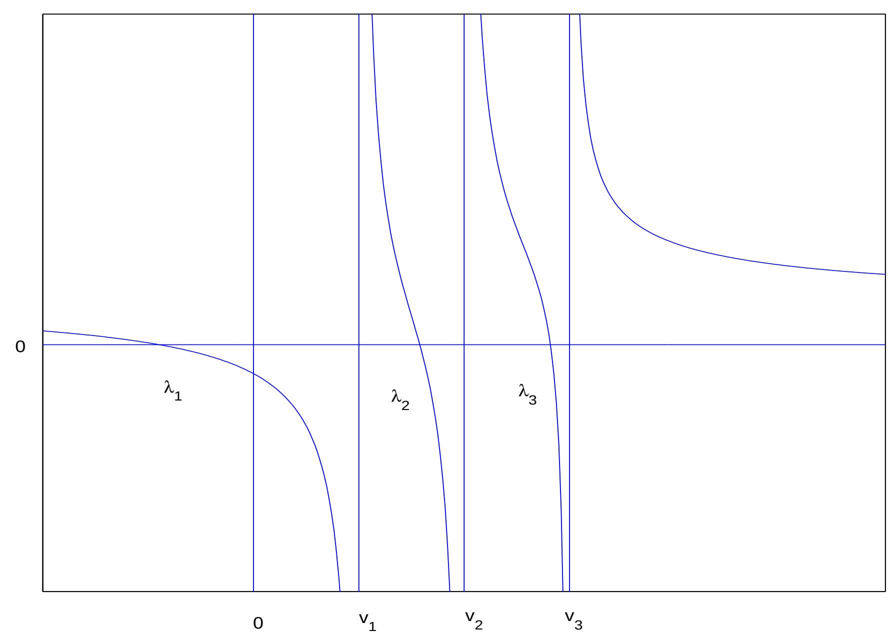
$$0 = 1 + \sum_{i=1}^{m} \frac{\rho_i v_i'(\rho)}{v_i(\rho) - \lambda} = R(\lambda)$$

Hyperbolicity of Multi-class LWR models

- Since $v_i' = \beta_i V'(\rho) < 0$, roots of $R(\lambda) = 1 + \sum_{i=1}^{m} \frac{\rho_i v_i'(\rho)}{v_i(\rho) - \lambda}$ verify:

$$v_1(\rho) + \sum_{i=1}^{m} \rho_i v_i'(\rho) < \lambda_1 < v_1(\rho) < \lambda_2 < \cdots < v_{m-1}(\rho) < \lambda_m < v_m(\rho)$$

- Strictly hyperbolic system, with only one possibly negative eigenvalue λ_1.



Eigen-structure of Multi-class LWR models

- Exploit structure of $DQ = D + ae^T$ to obtain its eigen-decomposition!
- **Theorem** Let $\rho_i \neq 0,\ \forall i$ and $0 < \beta_1 < \beta_2 < \ldots < \beta_n$.
  Then the eigenvalues of DQ are the real roots of the function

$$0 = 1 + \sum_{i=1}^{m} \frac{\rho_i v_i'(\rho)}{v_i(\rho) - \lambda} = R(\lambda)$$

- Right and Left (non-normalized) eigenvectors by

$$r_i(\lambda) = \frac{\rho_i v_i'(\rho)}{v_i(\rho) - \lambda}, \quad l_i(\lambda) = \frac{1}{v_i(\rho) - \lambda}.$$

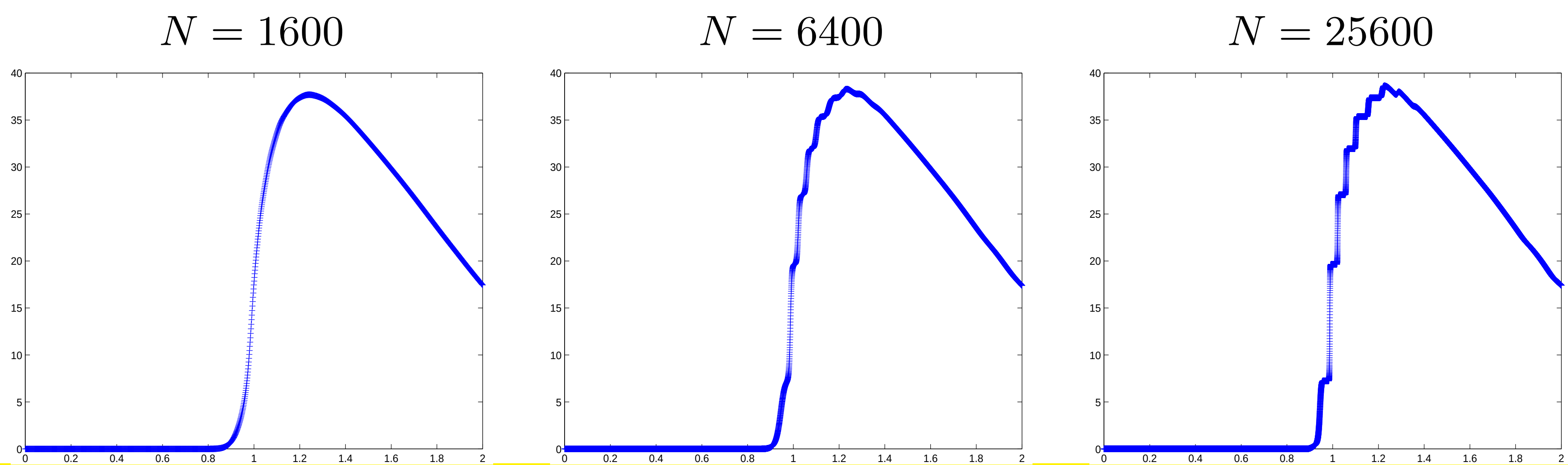
Numerical schemes for Multi-class LWR models

Conservative Numerical Schemes: $$\frac{dU_i}{dt} + \frac{1}{\Delta x}\left(\bar{Q}_{i+1/2} - \bar{Q}_{i-1/2}\right) = 0$$

First Numerical Results **[Wong, Wong]**: Global Lax-Friedrichs (GLF)

$$\bar{Q}_{j,i+\frac{1}{2}} = \frac{1}{2}(Q_j + \alpha\rho_j) + \frac{1}{2}(Q_{j+1} - \alpha\rho_{j+1}), \qquad \alpha = \max_{\rho}\{|v_1(\rho)|, \ldots, |v_m(\rho)|\}$$

Qualitative behaviour OK BUT Very Poor Resolution (First order scheme)



HRSC schemes for Multi-class LWR models

Shu-Osher HRSC framework:
Characteristic-based Local Lax FriedrichsLLF Numerical Flux function:

$$\bar{Q}_{i+1/2} = \sum_{k=1}^{m} r^k \left(\mathcal{R}^+(l^k \cdot \frac{Q+\alpha_k U}{2}; x_{i+1/2}) + \mathcal{R}^-(l^k \cdot \frac{Q-\alpha_k U}{2}; x_{i+1/2}) \right)$$

α_k estimate of the local k-th speed at the $i+1/2$ interface.

- BUT Eigen-structure of $\dfrac{\partial Q}{\partial U}$ WAS NOT explicitly known,

[Wong et al] propose to use a component-wise WENO5GLF scheme:

$$\bar{Q}_{j,i+\frac{1}{2}} = \mathcal{R}^+(\frac{1}{2}(Q_j + \alpha\rho_j); x_{i+\frac{1}{2}}) + \mathcal{R}^-(\frac{1}{2}(Q_j - \alpha\rho_j); x_{i+\frac{1}{2}}),$$

$\mathcal{R}^{\pm} \equiv \pm$-biased WENO5-rec.

Adaptive WENO schemes

[Burger, Kozakevicius, JCP]

- Same GLF-WENO5 Component-wise flux splitting

$$\bar{Q}_{j,i+\frac{1}{2}} = \mathcal{R}^{+}(\frac{1}{2}(Q_j + \alpha\rho_j); x_{i+\frac{1}{2}}) + \mathcal{R}^{-}(\frac{1}{2}(Q_j - \alpha\rho_j); x_{i+\frac{1}{2}}),$$

$\alpha = \max_{\rho}(|v_1(\rho) + \sum_{i=1}^{m} \rho_i v_i'(\rho)|, v_m(\rho));$

- Multiresolution-based Sparse-Point-Representation (MR-SPR) of numerical solution.
- Adaptive techniques optimize computational resources, while maintaining the HRSC properties of the basic underlying scheme.

Characteristic-Based Shu-Osher WENO

BUT Eigen-structure of $\dfrac{\partial Q}{\partial U}$ can be obtained by

- numerically solving $$R(\lambda) = 1 + \sum_{k=1}^{m} \frac{\rho_k v_k'(\rho)}{v_k(\rho) - \lambda} = 0.$$
- Given λ_k, Compute r^k, l^k (and normalize)

$$r_l^k = \frac{\rho_l v_l'(\rho)}{v_l(\rho) - \lambda_k}, \quad l_l^k = \frac{1}{v_l(\rho) - \lambda_k}.$$

Characteristic-based Local Lax FriedrichsLLF Numerical Flux function:

$$\bar{Q}_{i+1/2} = \sum_{k=1}^{m} r^k \left(\mathcal{R}^+(l^k \cdot \frac{Q + \alpha_k U}{2}; x_{i+1/2}) + \mathcal{R}^-(l^k \cdot \frac{Q - \alpha_k U}{2}; x_{i+1/2}) \right)$$

$$\alpha_k = \max\{|\lambda_k(\rho_{i-1/2})|, |\lambda_k(\rho_{i+1/2})|, |\lambda_k(\rho_{i+3/2})|\}$$

Characteristic-based versus component-wise HRSC

- LLF-Characteristic-based WENO5

$$\bar{Q}_{i+1/2} = \sum_{k=1}^{m} r^k \left(\mathcal{R}^+(l^k \cdot \frac{Q+\alpha_k U}{2}; x_{i+1/2}) + \mathcal{R}^-(l^k \cdot \frac{Q-\alpha_k U}{2}; x_{i+1/2}) \right)$$

$\alpha_k = \max\{|\lambda_k(\rho_{i-1/2})|, |\lambda_k(\rho_{i+1/2})|, |\lambda_k(\rho_{i+3/2})|\}$

- GLF- Component-wise WENO5

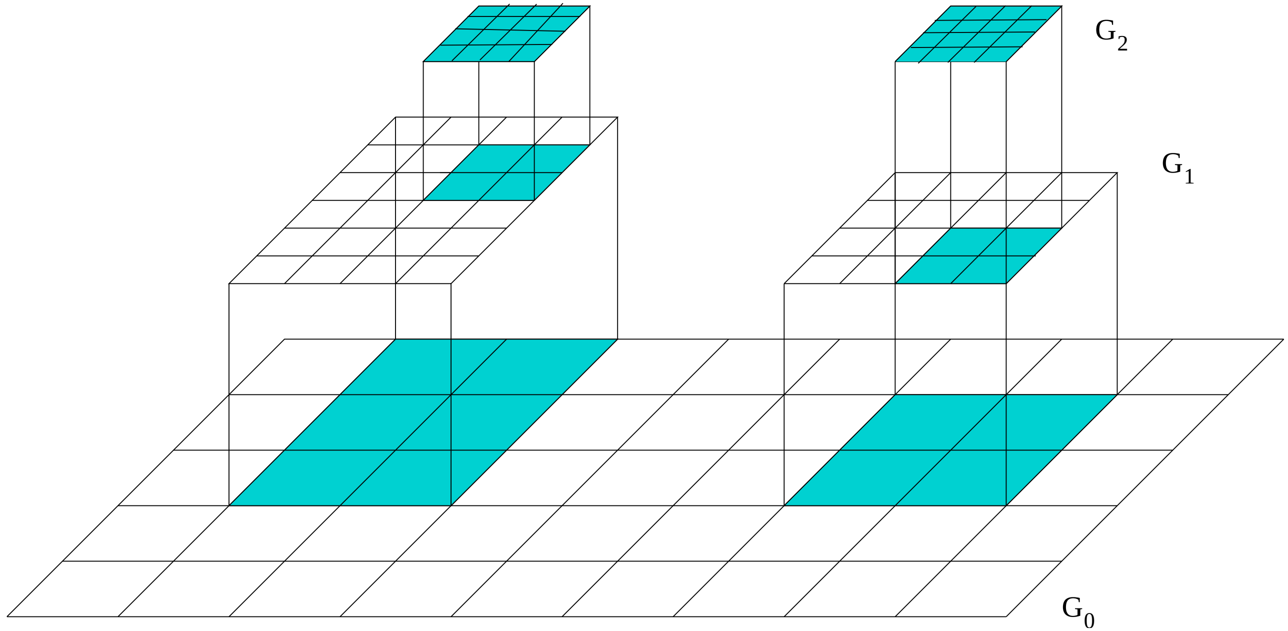
$$\bar{Q}_{j,i+\frac{1}{2}} = \mathcal{R}^+(\frac{1}{2}(Q_j + \alpha\rho_j); x_{i+\frac{1}{2}}) + \mathcal{R}^-(\frac{1}{2}(Q_j - \alpha\rho_j); x_{i+\frac{1}{2}}),$$

$\alpha = \max_\rho(|v_1(\rho) + \sum_{i=1}^m \rho_i v_i'(\rho)|, v_m(\rho));$

- **GLF-Component-wise PHM**:
  - $\mathcal{R}^\pm \equiv \pm$-biased **Piecewise hyperbolic reconstruction,** **[Marquina]**

- Numerical studies using Adaptive Mesh Refinement code **AMR [Baeza,Mulet, IJNMF06]** for basic Shu-Osher style scheme.

Memory Savings: AMR [Berger & Oliger]



- $G_l = \bigcup_{k=1}^{N_l} G_{l,k}$, $G_{l,k} = \prod_{i=1}^{d} [a_i, b_i]$, $h_l = h_{l-1}/r_l$
- Nestedness $G_L \subseteq G_{L-1} \in \cdots G_0$
- $u_l = u_l(G_l)$, solution on grid G_l
- Patches at same level can overlap, but information in u_l is maintained coherently.

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Main AMR-steps

- Adaption process:
  - **Flagging** (+ safety region) [Gradients (Quirk), Higher order interpolation (Baeza-Mulet)]
  - **Clustering**: Creations of rectangular patches
  - **Transfer** of solution between patches

Main AMR-steps

- Adaption process:
  - **Flagging** (+ safety region) [Gradients (Quirk), Higher order interpolation (Baeza-Mulet)]
  - **Clustering**: Creations of rectangular patches
  - **Transfer** of solution between patches
- Flow Integration and time refinement:

  Each single patch is integrated 'in isolation', using its own time step, so that $\Delta t_l / h_l =$ constant (indep. of l)

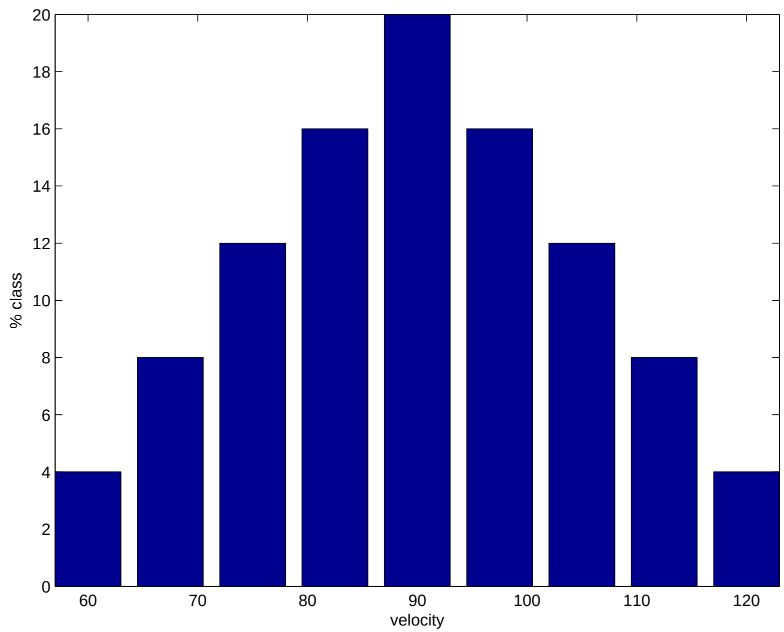
  - **Treatment of patch boundaries** (Filling up the ghost cells)
  - **Conservative update** (Flux Projection) (Sub-grid Transfer of Information. From fine to coarse).

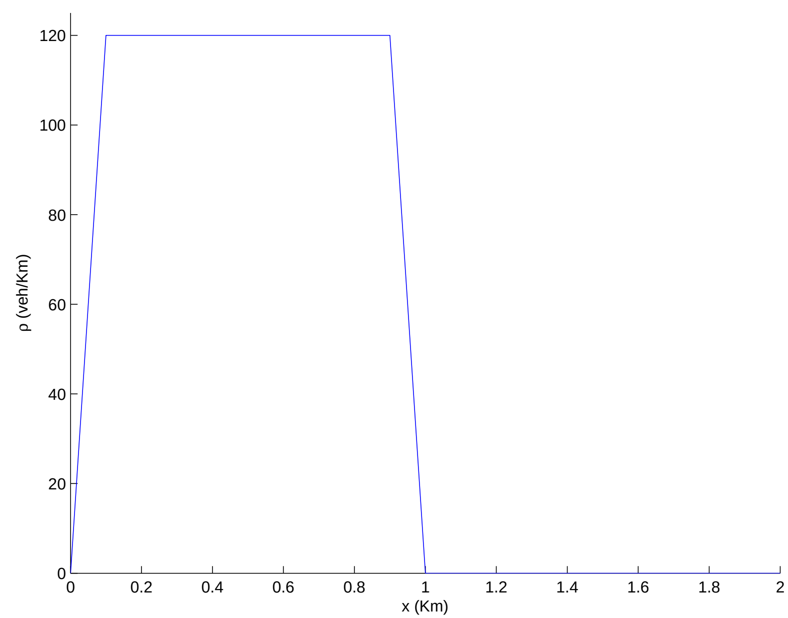
AMR-based Convergence Studies

- Drake model for traffic velocity:

$$V(\rho) = e^{-\frac{1}{2}(\frac{\rho}{\rho_0})^2}, \rho_0 = 50\text{veh}/\text{Km}$$

- Nine-class model with distribution and total density [Wong et al, JCP 06]:

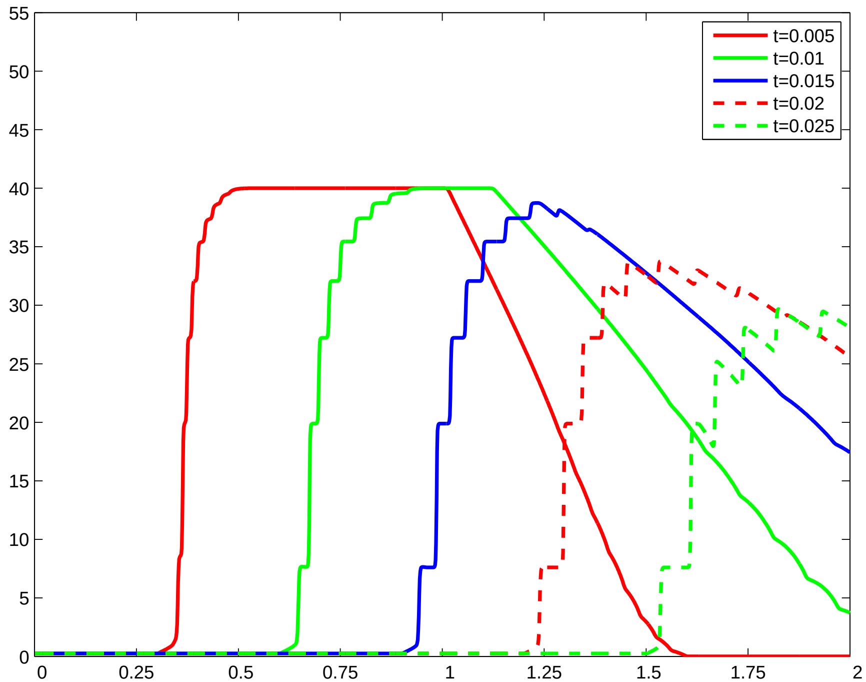




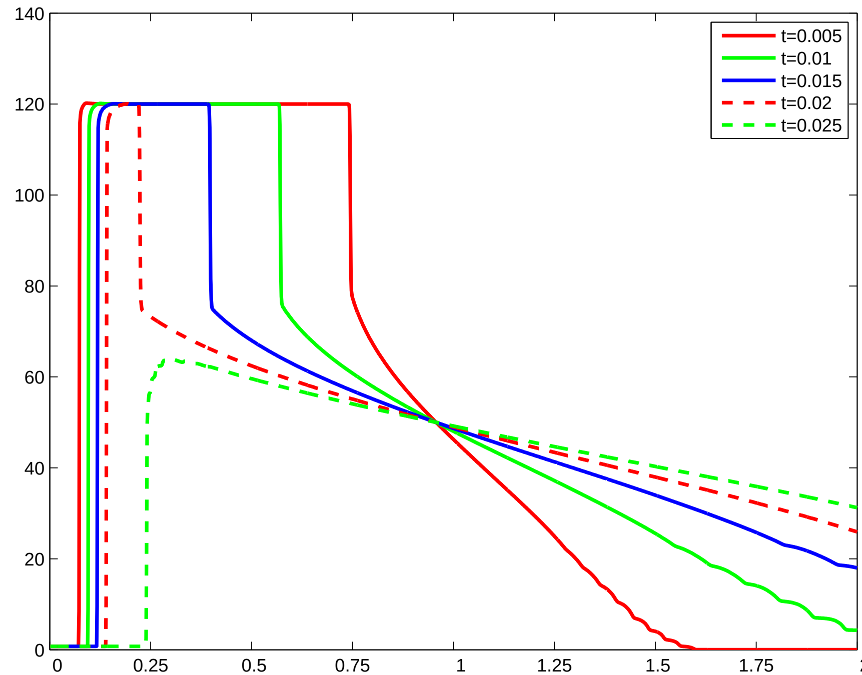
Non-Congested versus Congested traffic

Initial platoon with maximum density ρ_{max}

$\rho_{max} = 40$ veh/Km

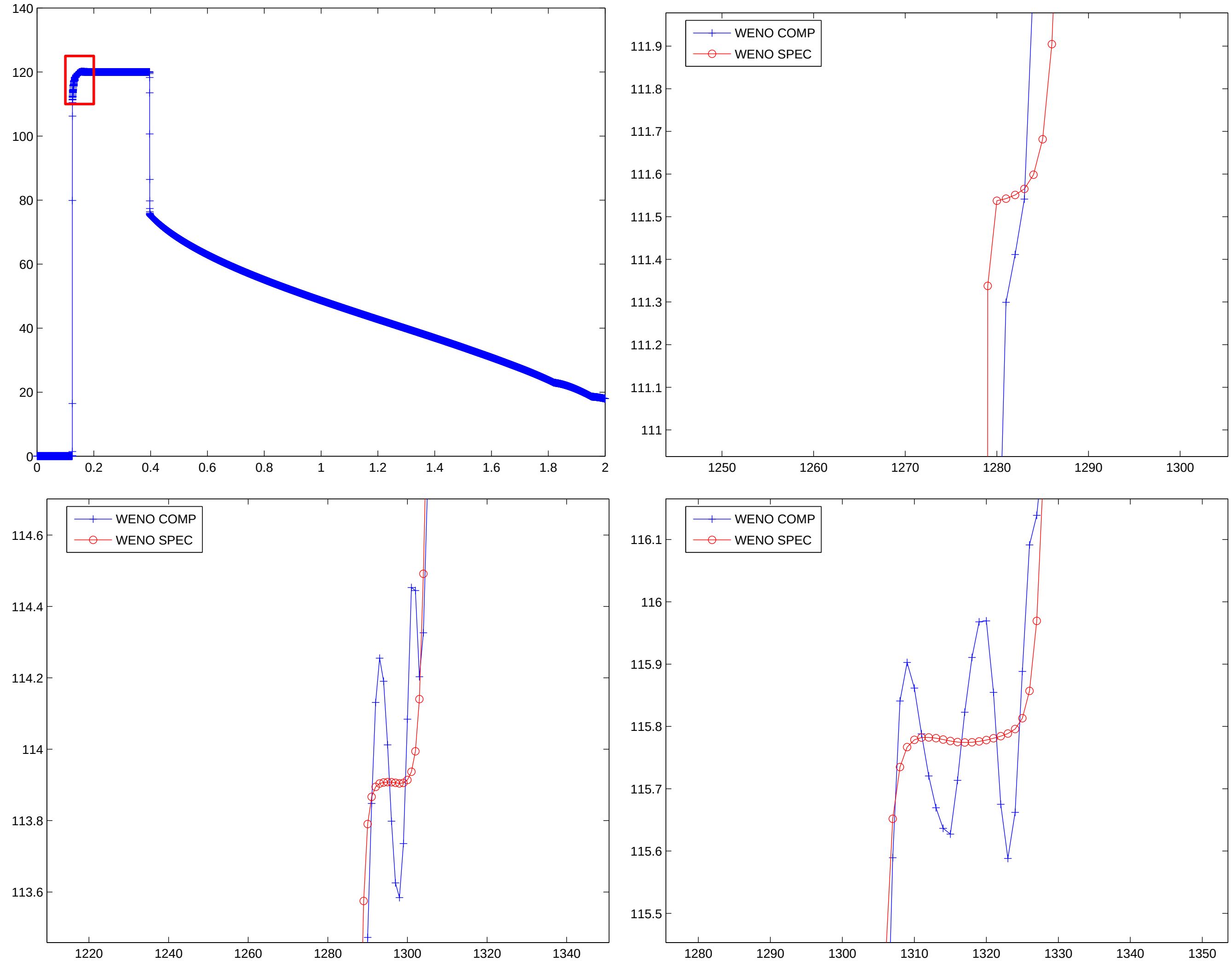


$\rho_{max} = 120$ veh/Km



Congested traffic $\rho_{max} = 120$ veh/Km

- $T = 0.015h = 54s$, $N_0 = 20$, Levels = 11 ($N = 20480$)



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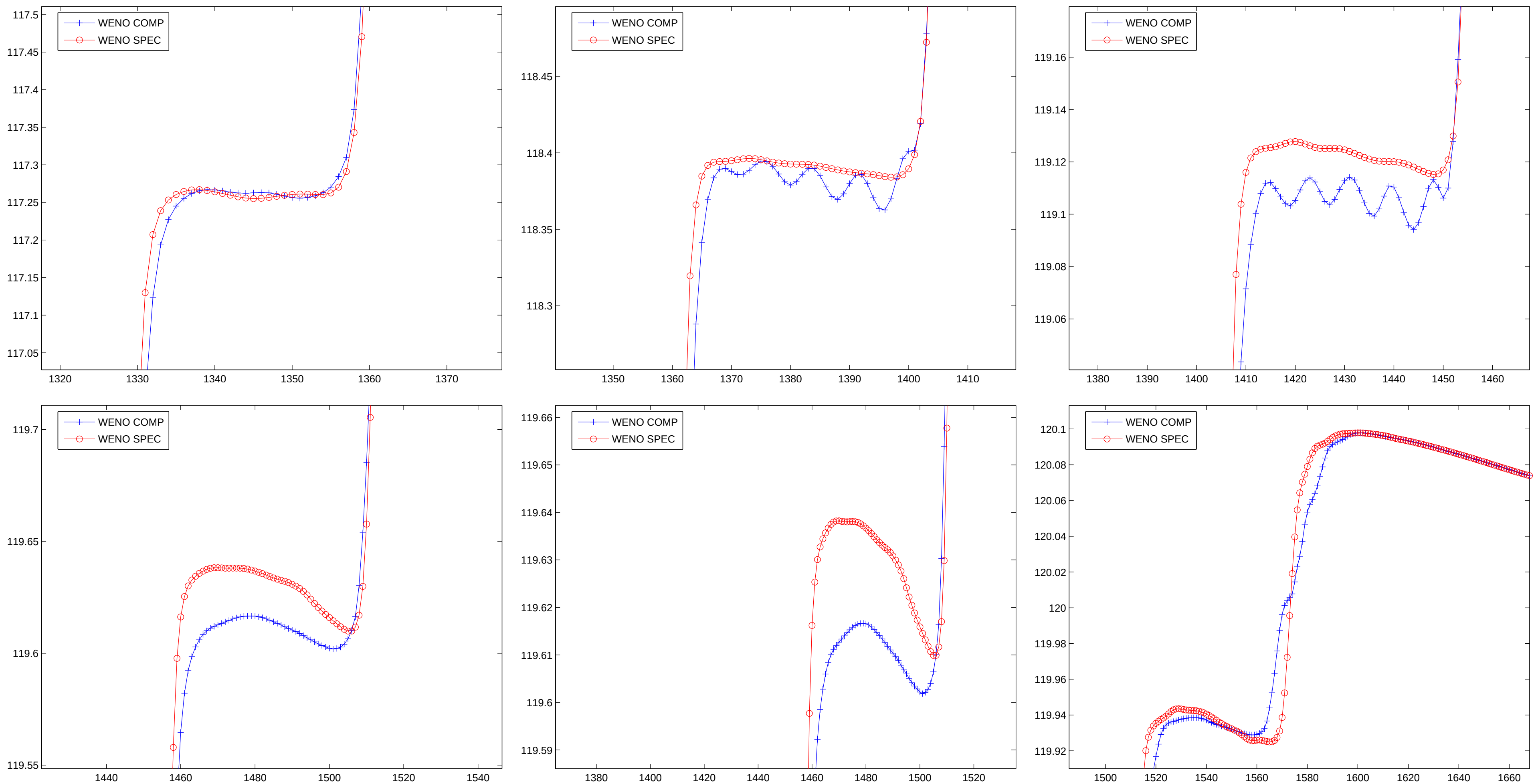
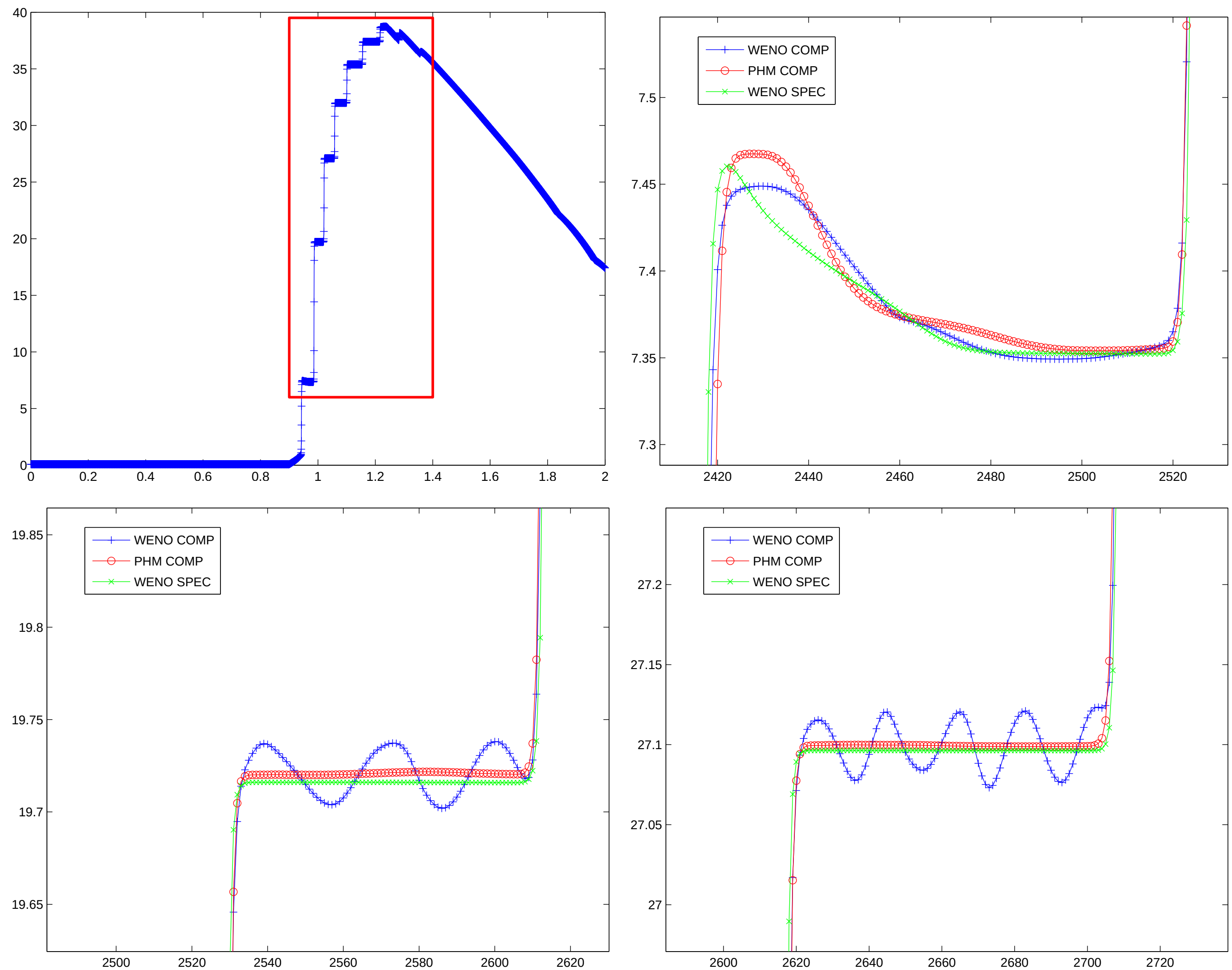
Congested traffic



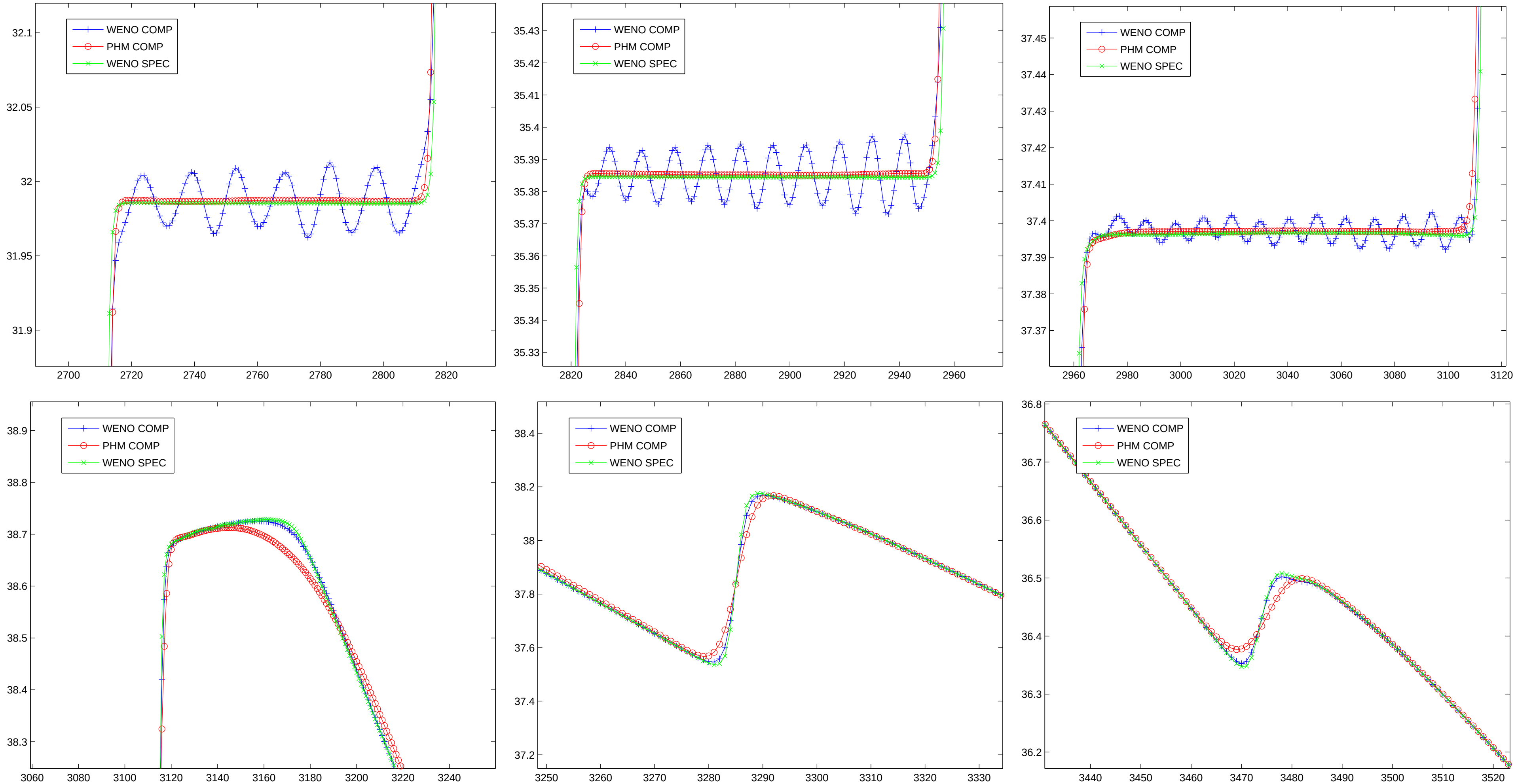
Non-congested traffic $\rho_{max} = 40$ veh/Km

- $T = 0.015h = 54s$, $N_0 = 20$, levels = 9 ($N = 5120$)

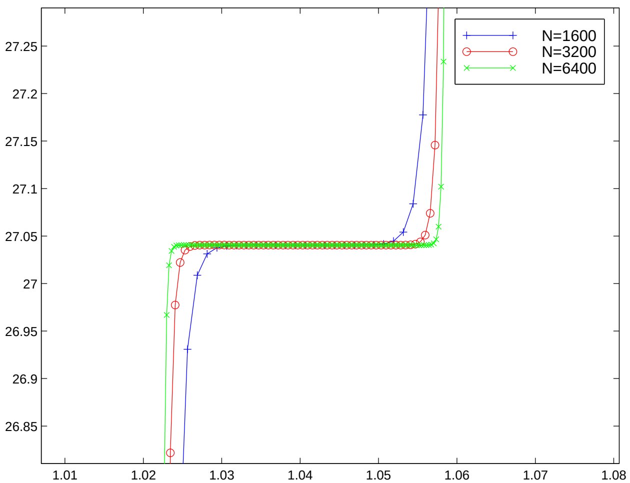


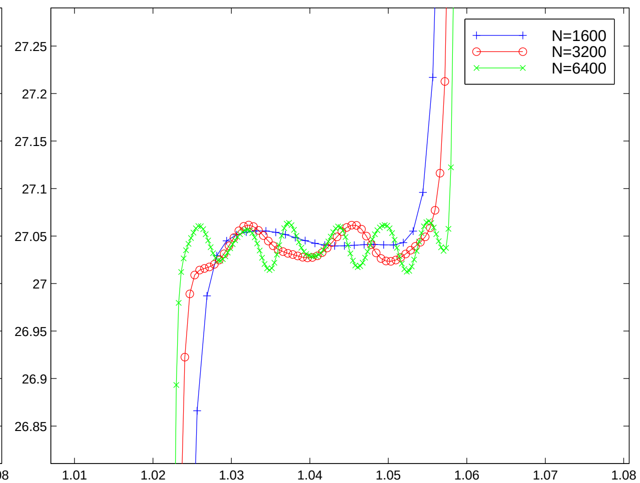
Non-Congested Traffic

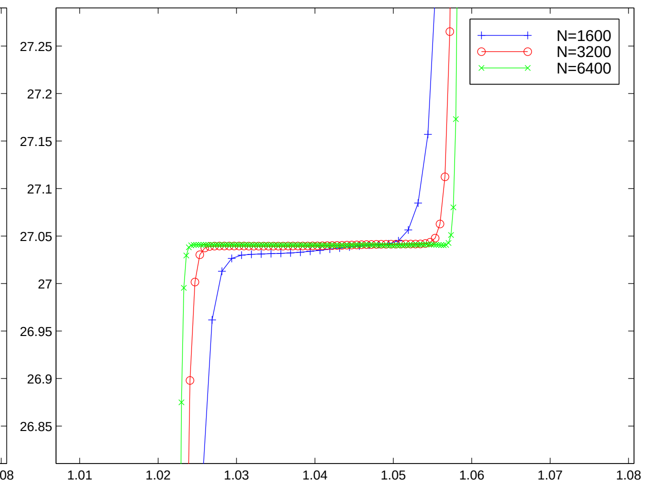
WENO5-GLF Comp-wise, PHM-GLF Comp-wise, WENO5-LLF Spec



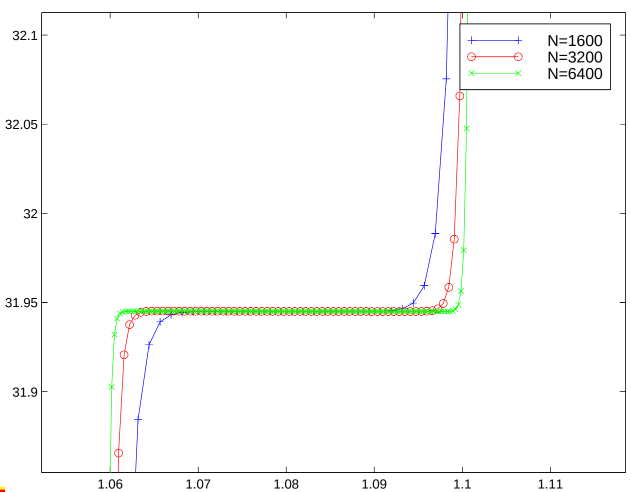
Convergence-Study: N=1600 , N=3200, N= 6400

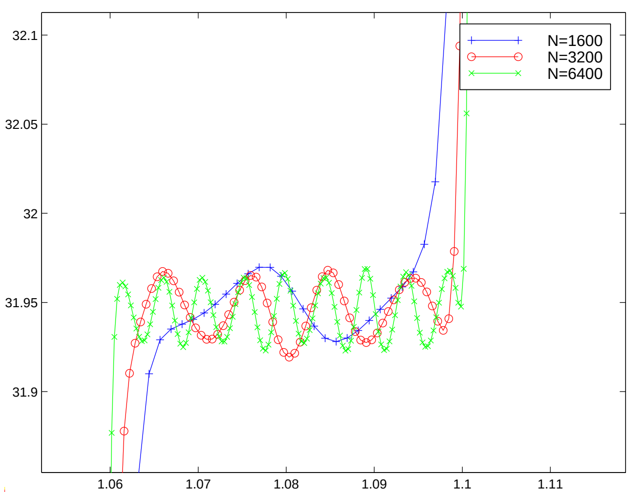


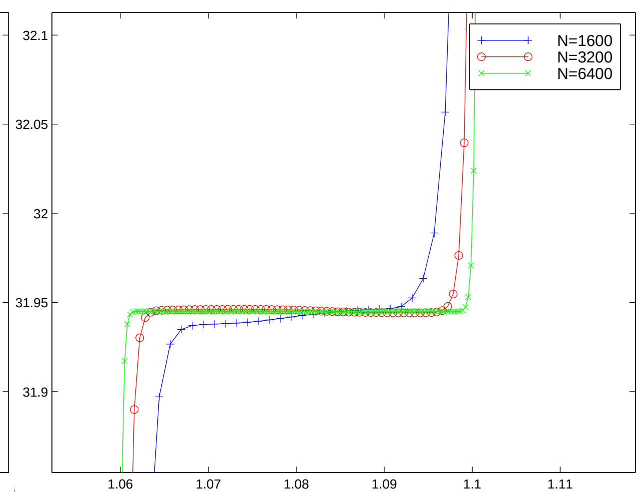




WENO SPEC | WENO COMP | PHM COMP







A Block-Spec Flux-Splitting

- Compute only the s first characteristic eigen-values/vectors, λ_k, r^k, l^k, $k = 1, \dots s$ at each interface ($\rho_{i+1/2} = .5(\rho_i + \rho_{i+1})$)
- Incorporate only this spectral information into the numerical flux:

$$\bar{Q}_{i+1/2} = \sum_{k=1}^{s} r^k \left(\mathcal{R}^+ (l^k \cdot \frac{Q + \alpha_k U}{2}; x_{i+1/2}) + \mathcal{R}^- (l^k \cdot \frac{Q - \alpha_k U}{2}; x_{i+1/2}) \right)$$
$$+ \mathcal{R}^+ \left((\sum_{k=s+1}^{m} r^k l^k) \frac{Q + \alpha U}{2}; x_{i+1/2} \right) + \mathcal{R}^- \left((\sum_{k=s+1}^{m} r^k l^k) \frac{Q - \alpha U}{2}; x_{i+1/2} \right)$$

$$\alpha_k = \max\{|\lambda_k(\rho_{i-1/2})|, |\lambda_k(\rho_{i+1/2})|, |\lambda_k(\rho_{i+3/2})|\}$$

$$\alpha = \max\{\alpha_1, |v_m(\rho_{i-1/2})|, |v_m(\rho_{i+1/2})|, |v_m(\rho_{i+3/2})|\}$$

- $\sum_{k=s+1}^{m} r^k l^k$, projector onto 'unknown characteristic fields'

  implemented as $I - \sum_{k=1}^{s} r^k l^k$

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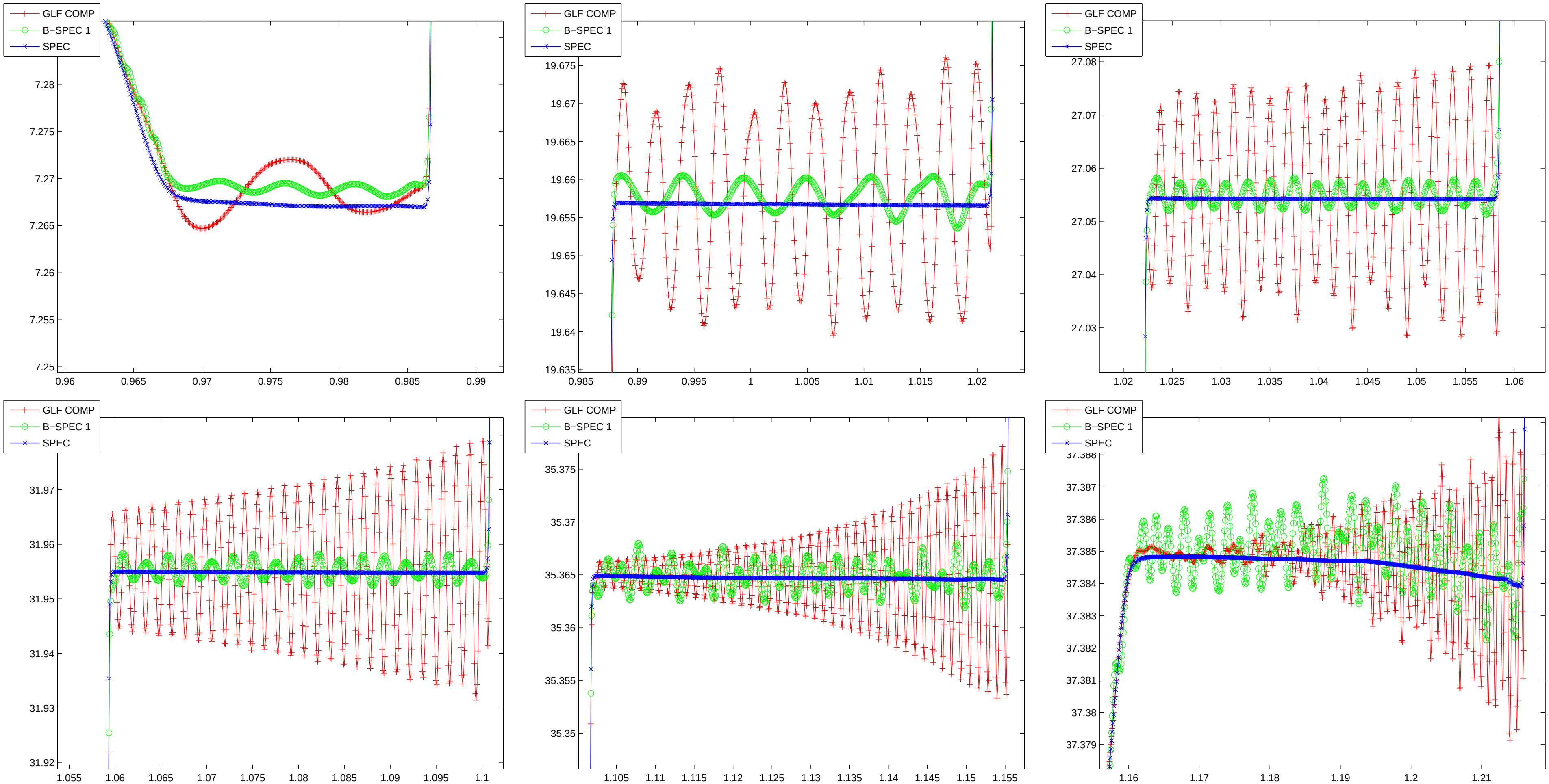
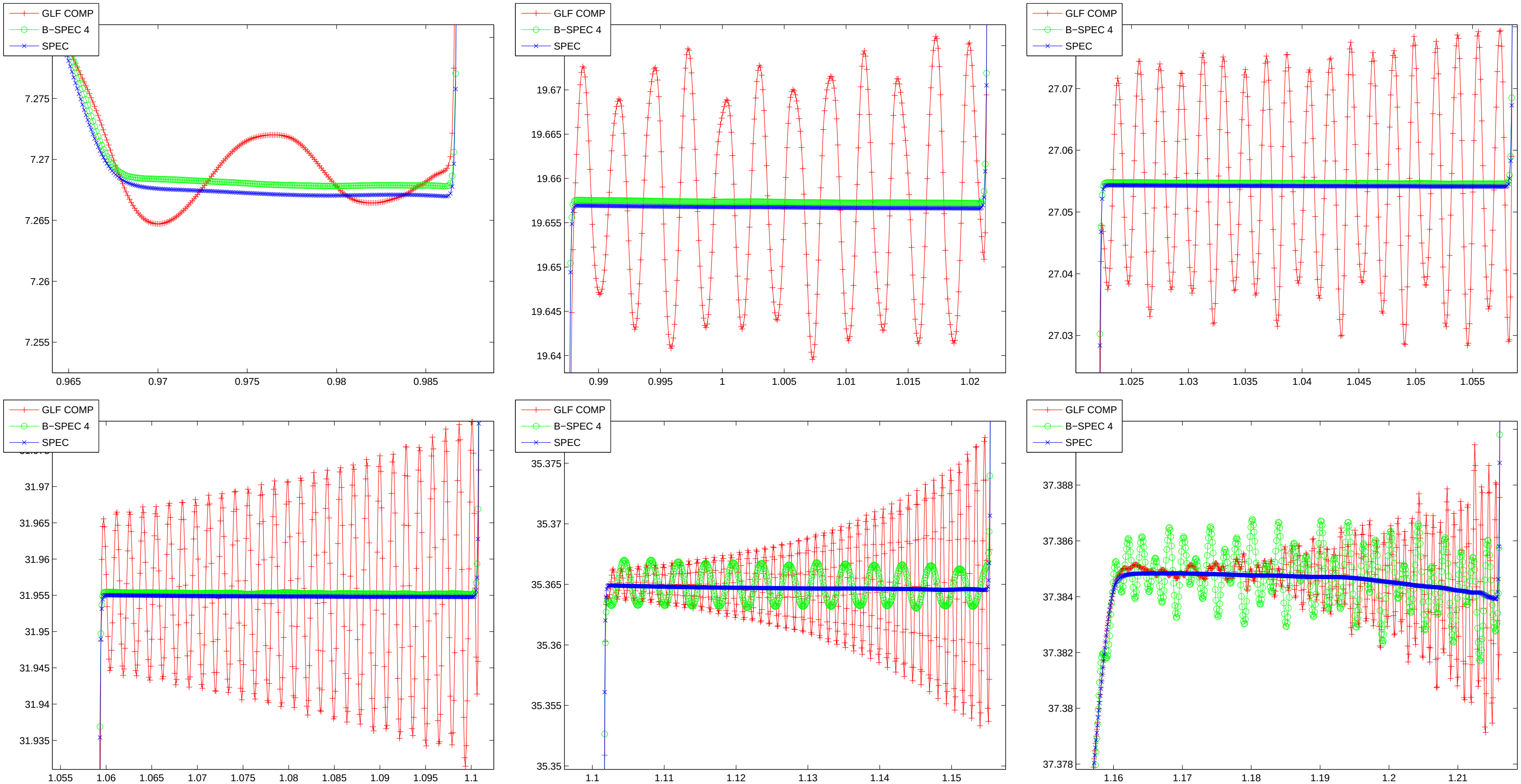
WENO5: GLF, Block-Spec–$s = 1$, Full-Spec

Non-congested traffic



WENO5: GLF, Block-Spec–$s = 4$, Full-Spec

Non-congested traffic



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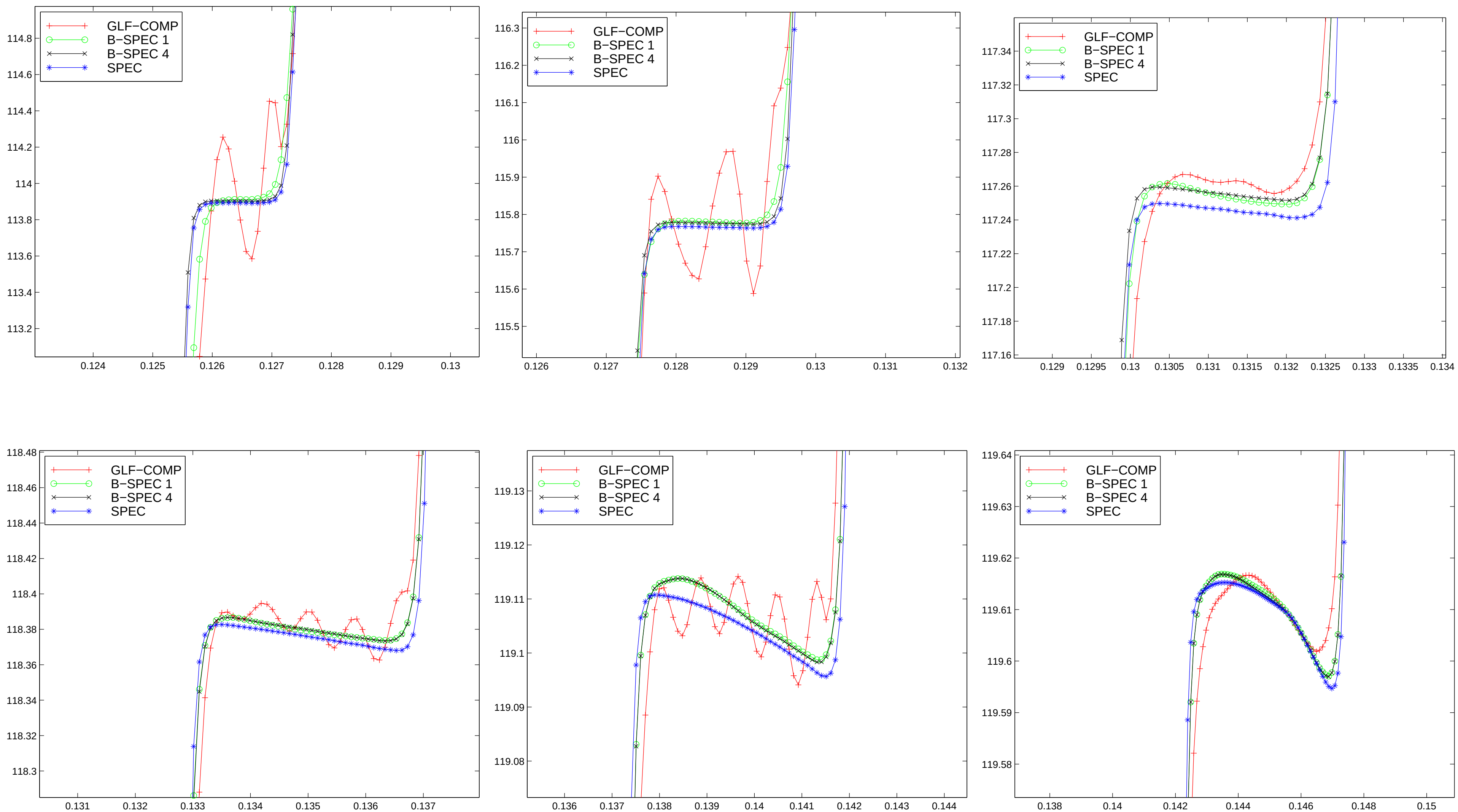
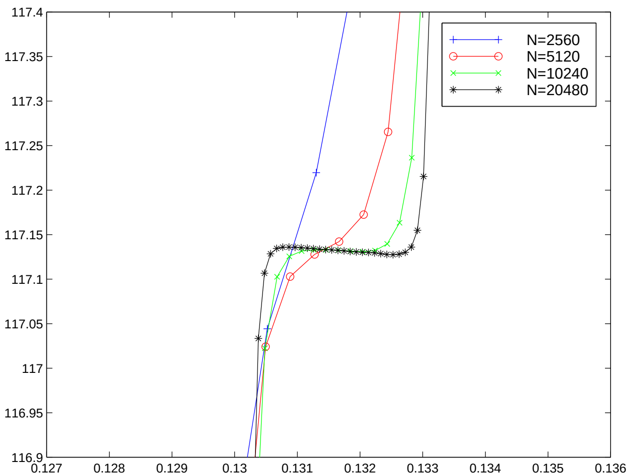
WENO5: GLF, BS-1, BS-4, Full-Spec

Congested traffic

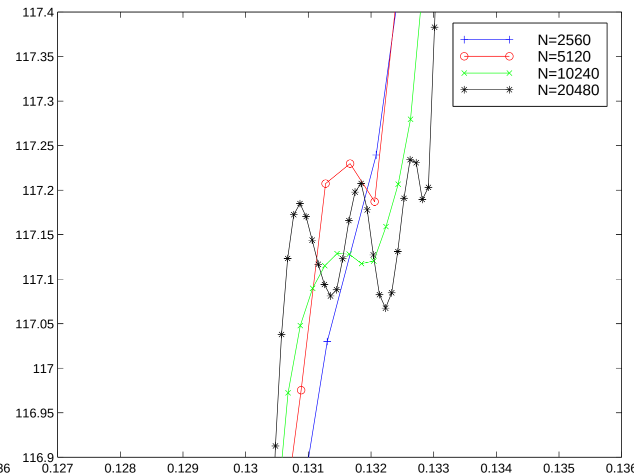


Convergence-Study: Congested case

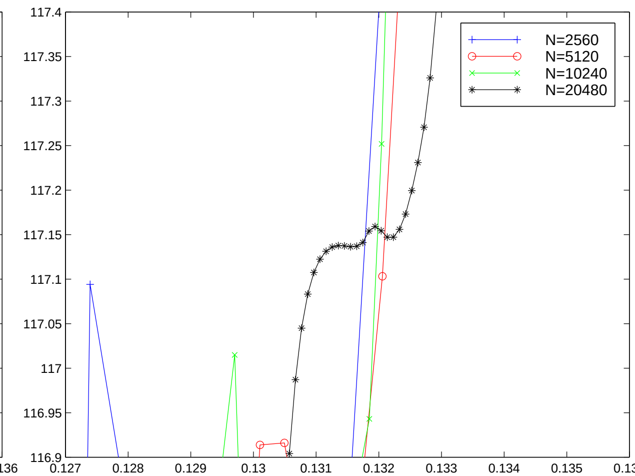
N=2560, N=5120, N=10240, N=20480



WENO SPEC



WENO COMP



PHM COMP

Conclusions

- Componentwise ENO/WENO reconstructions, slightly oscillatory. Oscillations do not diminish with grid refinement.
- Block-Spectral decomposition can help to obtain Essentially Oscillation Free solutions using only a part of the spectral information.
- Non-polynomial reconstructions with a smaller interpolatory stencil, such as PHM, better option for component-wise HRSC schemes.
- Adaptive Codes are essential to run these numerical studies.