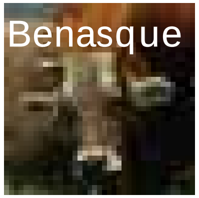


Accelerator Physics
Longutudinal motion

Elena Wildner

Acknowldements to
Rende Steerenberg,
Mats Lindroos,
for course material

Motion in longitudinal plane

- What happens when particle momentum increases?
  - $\Rightarrow$ particles follow longer orbit (fixed B field)
  - $\Rightarrow$ particles travel faster (initially)
- How does the **revolution frequency** change with the **momentum** ?

$$\frac{df}{f} = \frac{dv}{v} - \frac{dr}{r}$$

But

$$\frac{\Delta r}{r} = \alpha_p \frac{\Delta p}{p}$$

Change in velocity

Change in orbit length

Momentum compaction factor

Therefore:

$$\frac{df}{f} = \frac{dv}{v} - \alpha_p \frac{dp}{p}$$

The frequency - momentum relation

$$\frac{df}{f} = \frac{dv}{v} - \alpha_p \frac{dp}{p}$$

But

$$\frac{dv}{v} = \frac{d\beta}{\beta} \quad \left(\beta = \frac{v}{c} \right)$$

- The relativity theory says $\Rightarrow$ $p = \frac{E_0 \beta\gamma}{c}$

$$\frac{dp}{d\beta} = \frac{E_0 \gamma^3}{c}$$

$$\frac{dp}{p} = \gamma^2 \frac{d\beta}{\beta}$$

$$\frac{df}{f} = \left(\frac{1}{\gamma^2} - \alpha_p \right) \frac{dp}{p}$$

varies with momentum
$(\mathbf{E} = \mathbf{E}_0 \gamma)$

fixed by the quadrupoles

Transition

- **Lets look at the behaviour of a particle in a constant magnetic field.**
- **Low momentum** $(\beta \ll 1, \gamma \Rightarrow 1)$ $\longrightarrow$ $\dfrac{1}{\gamma^2} > \alpha_p$
- The revolution frequency increases as momentum increases
- **High momentum** $(\beta \approx 1, \gamma \gg 1)$ $\longrightarrow$ $\dfrac{1}{\gamma^2} < \alpha_p$
- The revolution frequency decreases as momentum increases
- For one particular momentum or energy we have:

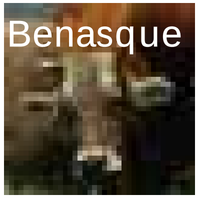
$$\frac{1}{\gamma^2} = \alpha_p$$

- This particular energy is called the **Transition energy**

The frequency slip factor

- We found $$\frac{df}{f} = \left(\frac{1}{\gamma^2} - \alpha_p\right)\frac{dp}{p} = \left(\frac{1}{\gamma^2} - \frac{1}{\gamma_{tr}^2}\right)\frac{dp}{p}$$ η

- $\frac{1}{\gamma^2} > \alpha_p \longrightarrow$ **Below transition** $\longrightarrow$ $\eta =$ **positive**
- $\frac{1}{\gamma^2} = \alpha_p \longrightarrow$ **Transition** $\longrightarrow$ $\eta =$ **zero**
- $\frac{1}{\gamma^2} < \alpha_p \longrightarrow$ **Above transition** $\longrightarrow$ $\eta =$ **negative**
- Transition is very important in proton machines.
  - A little later we will see why....
- In the PS machine : $\gamma_{tr} \approx 6$ GeV/c
- Transition does not exist in leptons machines, Why?



Radio Frequency System

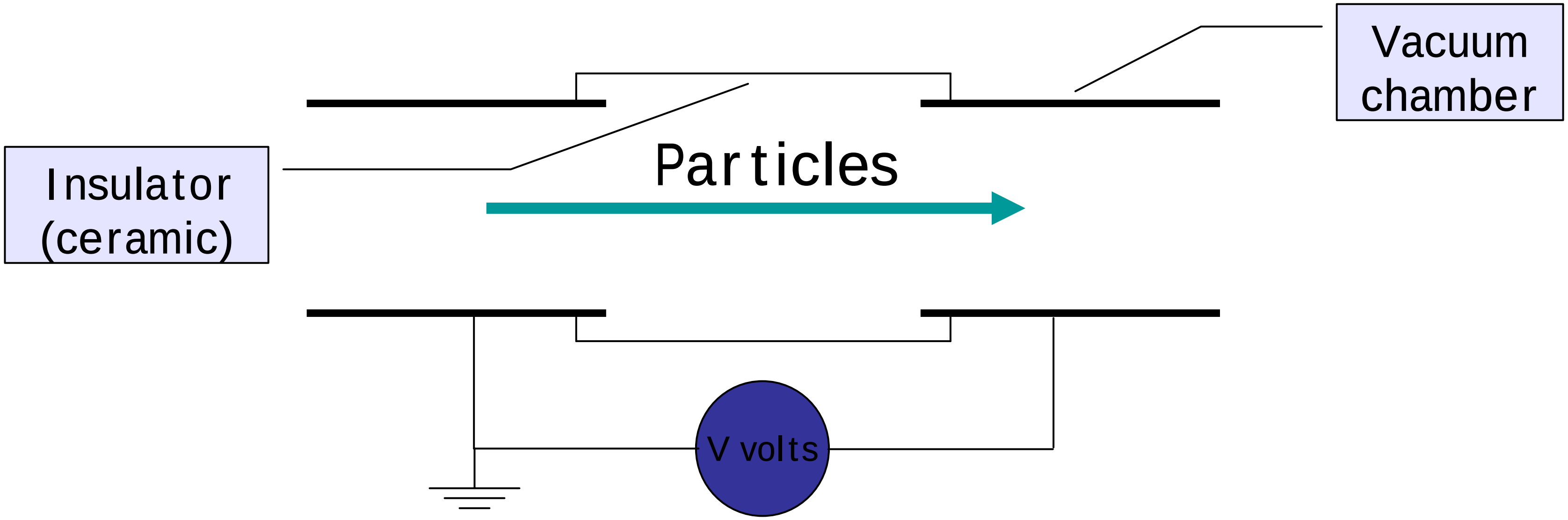
- Hadron machines:
  - Accelerate / Decelerate beams
  - Beam shaping
  - Beam measurements
  - Increase luminosity in hadron colliders
- Lepton machines:
  - Accelerate beams
  - Compensate for energy loss due to synchrotron radiation.

    (see lecture on Synchrotron Radiation)

Benasque

RF Cavity

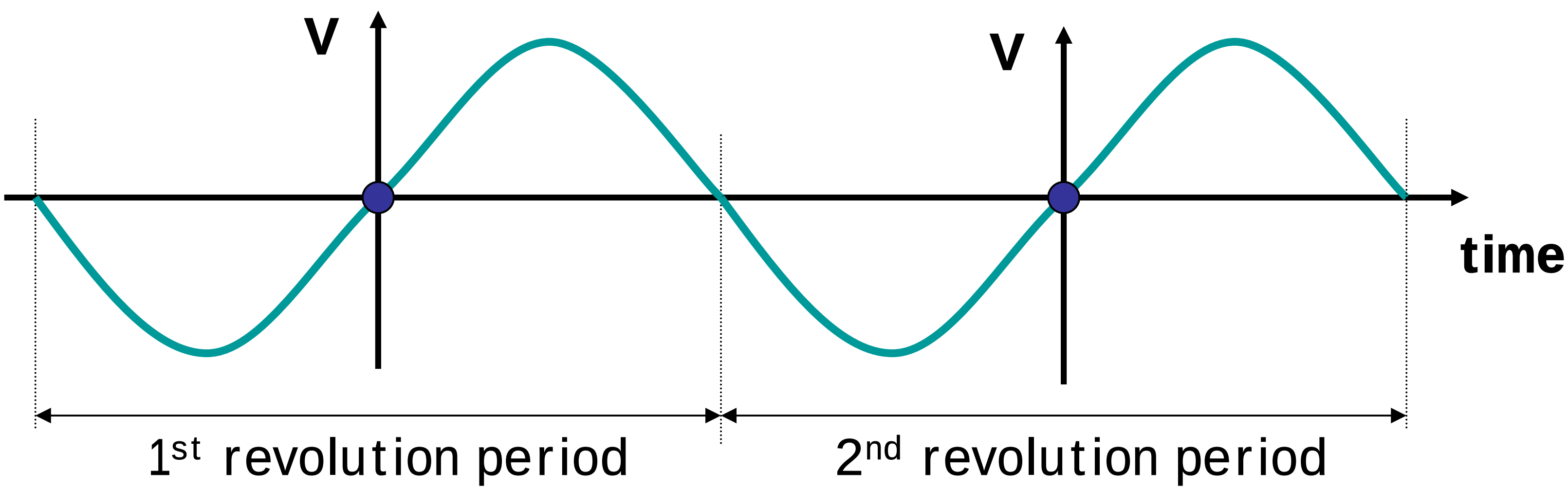
- To accelerate charged particles we need a longitudinal electric field.
- Magnetic fields deflect particles, but do not accelerate them.



- If the voltage is DC then there is no acceleration !
  - The particle will accelerate towards the gap but decelerate after the gap.
- Use an **Oscillating Voltage** with the right Frequency

A Single particle in a longitudinal electric field

- Lets see what a low energy particle does with this oscillating voltage in the cavity.

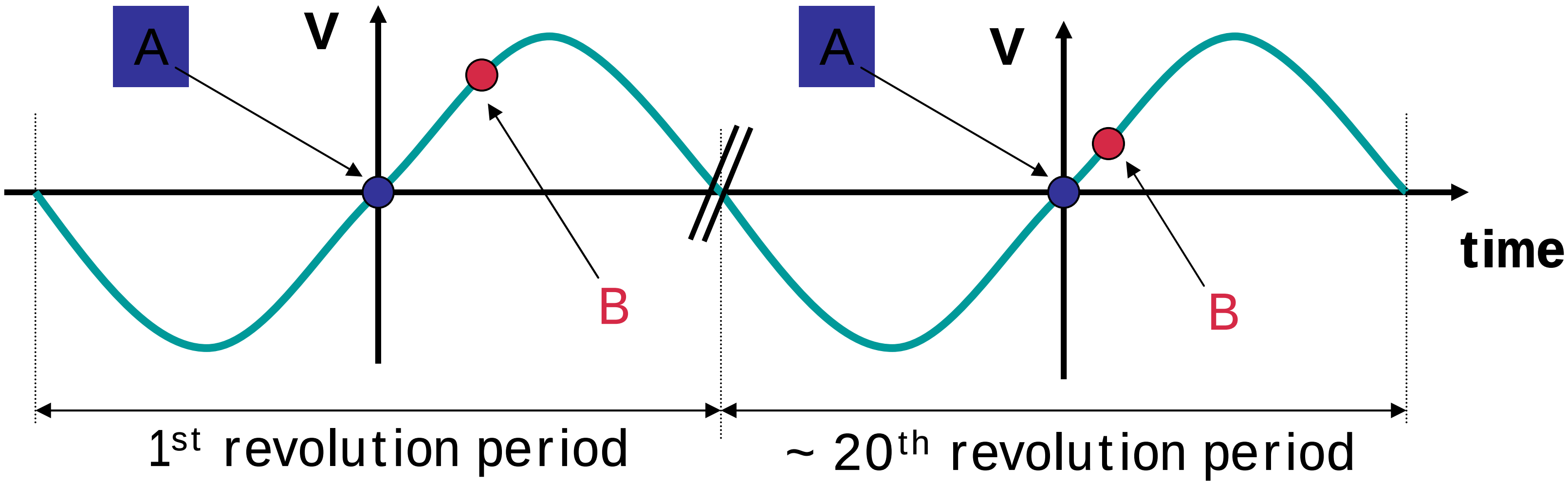


- Set the oscillation frequency so that the period is exactly equal to one revolution period of the particle.

Benasque

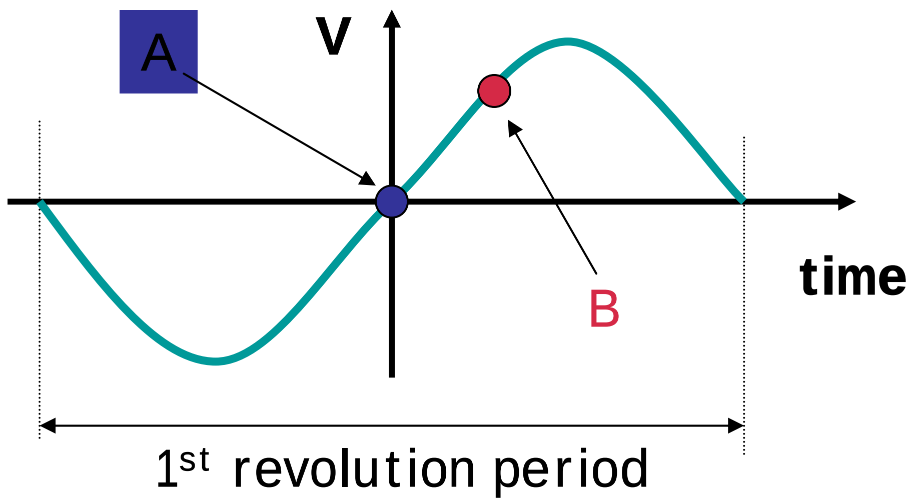
Add a second particle to the first one

- Lets see what a second low energy particle, which arrives later in the cavity, does with respect to our first particle.

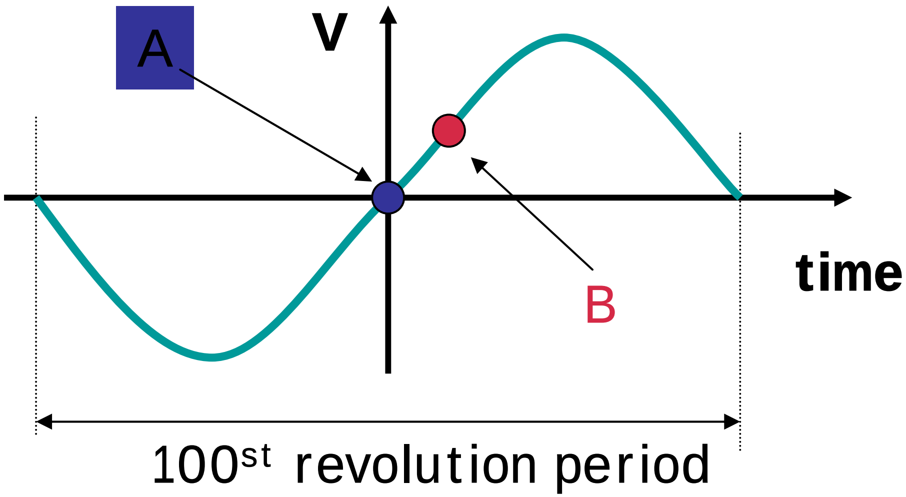


- B arrives late in the cavity w.r.t. A
- B sees a higher voltage than A and will therefore be accelerated
- After many turns B approaches A
- B is still late in the cavity w.r.t. A
- B still sees a higher voltage and is still being accelerated

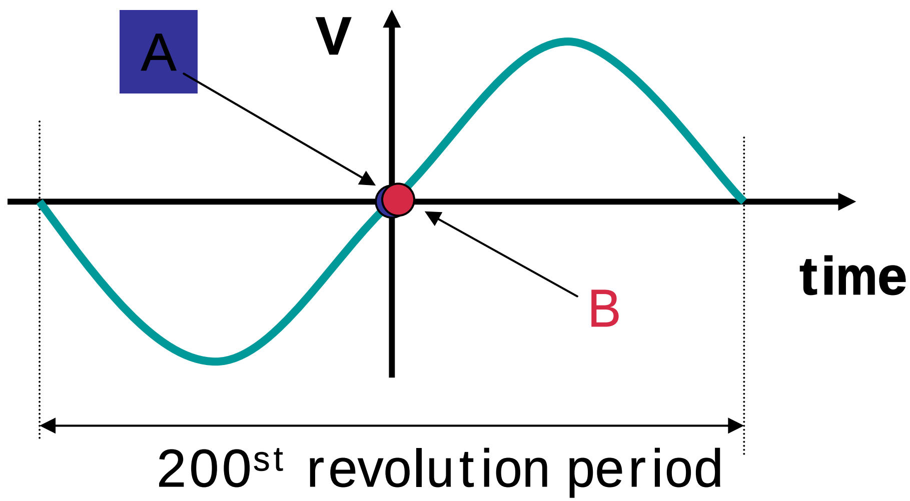
Lets see what happens after many turns



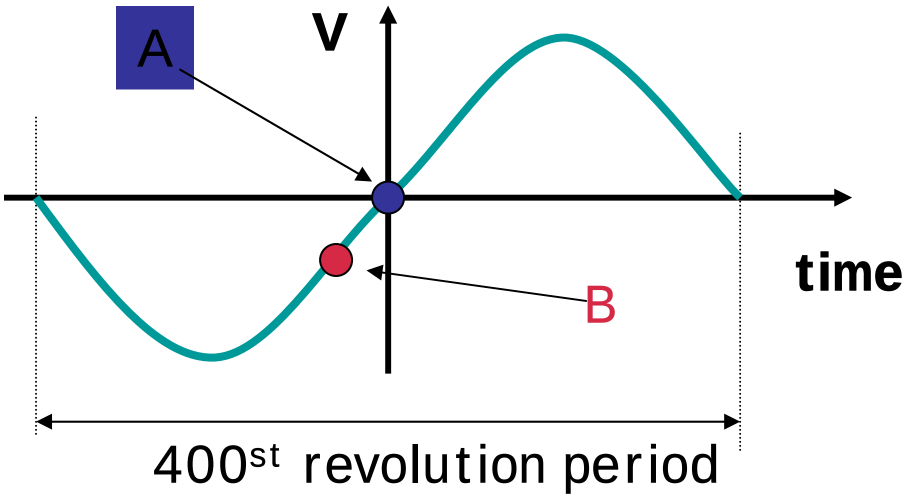
Lets see what happens after many turns



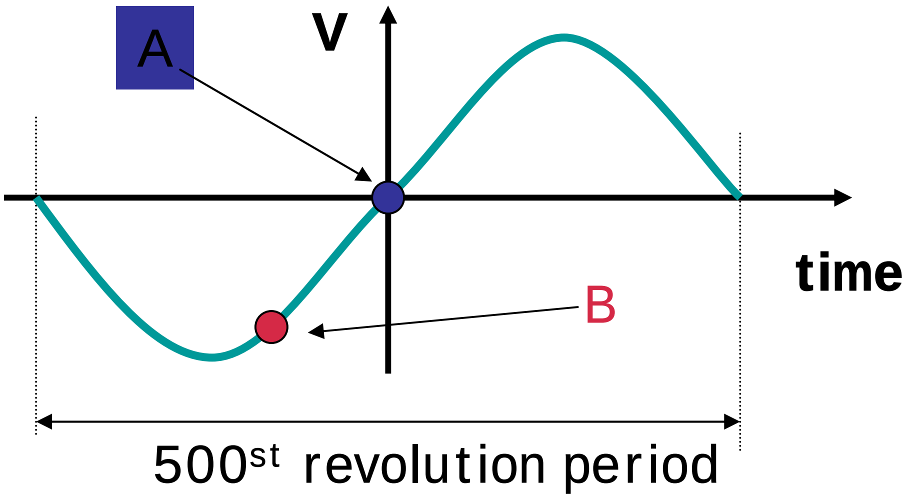
Lets see what happens after many turns



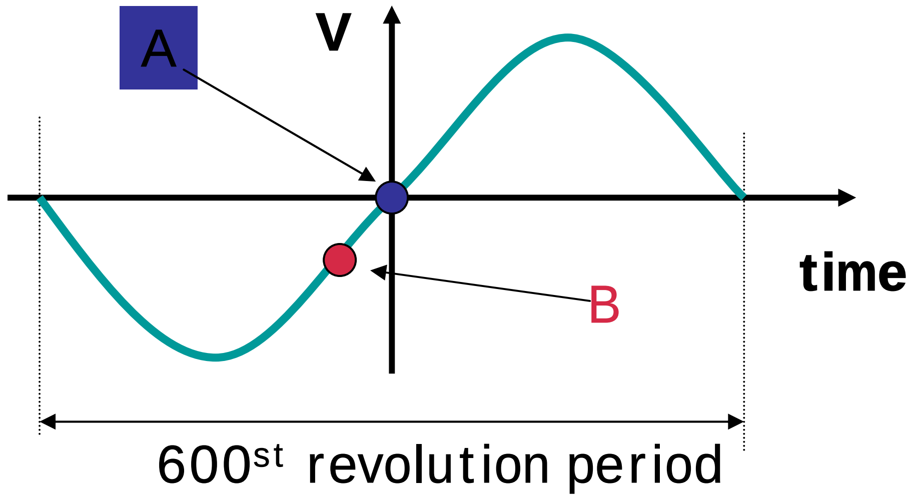
Lets see what happens after many turns



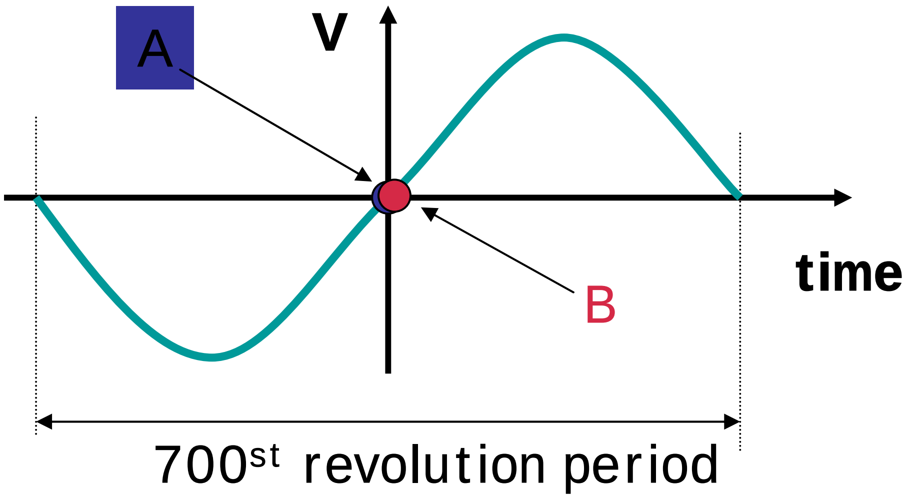
Lets see what happens after many turns



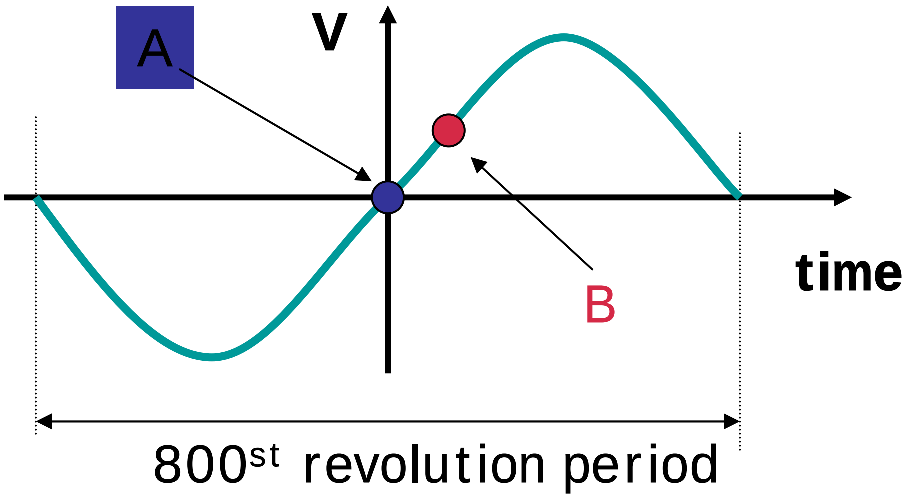
Lets see what happens after many turns

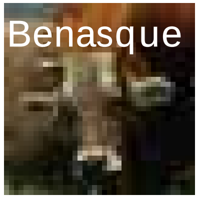


Lets see what happens after many turns

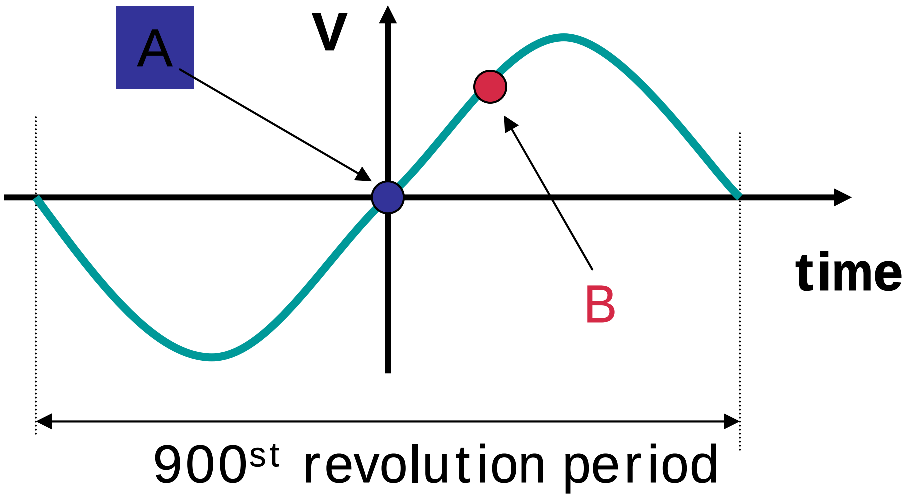


Lets see what happens after many turns



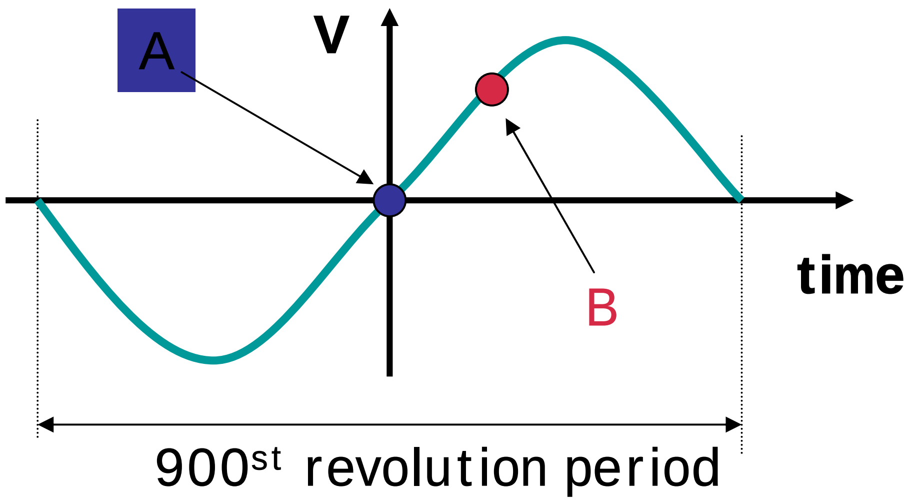


Lets see what happens after many turns



Benasque

Synchrotron Oscillations



- Particle B has made 1 full oscillation around particle A.
- The amplitude depends on the initial phase.

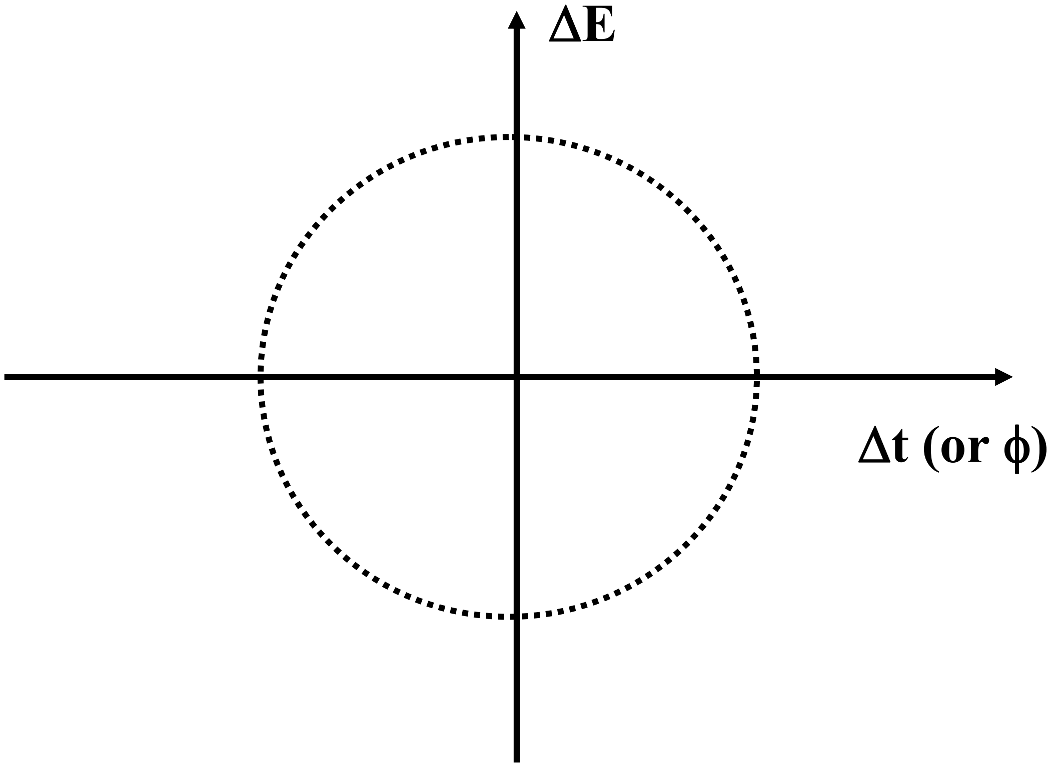
Exactly like the pendulum

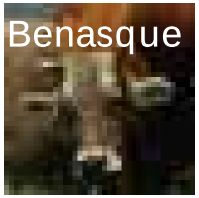
- We call this oscillation:

Synchrotron Oscillation

Longitudinal Phase Space

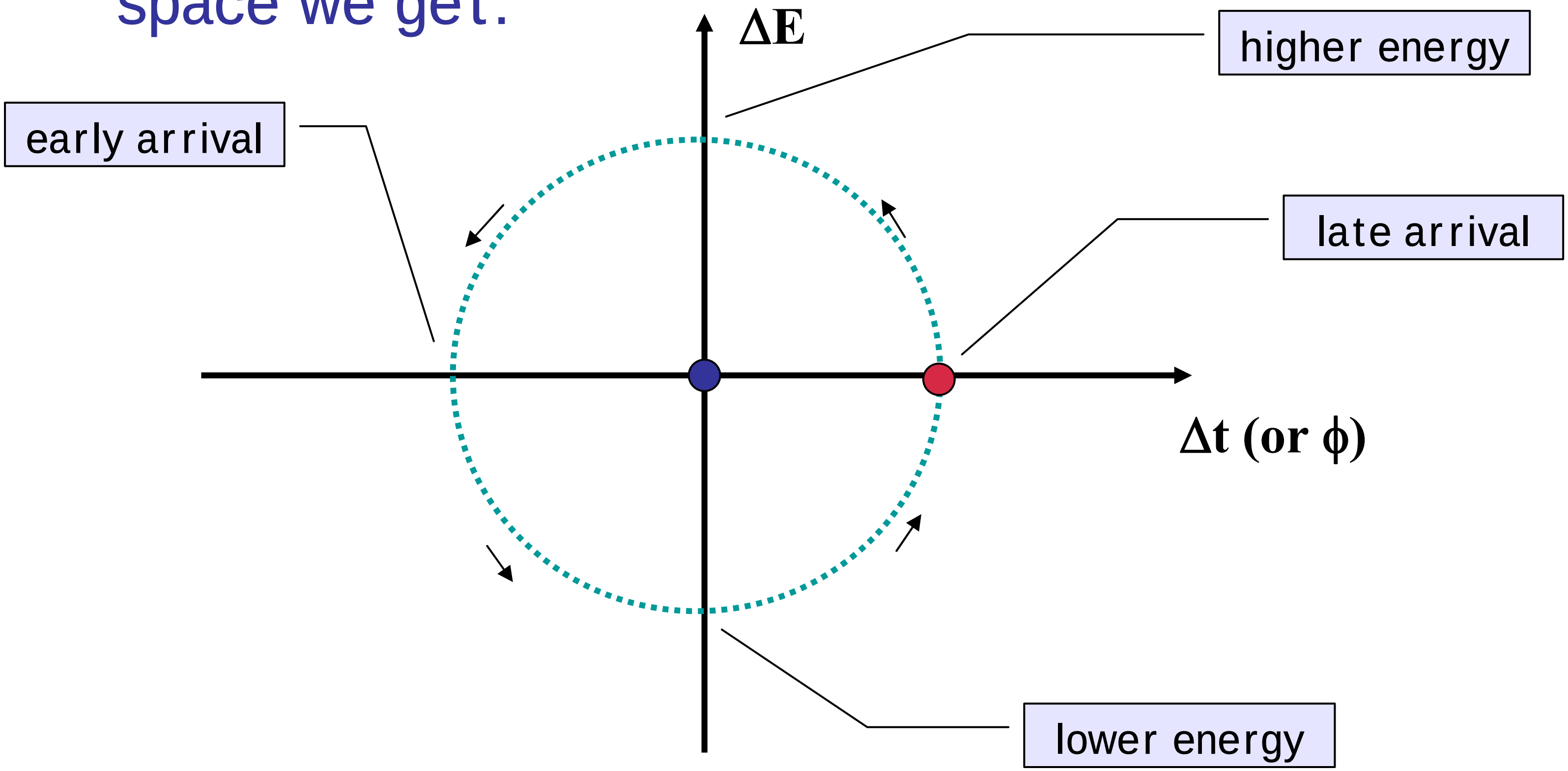
- In order to be able to visualize the motion in the longitudinal plane we define the longitudinal phase space (like we did for the transverse phase space)





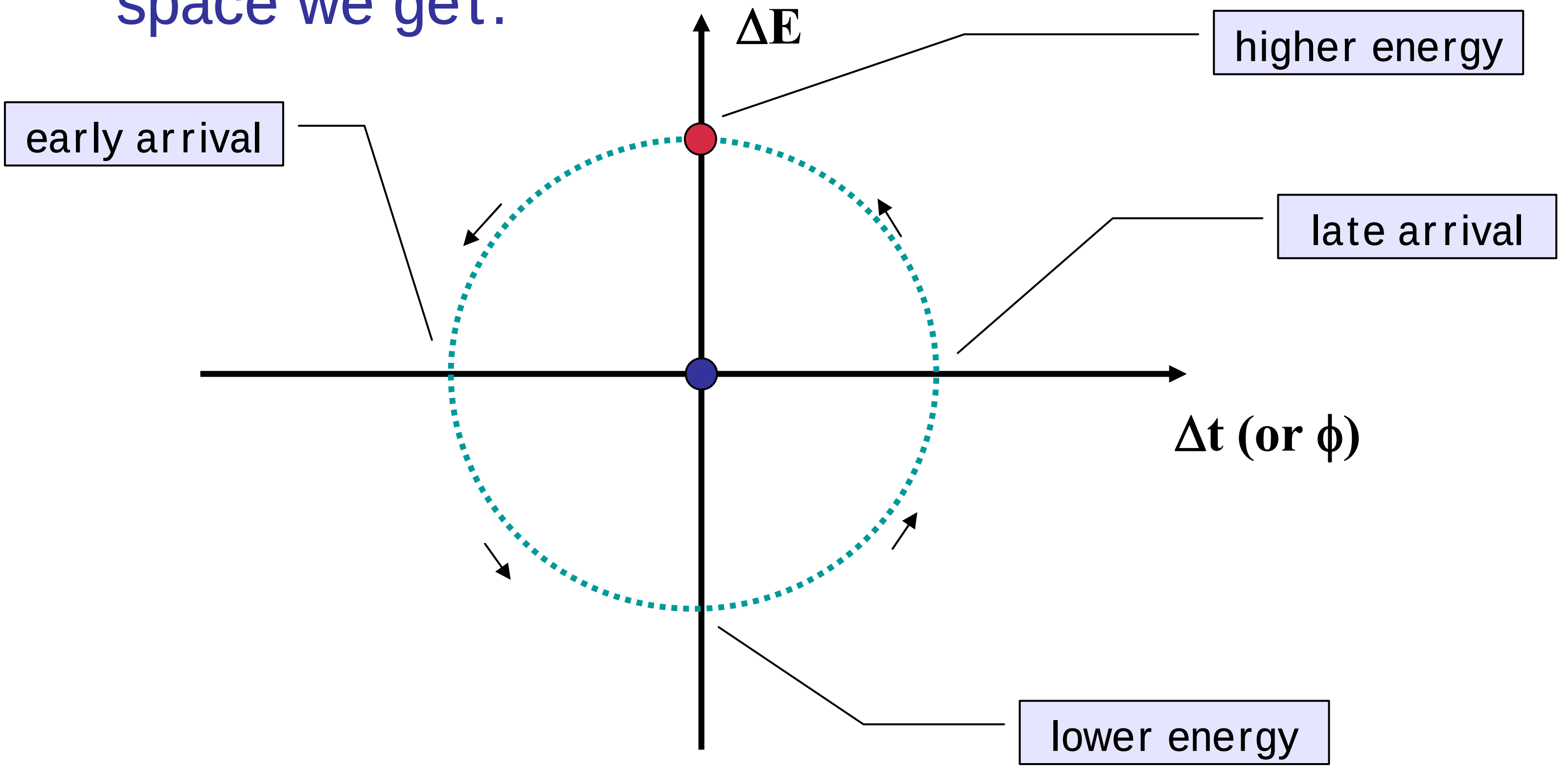
Phase Space motion (1)

- Particle B oscillates around particle A
  - This is synchrotron oscillation
- When we plot this motion in our longitudinal phase space we get:



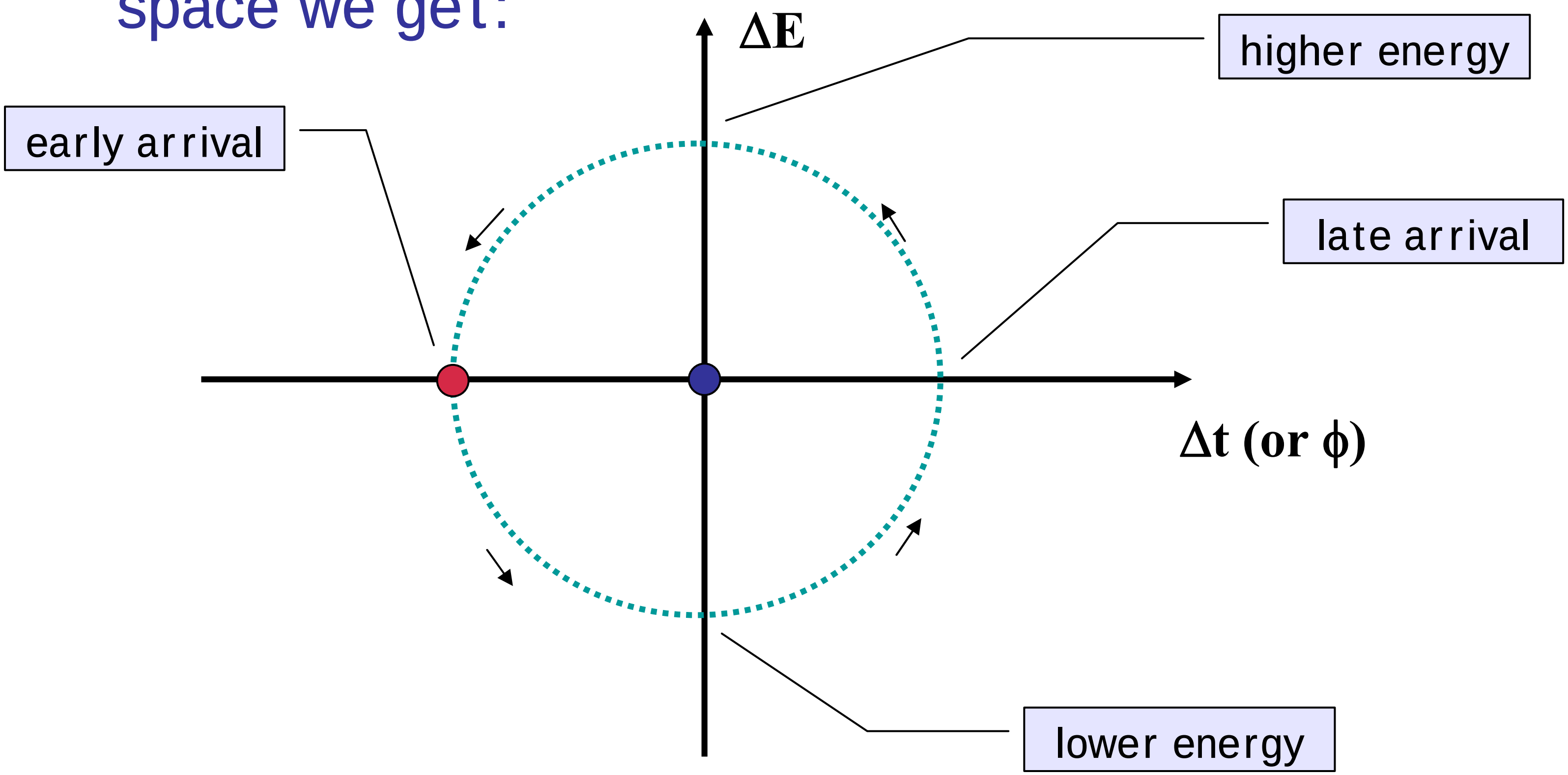
Phase Space motion (2)

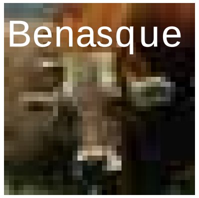
- Particle B oscillates around particle A
  - This is synchrotron oscillation
- When we plot this motion in our longitudinal phase space we get:



Phase Space motion (3)

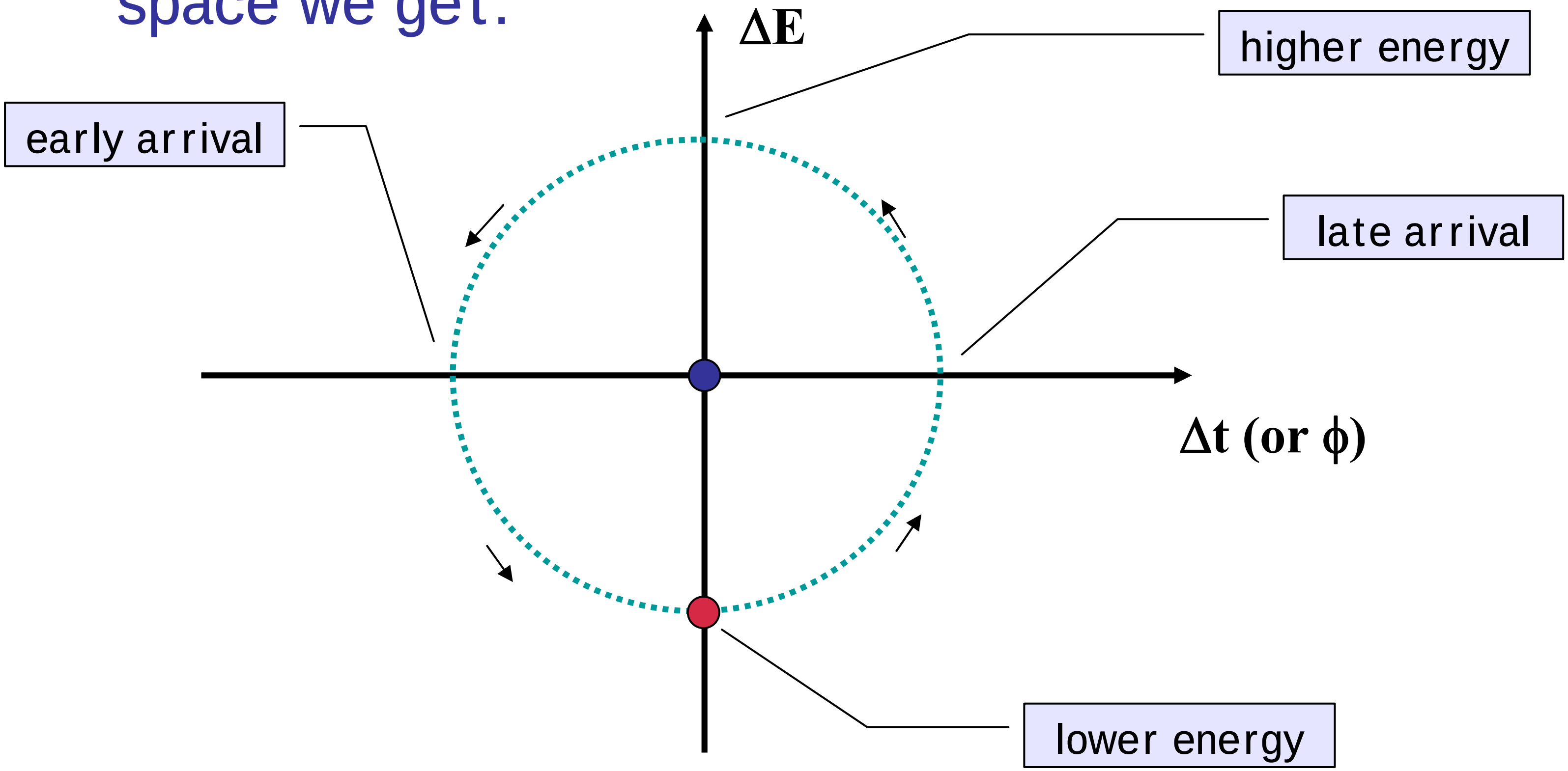
- Particle B oscillates around particle A
  - This is synchrotron oscillation
- When we plot this motion in our longitudinal phase space we get:

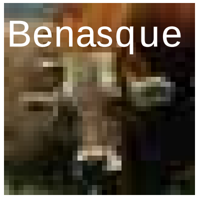




Phase Space motion (4)

- Particle B oscillates around particle A
  - This is synchrotron oscillation
- When we plot this motion in our longitudinal phase space we get:

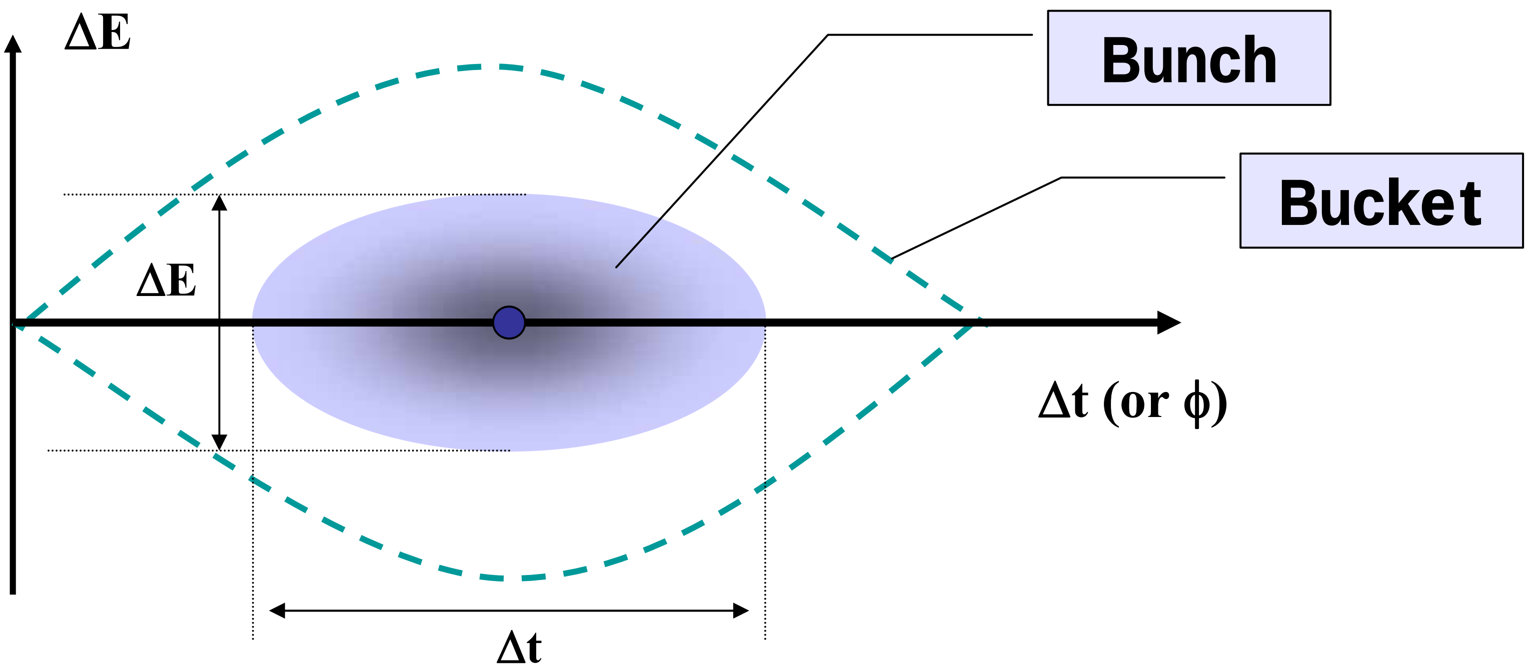




Quick intermediate summary...

- We have seen that:
  - The RF system forms a potential well in which the particles oscillate (synchrotron oscillation).
  - We can describe this motion in the longitudinal phase space (energy versus time or phase).
  - This works for particles below transition.
- However,
  - Due to the shape of the potential well, the oscillation is a non-linear motion.
  - The phase space trajectories are therefore no circles nor ellipses.
  - What when our particles are **above transition** ?

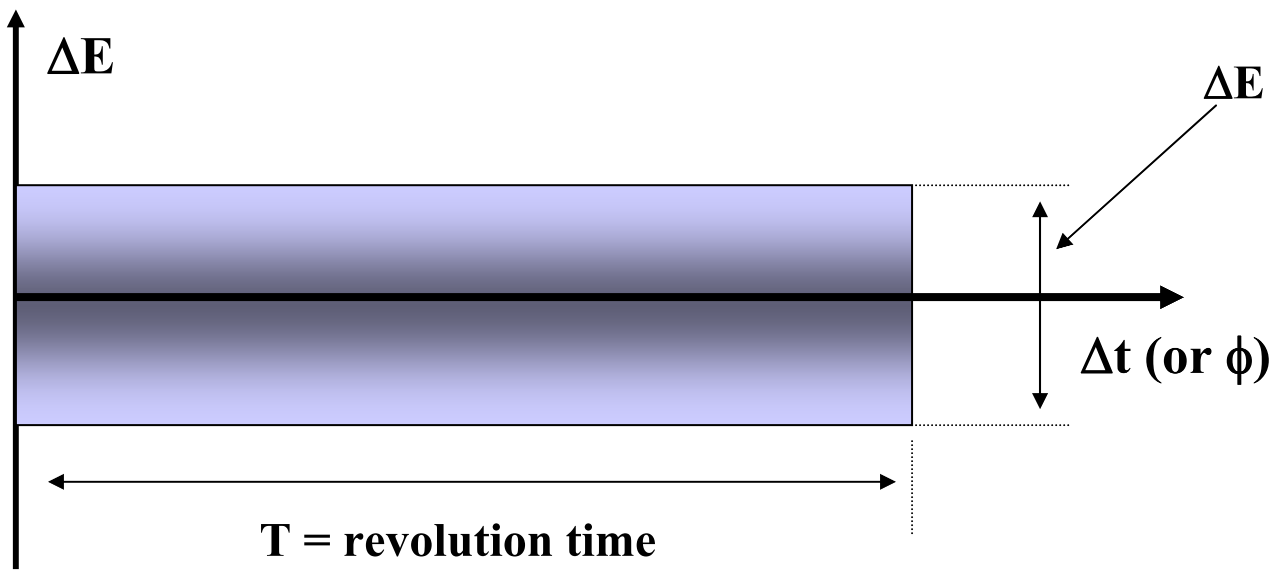
Stationary bunch & bucket

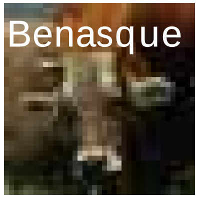


- Bucket area = **longitudinal Acceptance** [eVs]
- Bunch area = **longitudinal beam emittance** = $\pi.\Delta E.\Delta t/4$ **[eVs]**

Unbunched (coasting) beam

- The emittance of an unbunched beam is just **ΔET eVs**
  - ΔE is the energy spread [eV]
  - T is the revolution time [s]





What happens beyond transition ?

- Until now we have seen how things look like below transition

$\eta =$ **positive**

Higher energy ⇨ faster orbit ⇨ higher F_{rev} ⇨ next time particle will be **earlier**.

Lower energy ⇨ slower orbit ⇨ lower F_{rev} ⇨ next time particle will be **later**.

- What will happen above transition ?

$\eta =$ **negative**

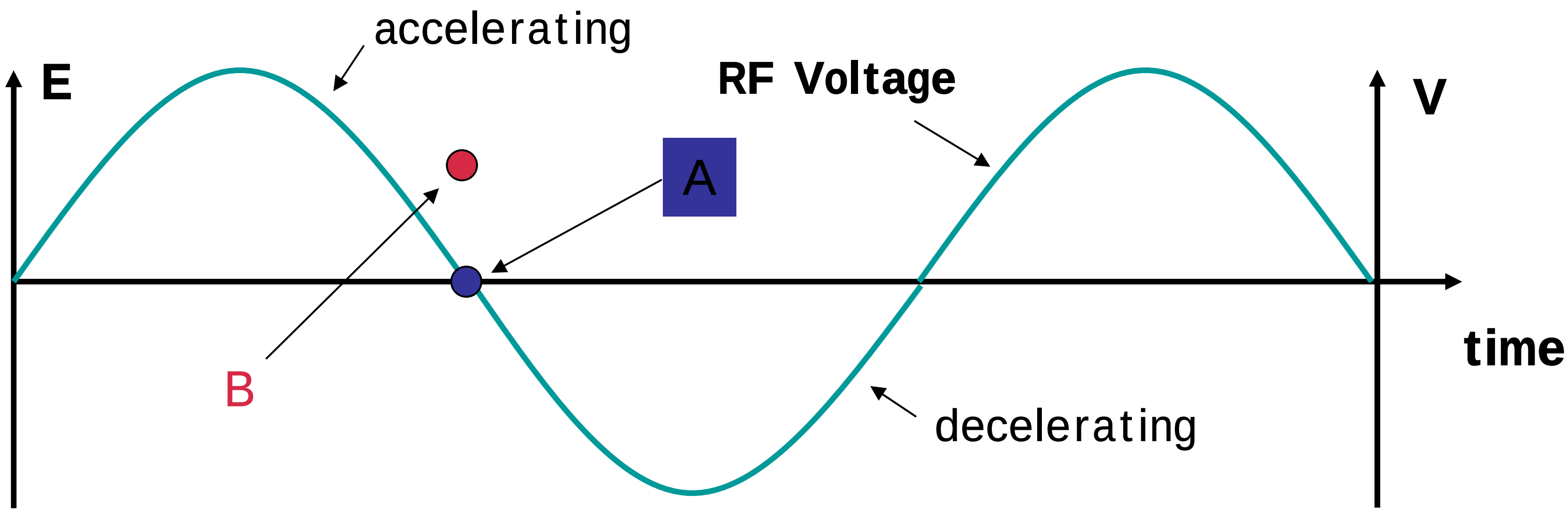
Higher energy ⇨ longer orbit ⇨ lower F_{rev} ⇨ next time particle will be **later**.

Lower energy ⇨ shorter orbit ⇨ higher F_{rev} ⇨ next time particle will be **earlier**.

What are the implication for the RF ?

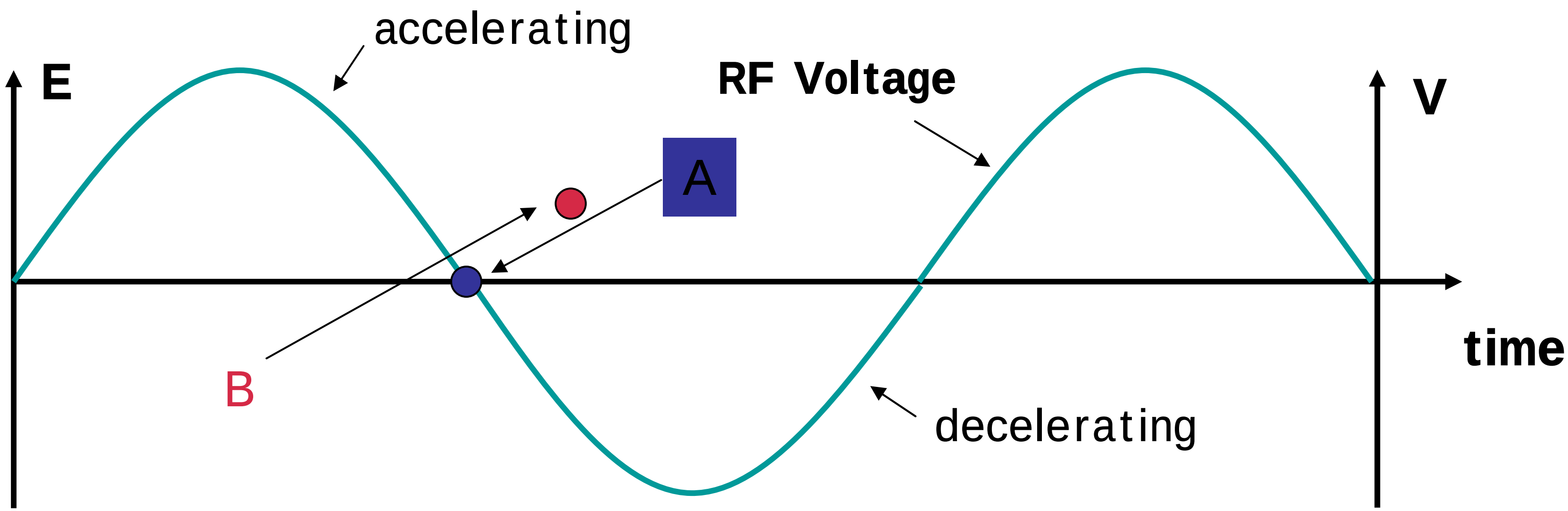
- For particles below transition we worked on the **rising edge** of the sine wave.
- For Particles above transition we will work on the **falling edge** of the sine wave.
- We will see why........

Longitudinal motion beyond transition (1)



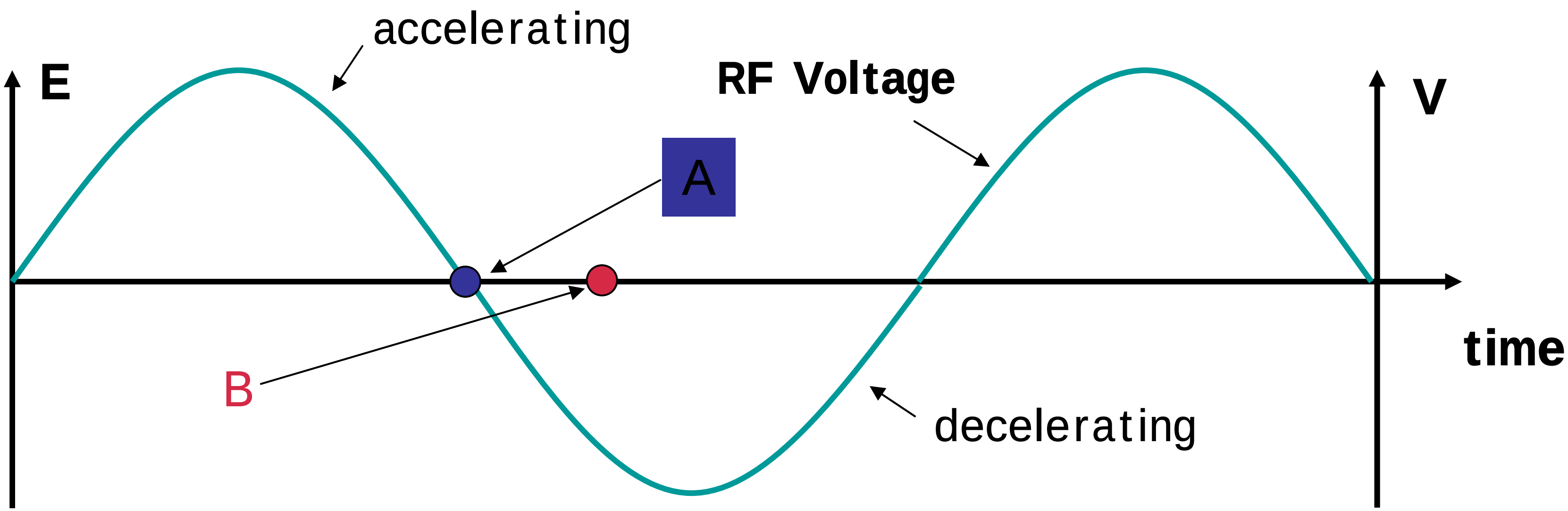
- Imagine two particles A and B, that arrive at the same time in the accelerating cavity (when $V_{rf} = 0V$)
  - For A the energy is such that $F_{rev\ A} = F_{rf}$.
  - The energy of B is higher → $F_{rev\ B} < F_{rev\ A}$

Longitudinal motion beyond transition (2)



- Particle B arrives after A and experiences a decelerating voltage.
  - The energy of B is still higher, **but less** → $F_{rev\ B} < F_{rev\ A}$

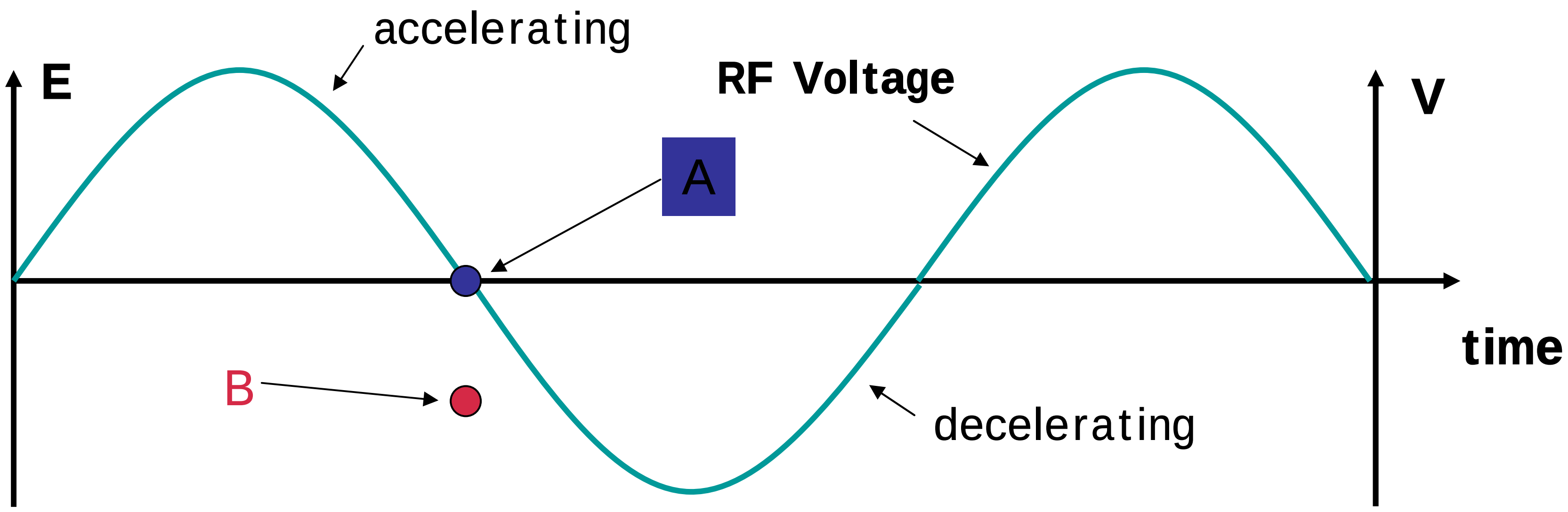
Longitudinal motion beyond transition (3)



- B has now the same energy as A, but arrives still later and experiences therefore a decelerating voltage.

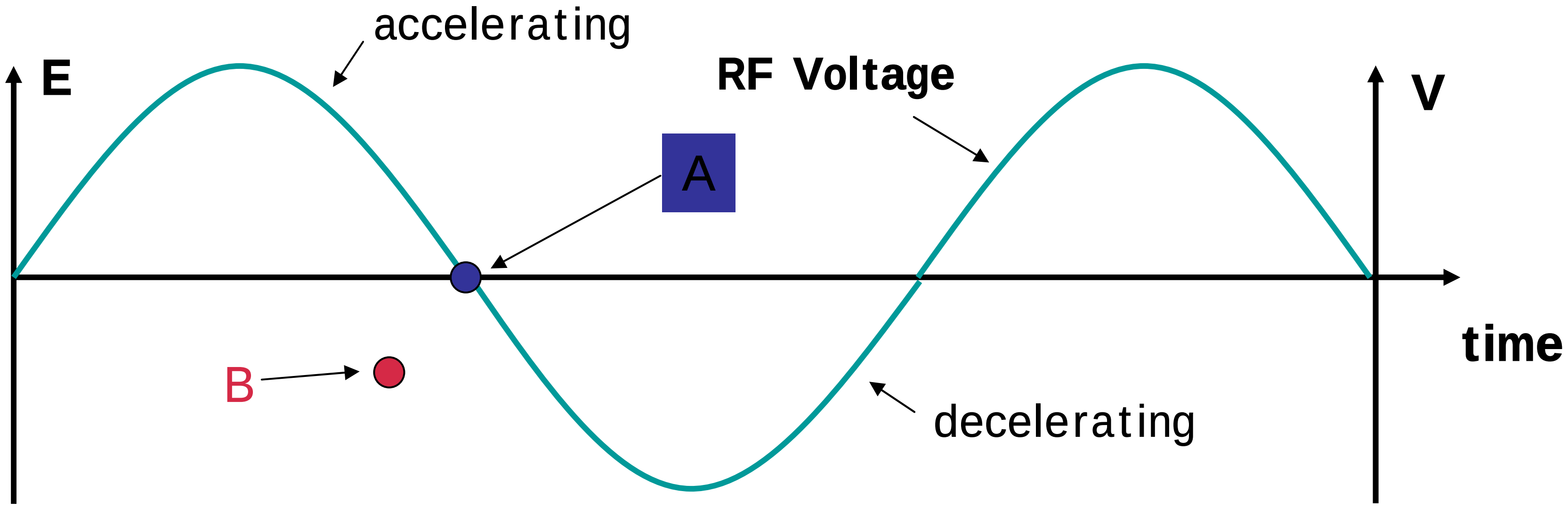
  – $F_{rev\ B} = F_{rev\ A}$

Longitudinal motion beyond transition (4)



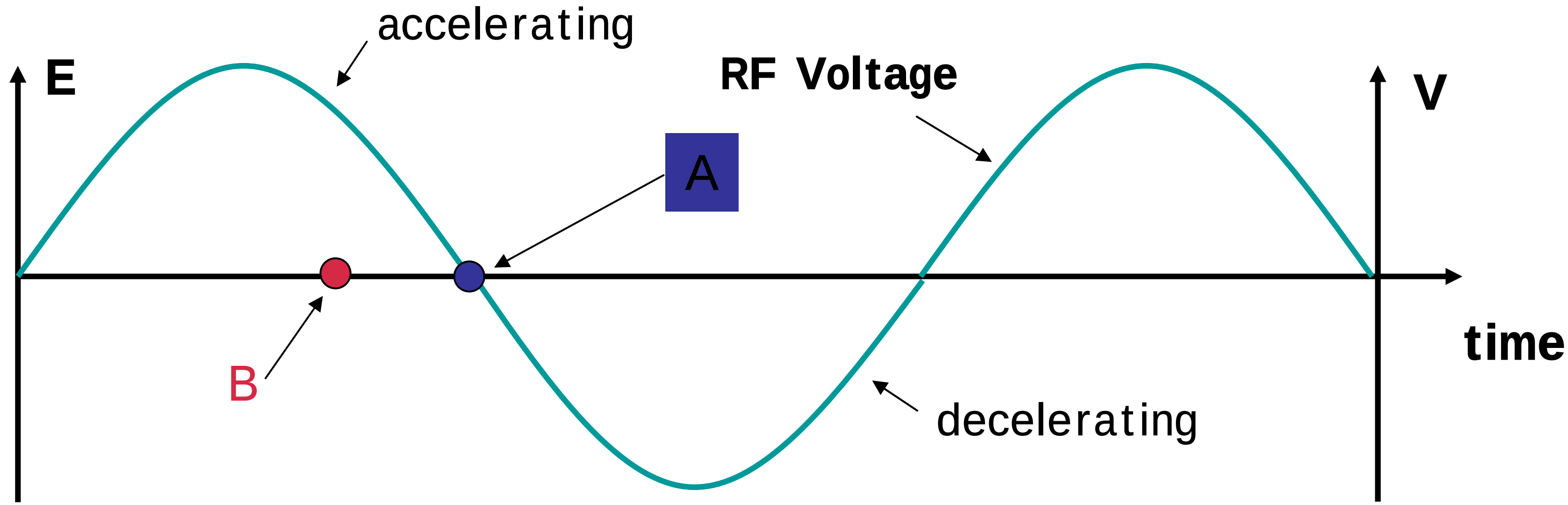
- Particle B has now a lower energy as A, but arrives at the same time
  - $F_{rev\ B} > F_{rev\ A}$

Longitudinal motion beyond transition (5)



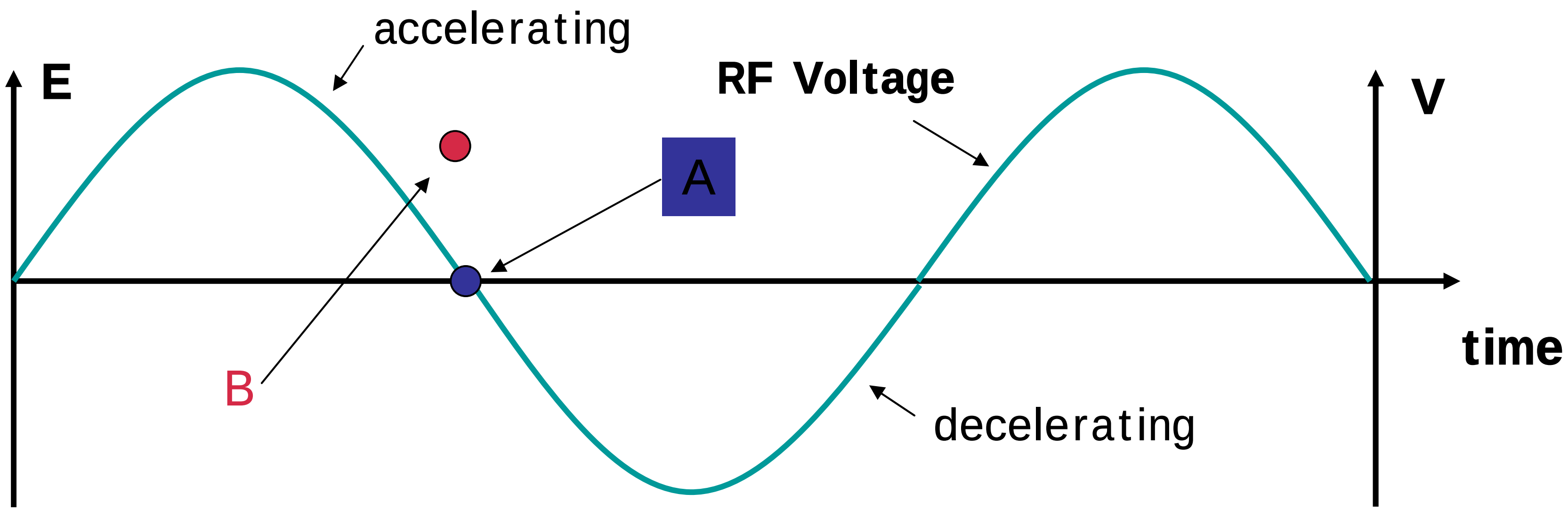
- Particle B has now a lower energy as A, but B arrives before A and experiences an accelerating voltage.
  - $F_{rev\ B} > F_{rev\ A}$

Longitudinal motion beyond transition (6)



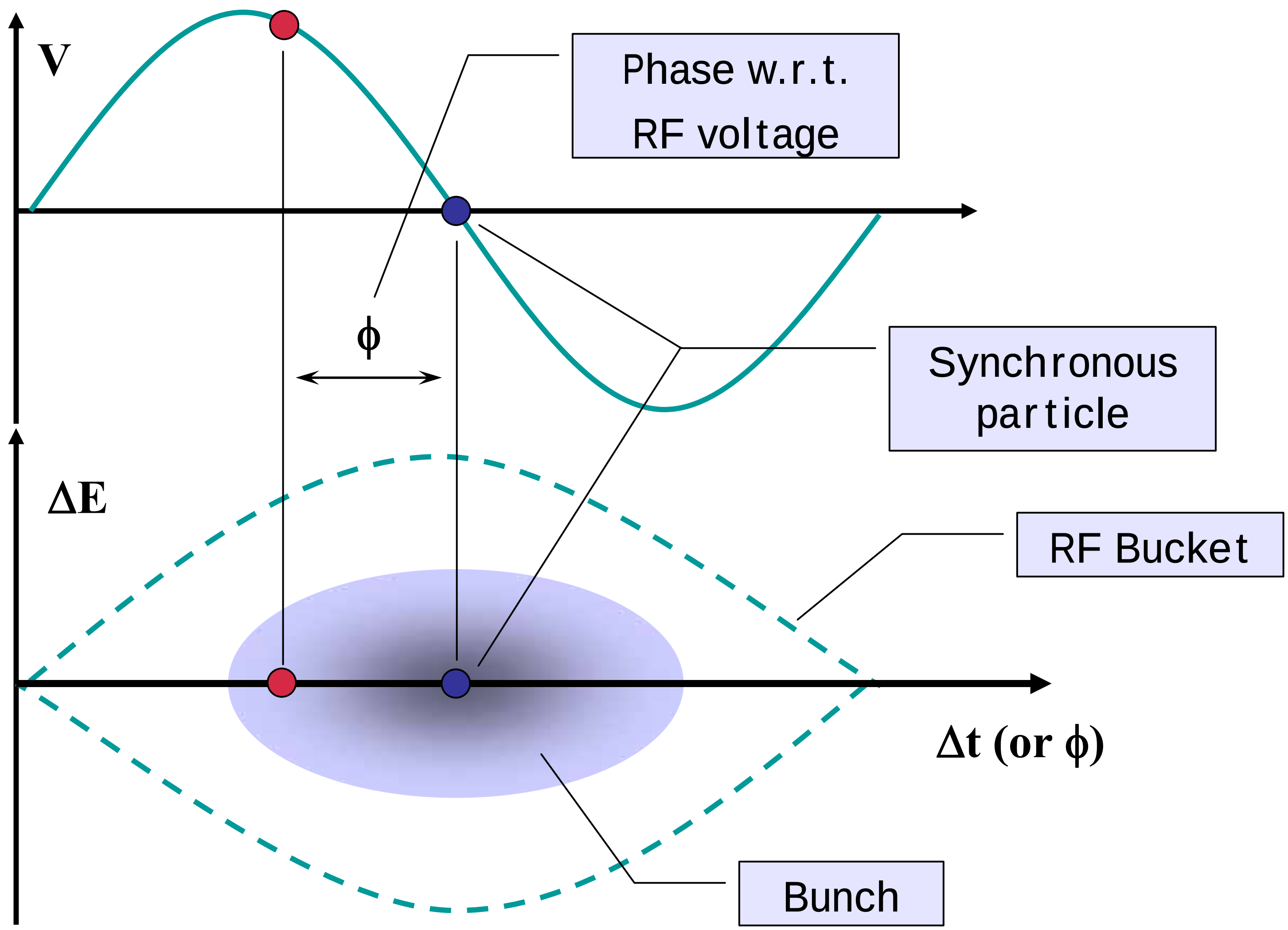
- Particle B has now the same energy as A, but B still arrives before A and experiences an accelerating voltage.
  - $F_{rev\ B} > F_{rev\ A}$

Longitudinal motion beyond transition (7)

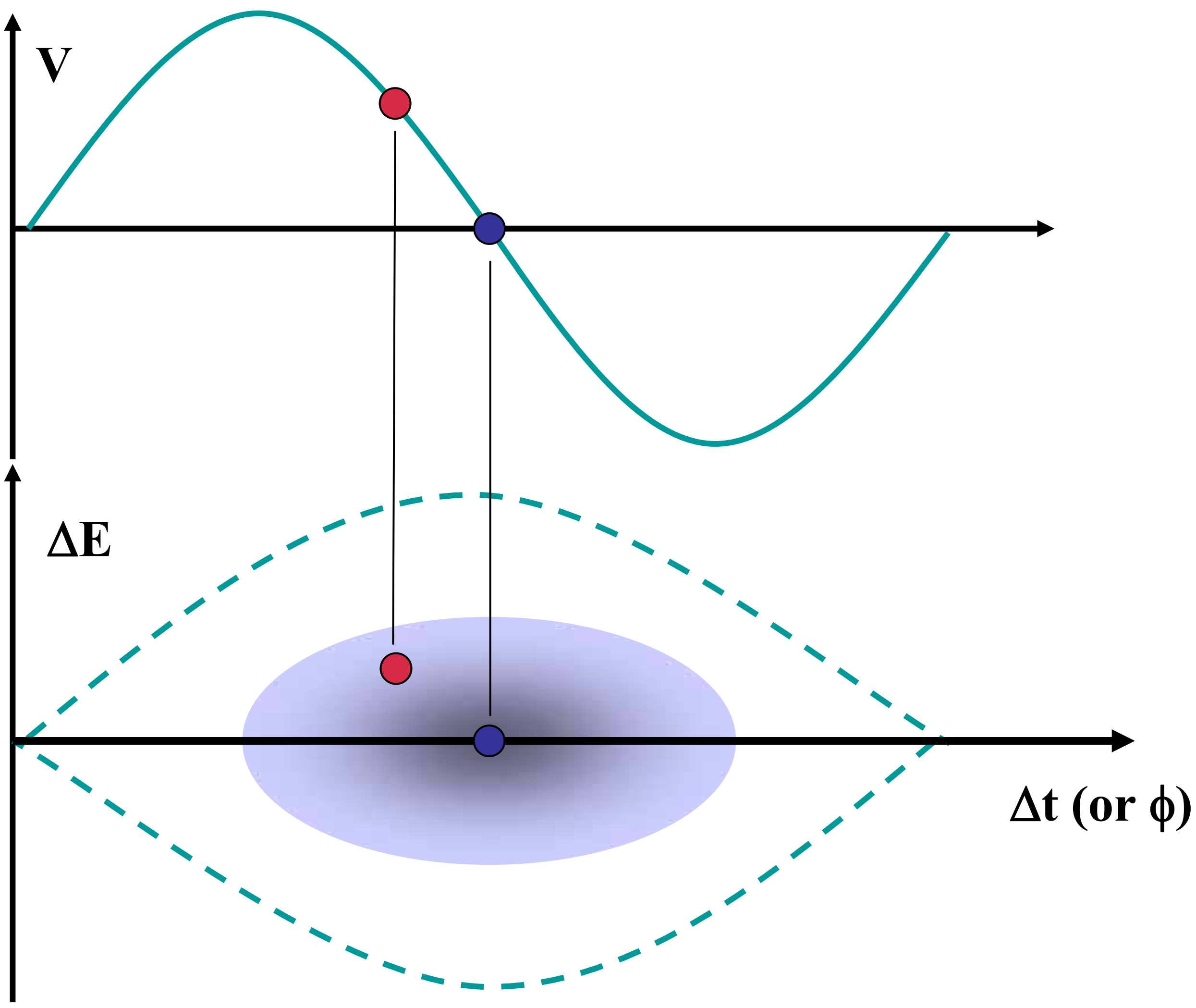


- Particle B has now a higher energy as A and arrives at the same time again....
  - $F_{rev\ B} < F_{rev\ A}$

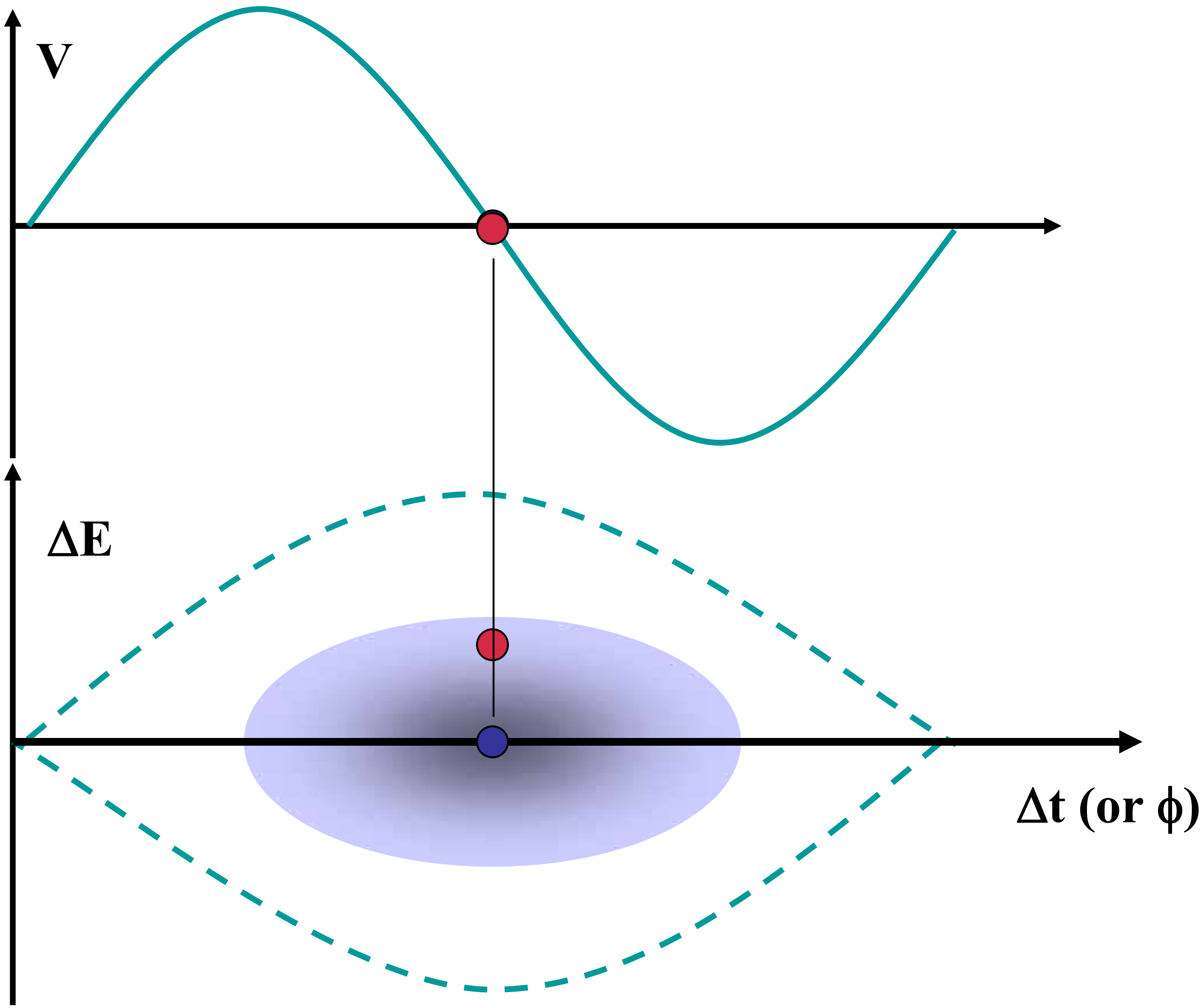
The motion in the bucket (1)



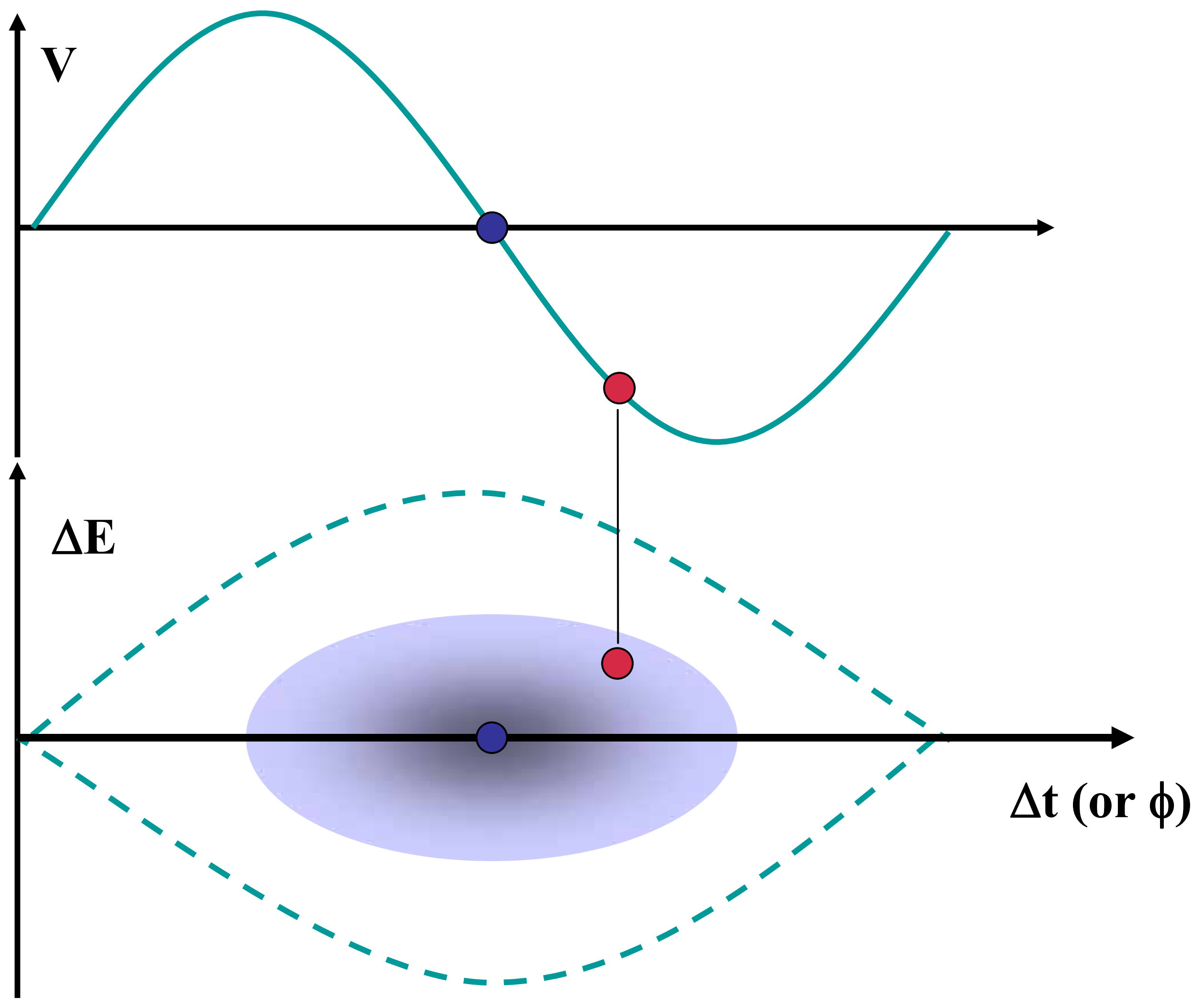
The motion in the bucket (2)



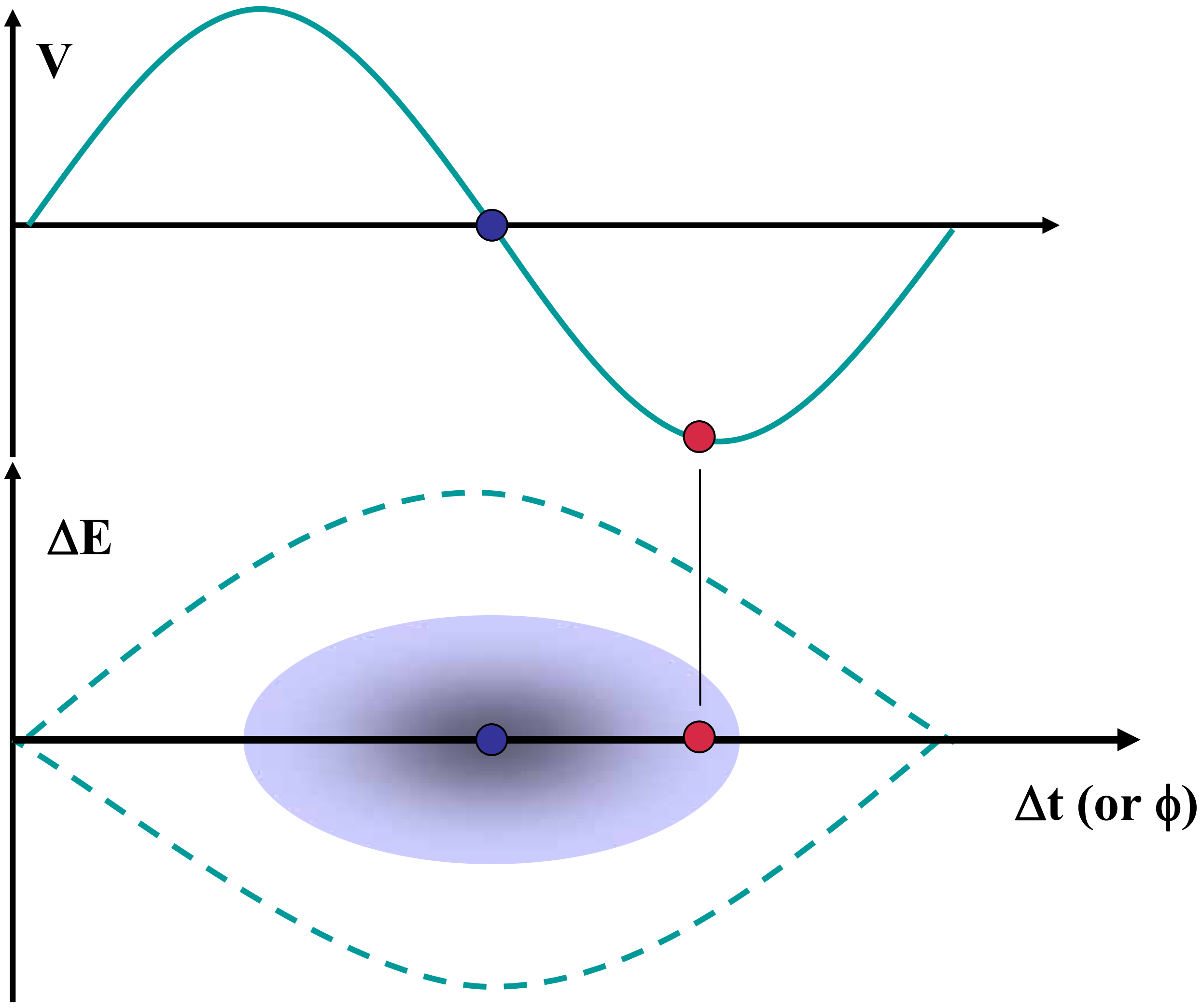
The motion in the bucket (3)



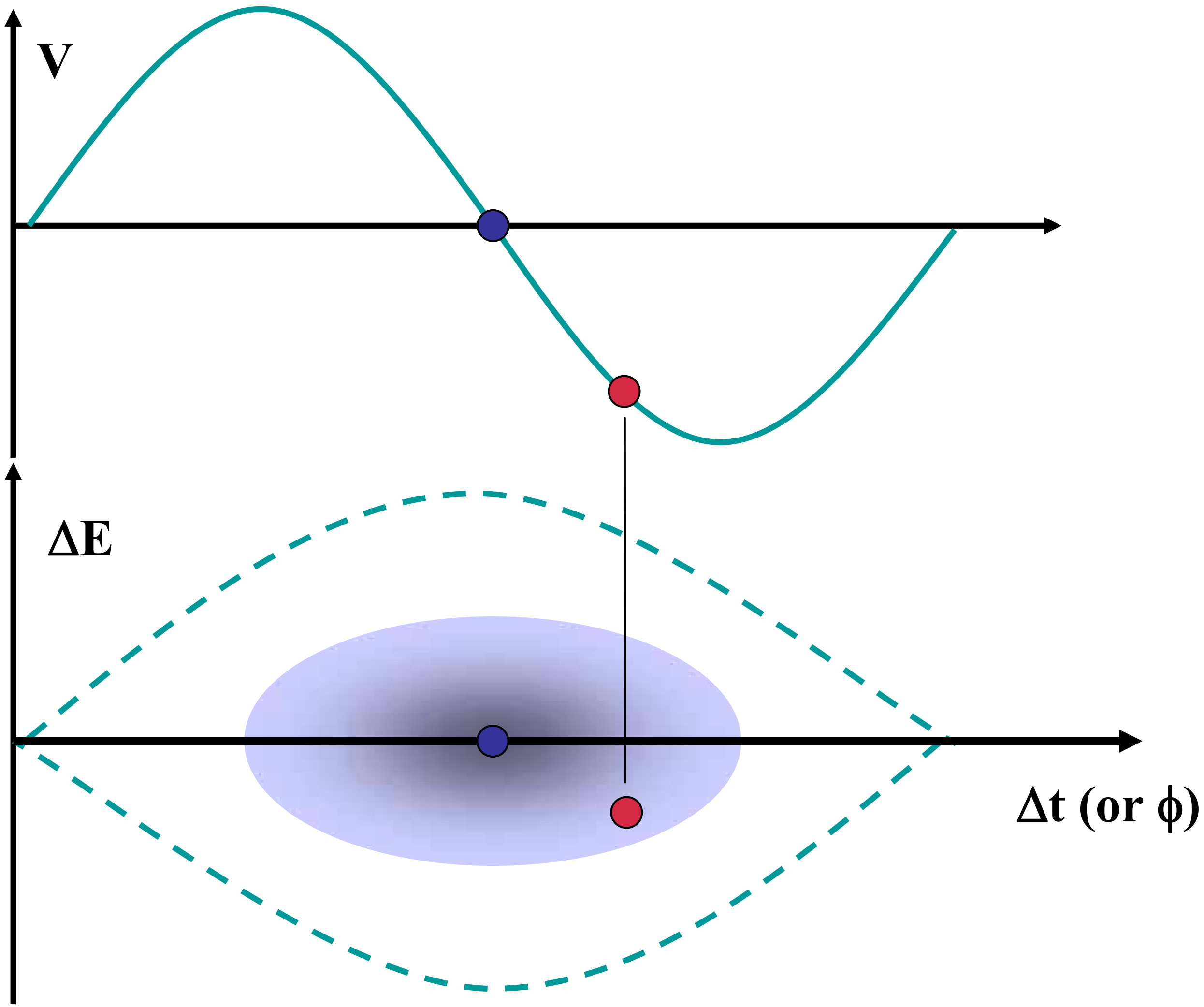
The motion in the bucket (4)



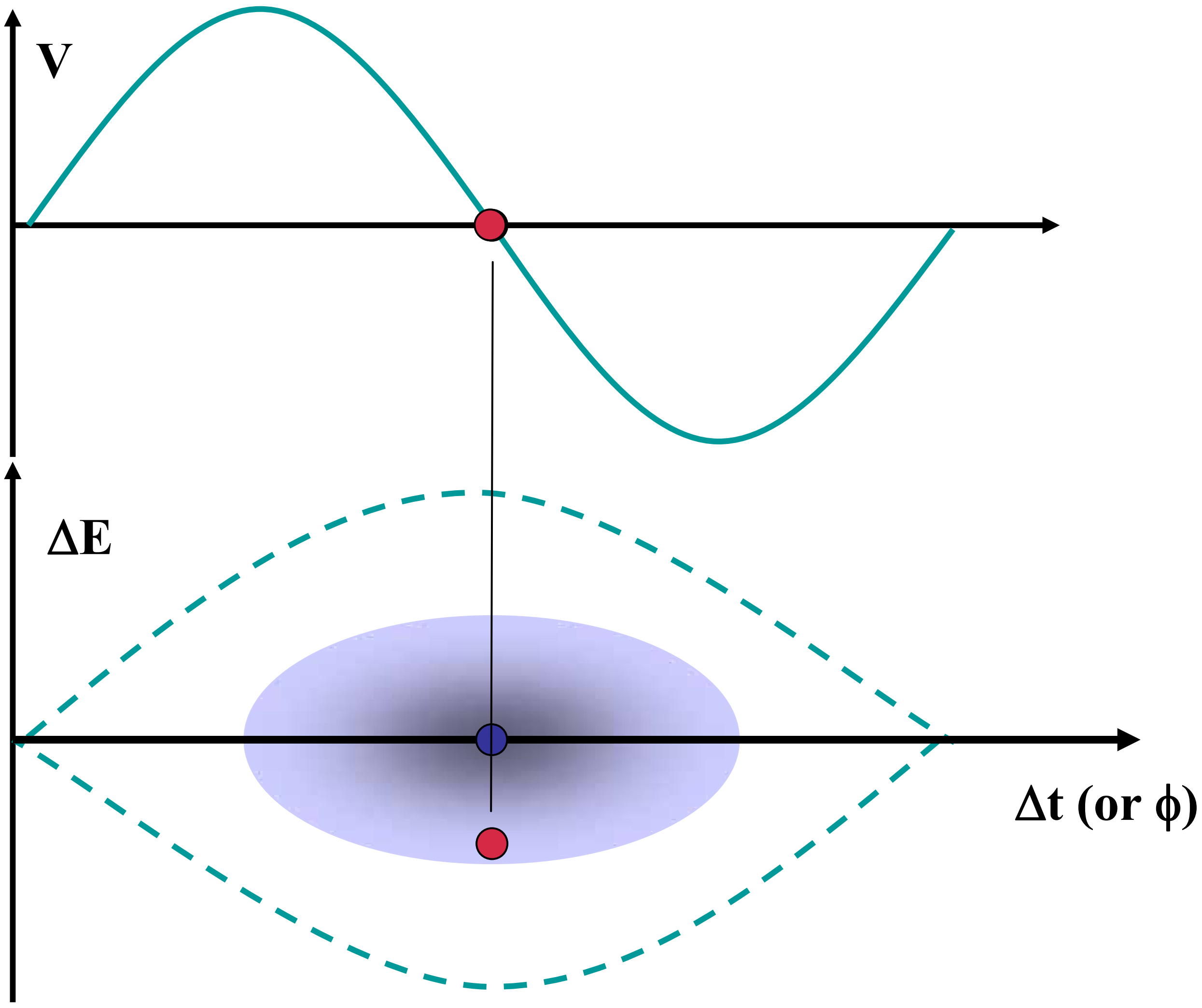
The motion in the bucket (5)



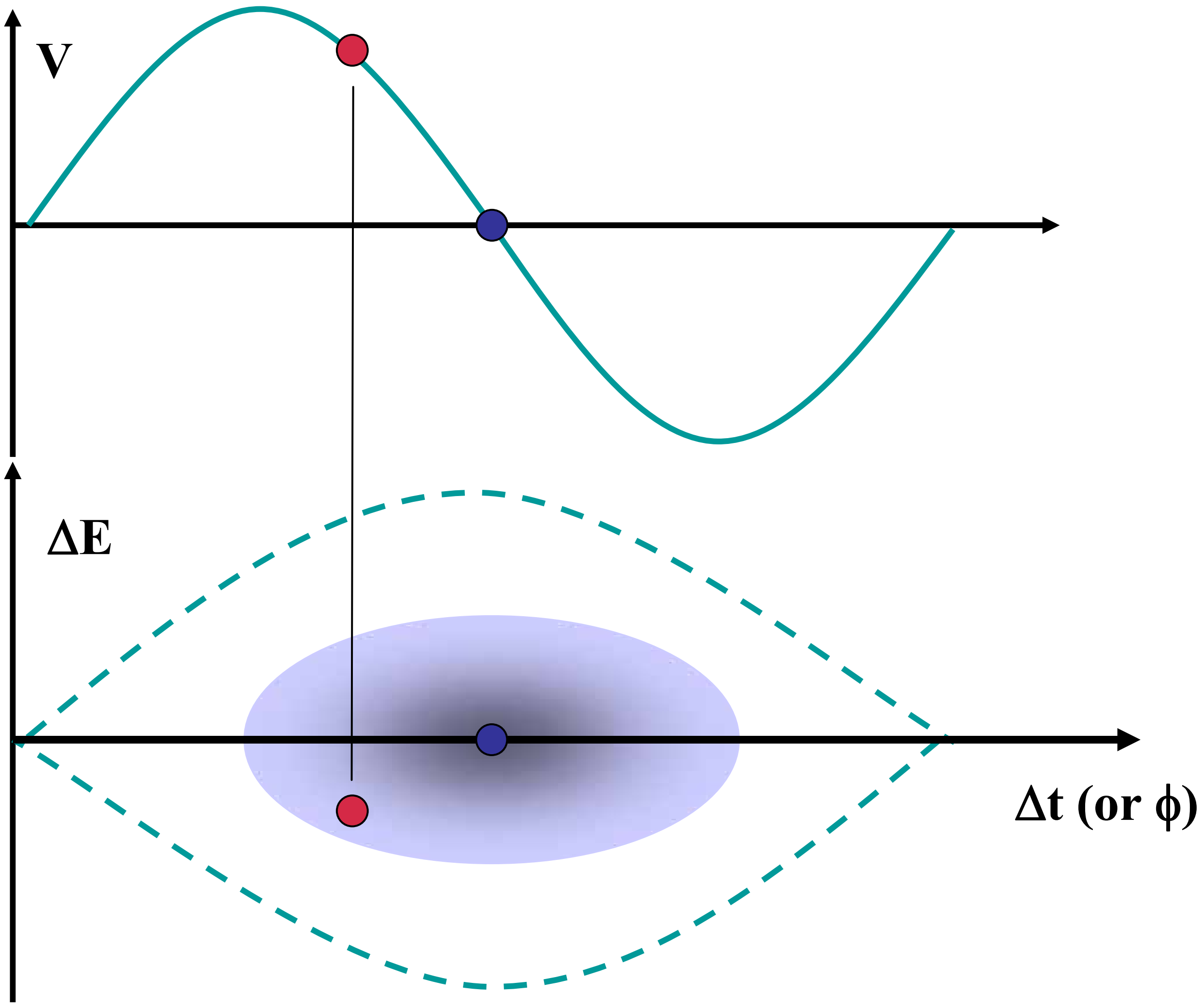
The motion in the bucket (6)



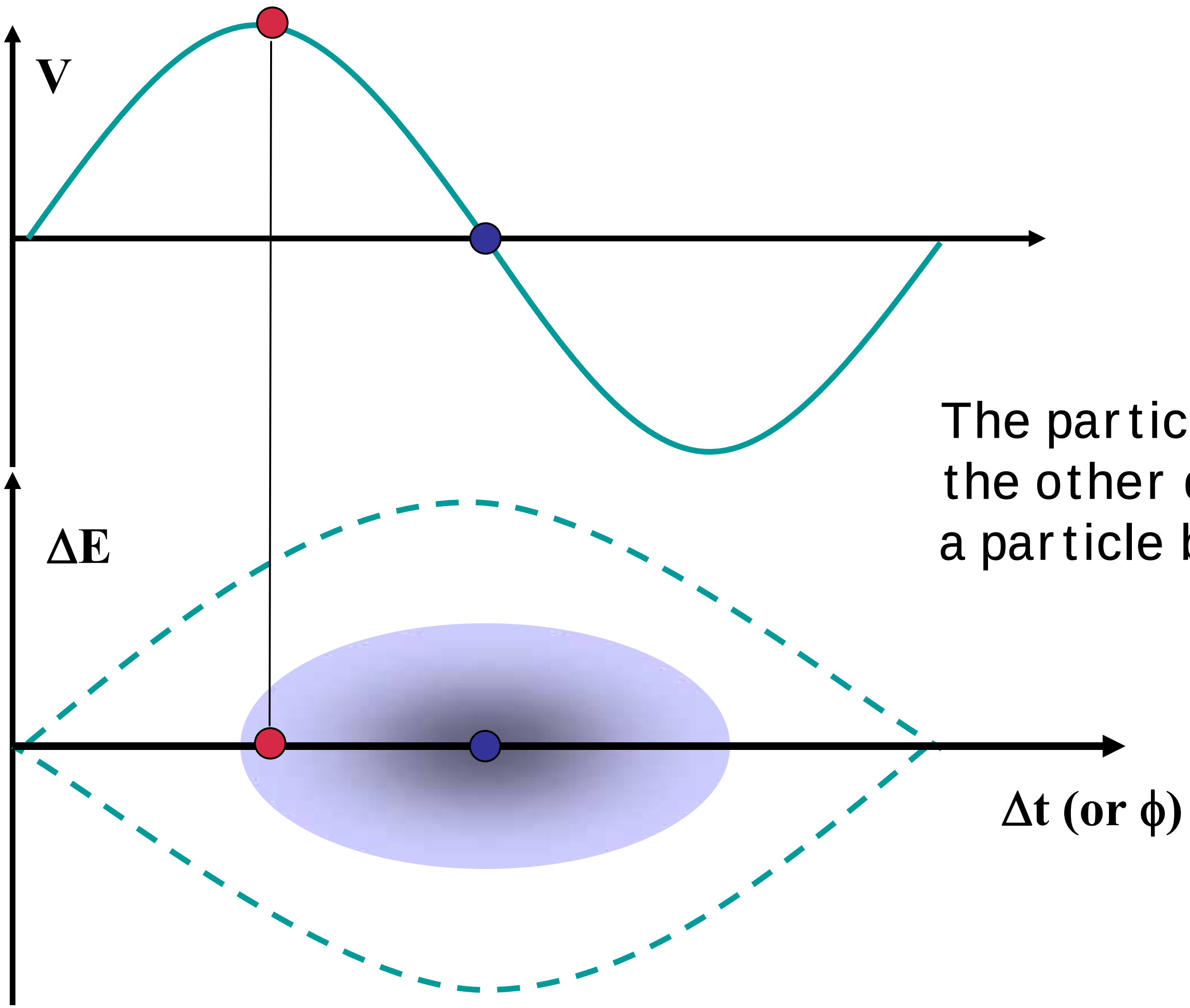
The motion in the bucket (7)



The motion in the bucket (8)

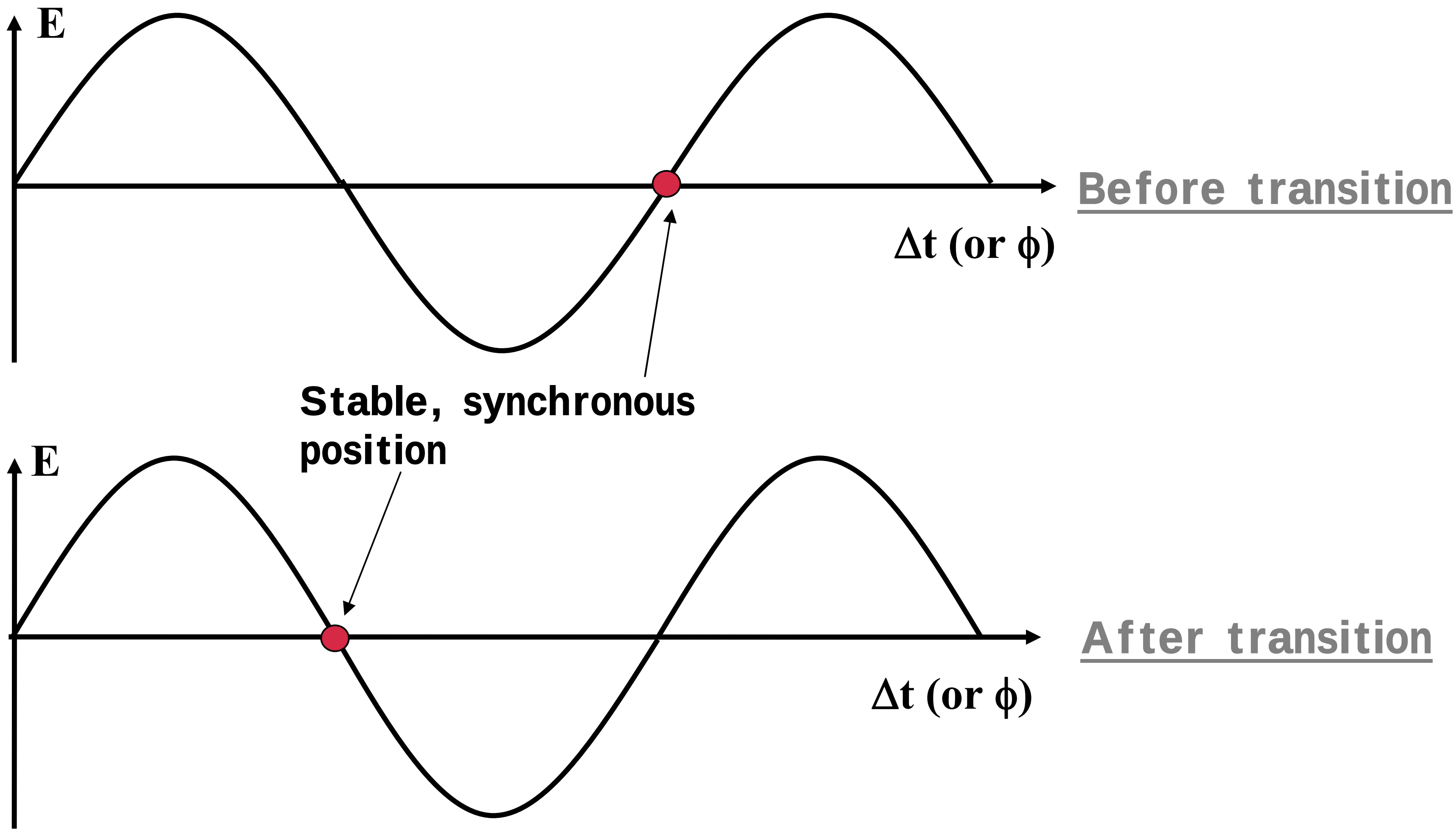


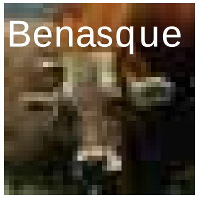
The motion in the bucket (9)



The particle now turns in the other direction w.r.t. a particle below transition

Before and After Transition



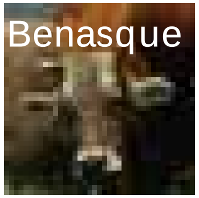


Transition crossing in the PS

- Transition in the PS occurs around 6 GeV/c
  - Injection happens at 2.12 GeV/c
  - Ejection can be done at 3.5 GeV/c up to 26 GeV/c

- Therefore the particles in the PS must nearly always cross transition.
- The beam must stay bunched

- Therefore the phase of the RF must "**jump**" by π at transition



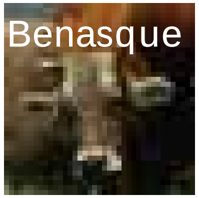
Harmonic number

- Until now we have applied an oscillating voltage with a frequency equal to the revolution frequency.

$$\boxed{F_{rf} = F_{rev}}$$

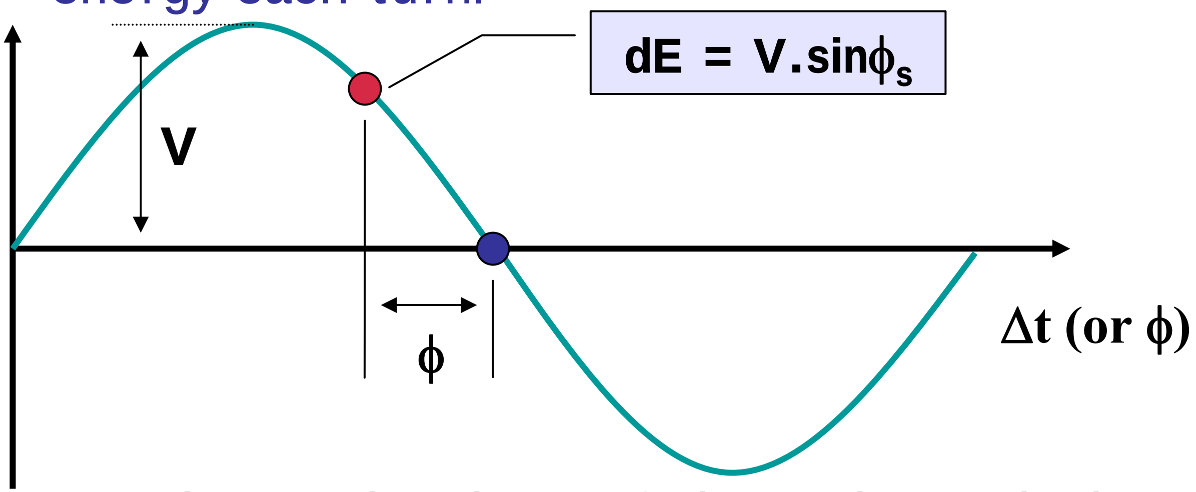
- What will happen when F_{rf} is a multiple of f_{rev} ???

$$\boxed{F_{rf} = h \times F_{rev}}$$



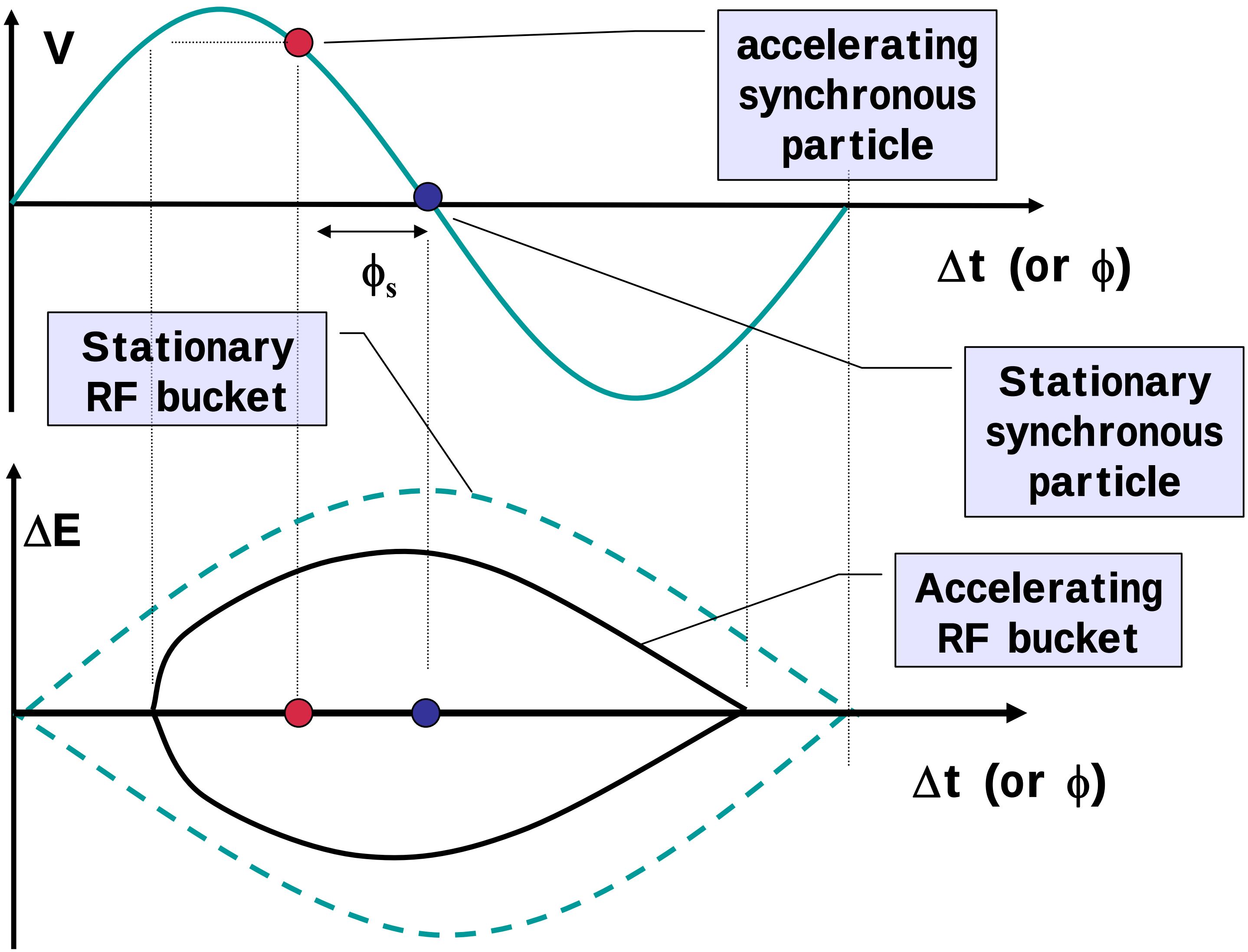
Acceleration

- Increase the magnetic field slightly on each turn.
- The particles will follow a shorter orbit. (**F_{rev} < F_{synch}**)
- Beyond transition, early arrival in the cavity causes a gain in energy each turn.

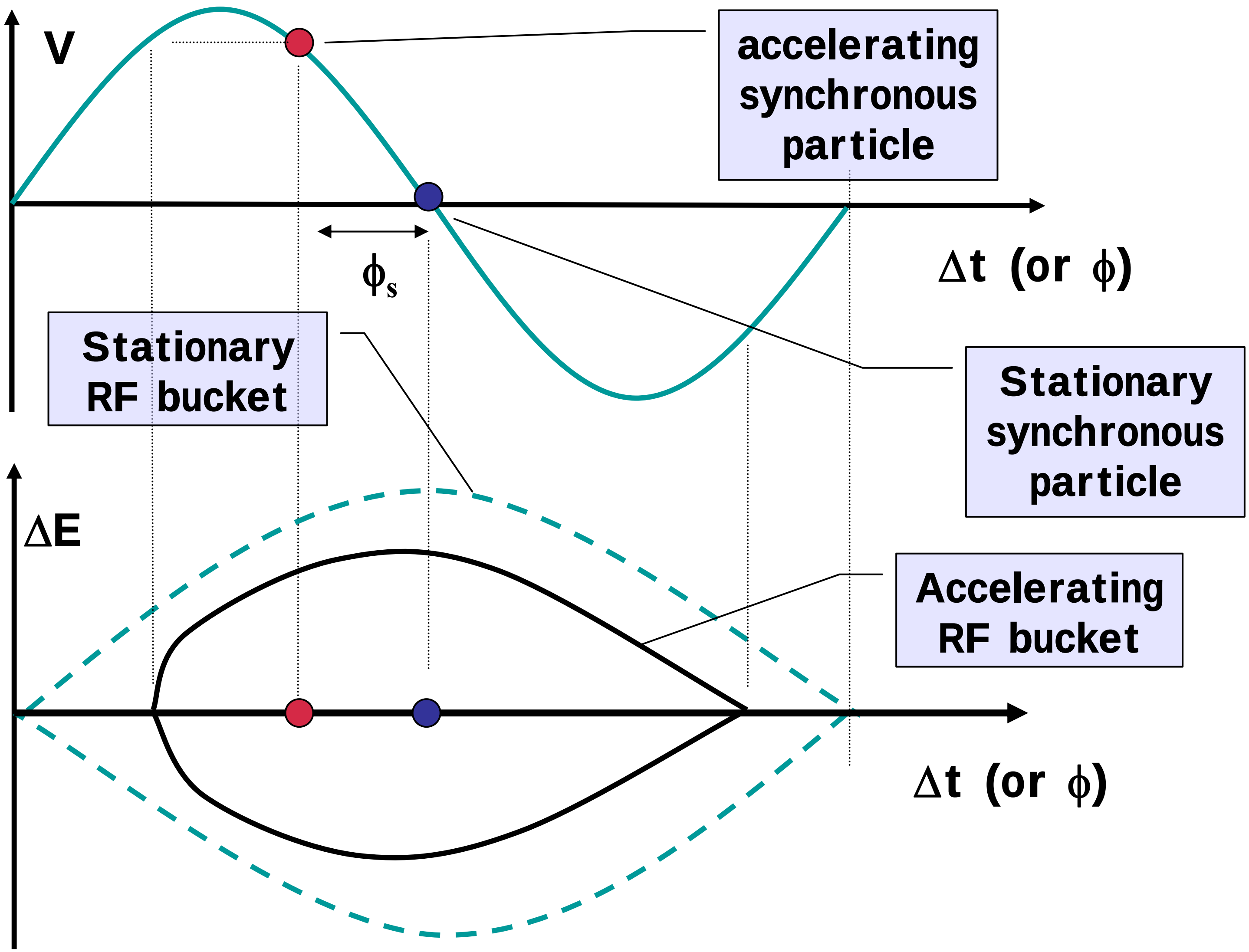


- We change the phase of the cavity such that the new synchronous particle is at ϕ_s and therefore always sees an accelerating voltage
- **V_s = V**sinϕ_s = VΓ = energy gain/turn = **dE**

Acceleration & RF bucket shape (1)



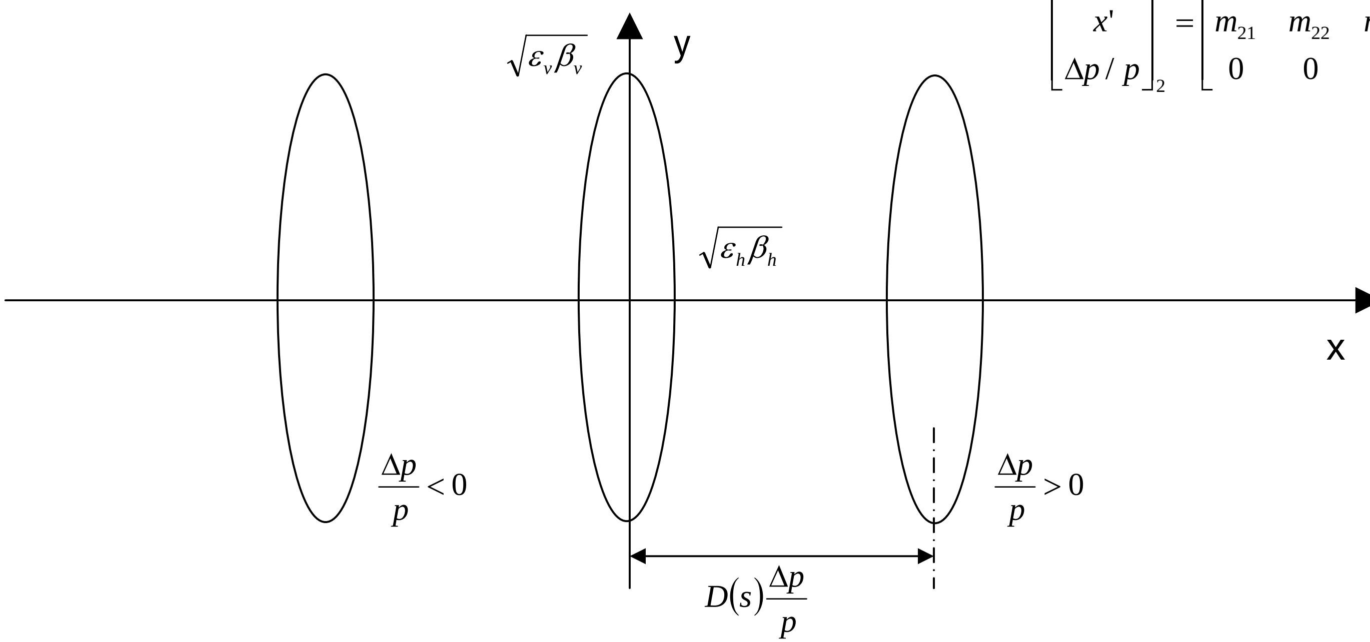
Acceleration & RF bucket shape (1)



Dispersion

- Lets revisit transverse motion with our knew knowledge of momentum and momentum spread!

$$\begin{bmatrix} x \\ x' \\ \Delta p / p \end{bmatrix}_2 = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ x' \\ \Delta p / p \end{bmatrix}_1$$



Chromaticity

The focusing strength of our quadrupoles depends on the beam momentum, “**p**”

$$k = \frac{dBy}{dx} \times \frac{1}{B\rho} \qquad 3.3356.p$$

Therefore a spread in momentum causes a spread in focusing strength

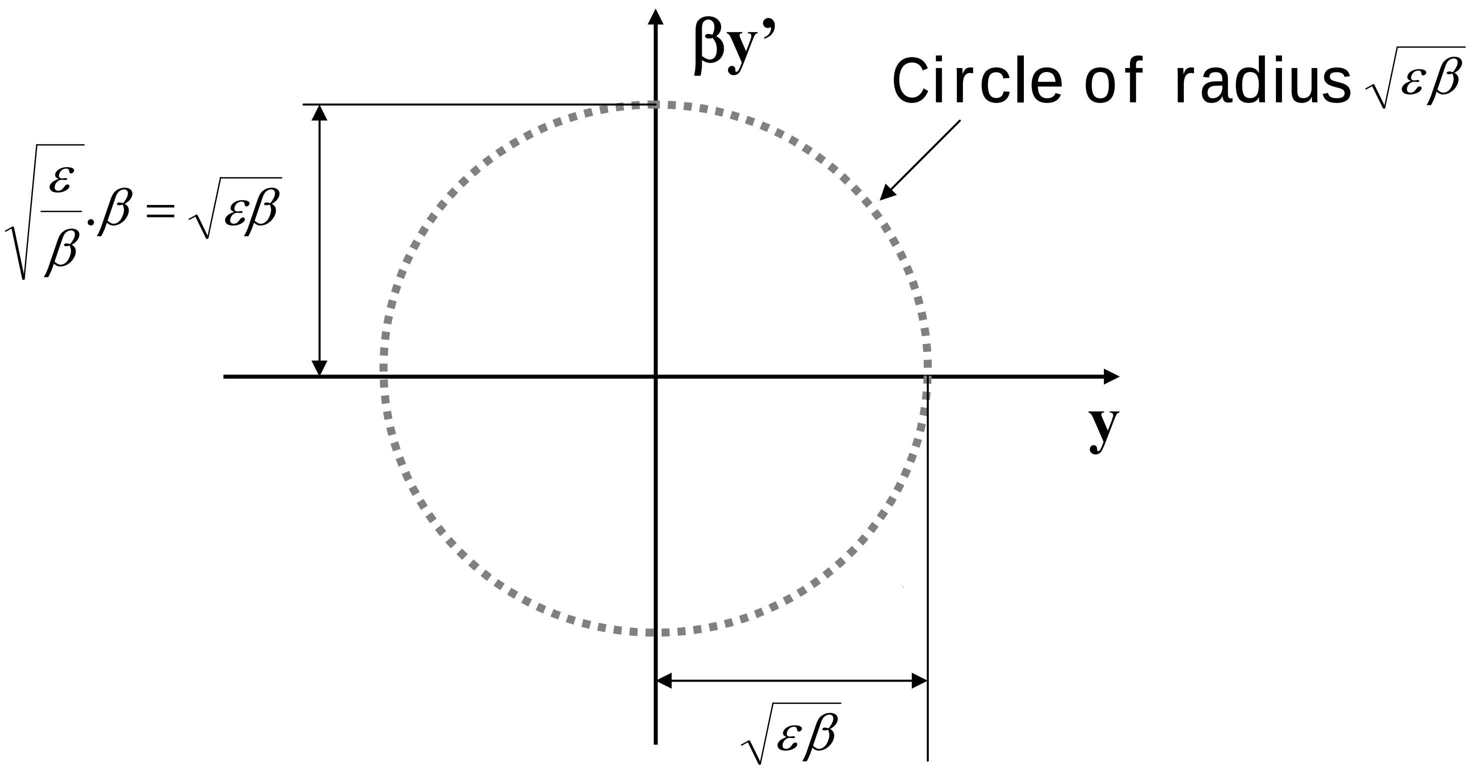
$$\frac{\Delta k}{k} = -\frac{\Delta P}{P}$$

But **Q** depends on the “**k**” of the quadrupoles

$$\frac{\Delta Q}{Q} \alpha \frac{\Delta p}{p} \longrightarrow \frac{\Delta Q}{Q} = \xi \frac{\Delta p}{p}$$

The constant here is called **Chromaticity**

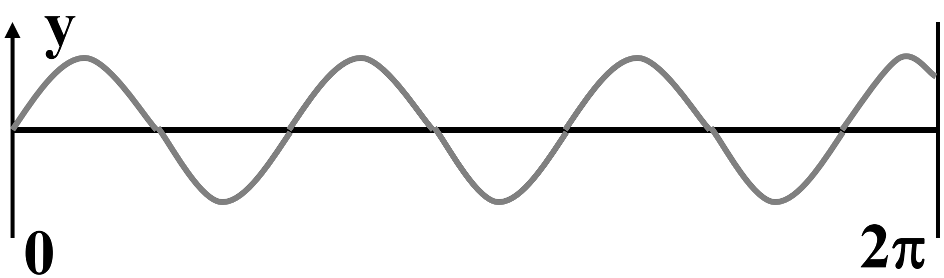
Normalised Phase Space



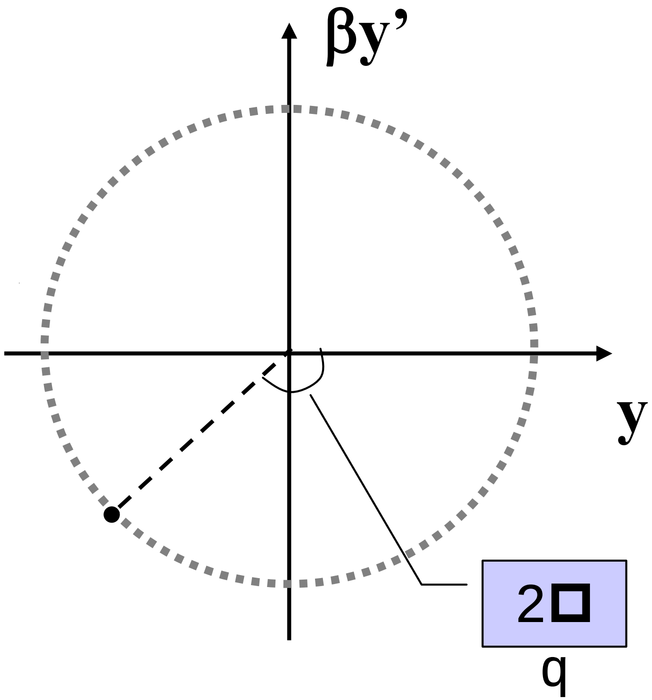
- ✓ By multiplying the y-axis by β the phase space is normalised and the ellipse turn into a circle.

Phase Space & Betatron Tune

- ✓ If we fold out a trajectory of a particle that makes one turn in our machine with a tune of Q = 3.333, we get:

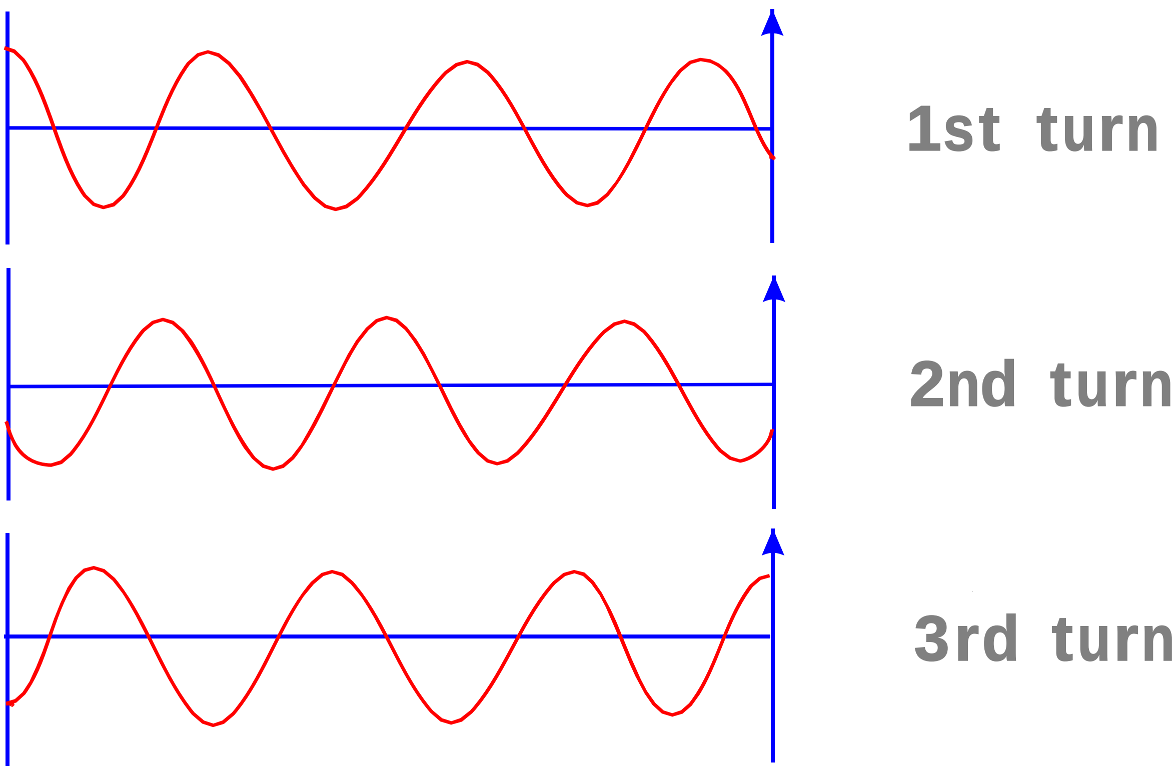


- ✓ This the same as going 3.333 time around on the circle in phase space.
- ✓ The net result is 0.333 times around on the circle.
- ✓ q is the fractional part of Q
- ✓ So here q = 0.333.



Benasque

Q = 3.333 in more detail

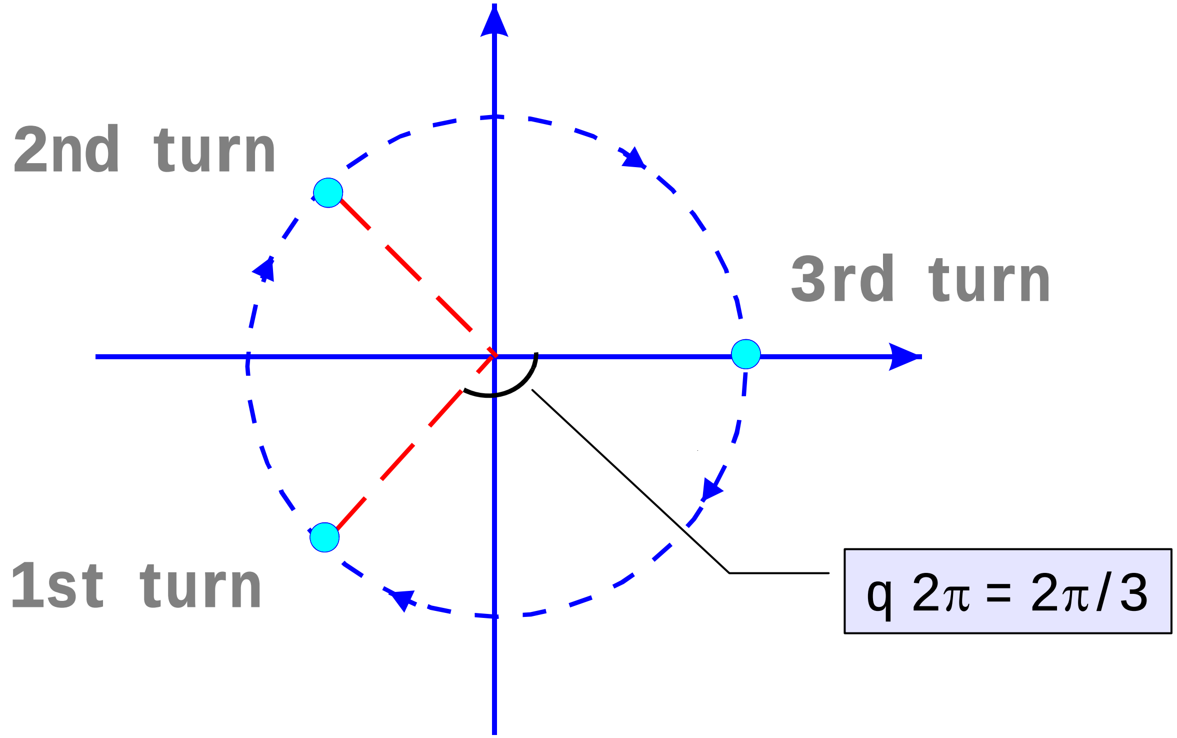


Third order resonant betatron oscillation

3Q = 10, Q = 3.333, q = 0.333

Q = 3.333 in Phase Space

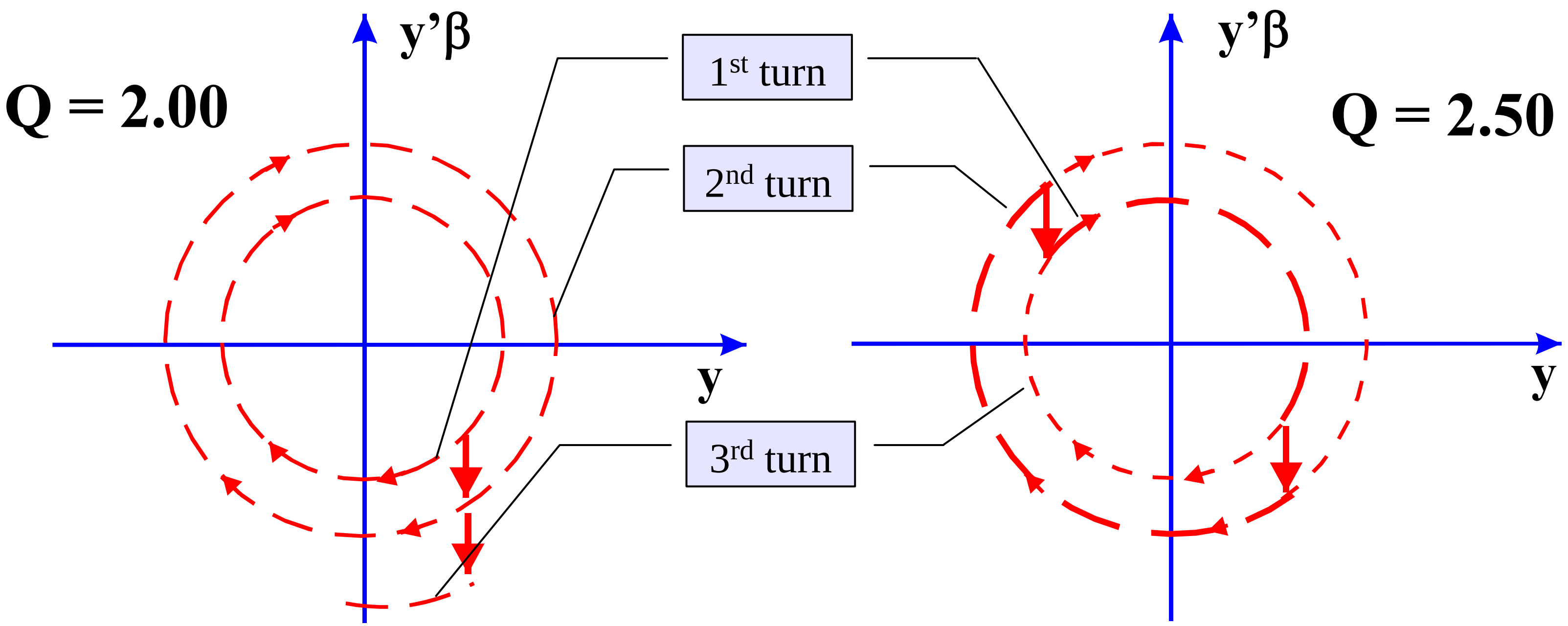
✓ Third order resonance on a normalised phase space plot



Machine imperfections

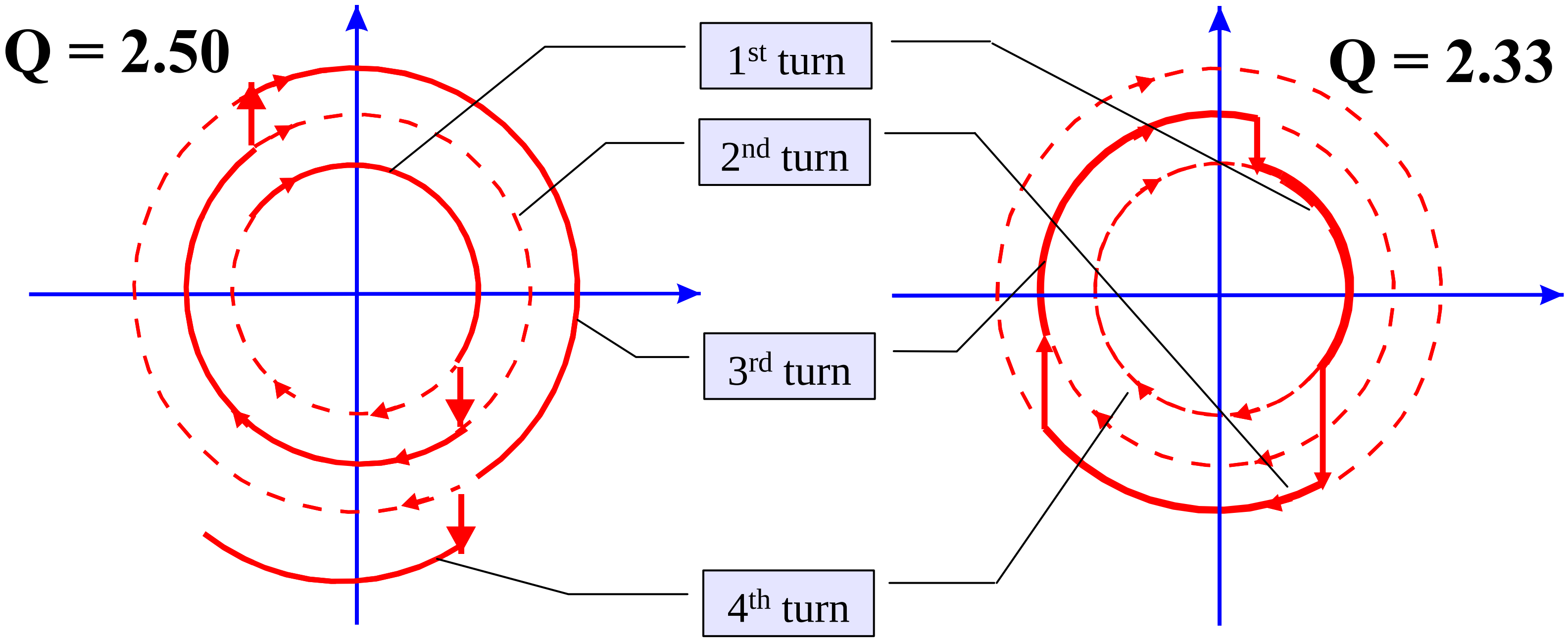
- It is not possible to construct a perfect machine.
  - Magnets can have imperfections
  - The alignment in the de machine has not zero tolerance.
  - Etc...
- So, we have to ask ourselves:
  - What will happen to the betatron oscillation due to the different field errors.
  - And therefore consider errors in dipoles, quadrupoles, sextupoles, etc...
- We will have a look at the beam behaviour as a function of ‘Q’
- How is it influenced by these resonant conditions?

Dipole (deflection independent of position)



- ✓ For <u>**Q = 2.00**</u>: Oscillation induced by the <u>**dipole kick**</u> grows on each turn and the particle is lost (<u>**1st order resonance Q = 2**</u>).
- ✓ For <u>**Q = 2.50**</u>: Oscillation is cancelled out <u>**every second turn**</u>, and therefore the particle <u>**motion is stable**</u>.

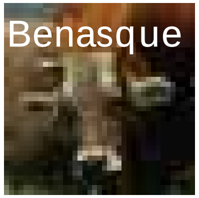
Quadrupole (deflection ∝ position)



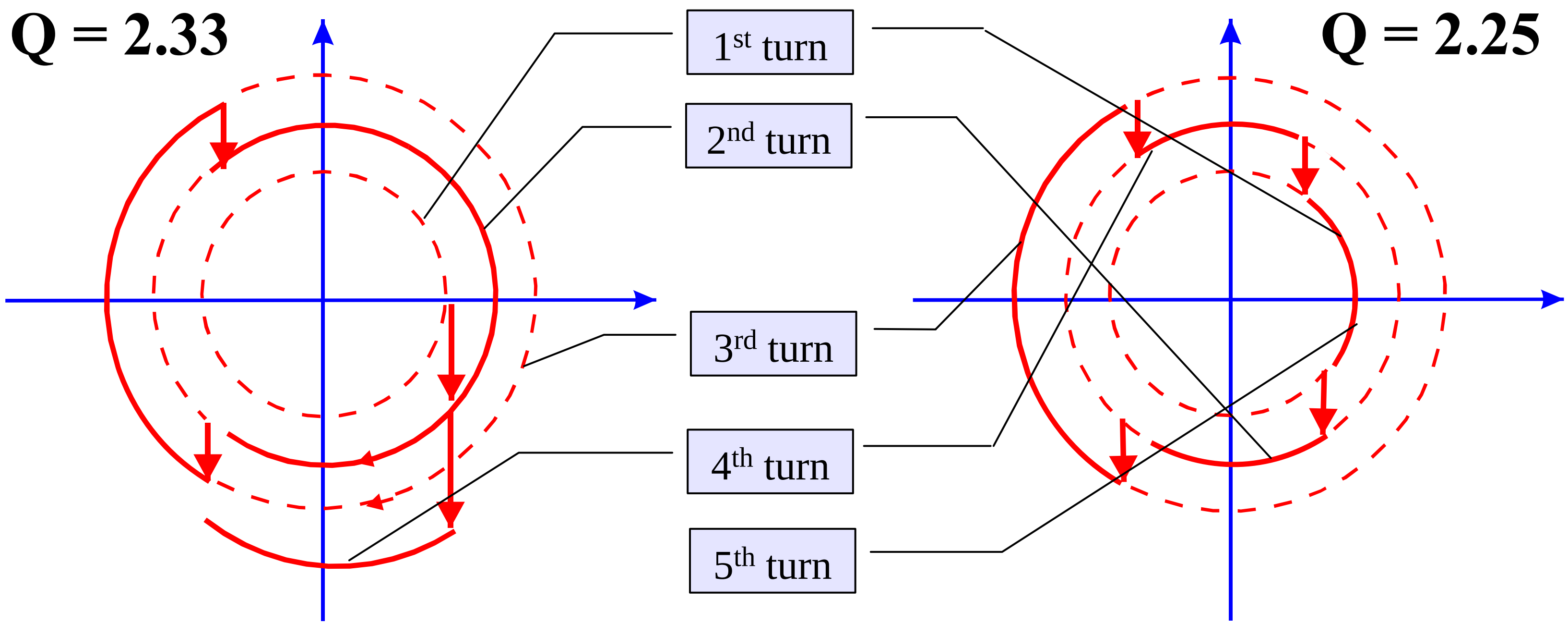
- ✓ For **Q = 2.50**: Oscillation induced by the **quadrupole kick** grows on each turn and the particle is lost

  (2nd order resonance 2Q = 5)

- ✓ For **Q = 2.33**: Oscillation is cancelled out **every third turn**, and therefore the particle **motion is stable**.



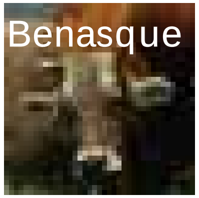
Sextupole (deflection ∝ position²)



- ✓ For Q = 2.33: Oscillation induced by the sextupole kick grows on each turn and the particle is lost

  (3rd order resonance 3Q = 7)

- ✓ For Q = 2.25: Oscillation is cancelled out every fourth turn, and therefore the particle motion is stable.



Instabilities (1)

- Until now we have only considered independent particle motion.
- We call this incoherent motion.
  - single particle synchrotron/betatron oscillations
  - each particle moves independently of all the others

- Now we have to consider what happens if all particles move in phase, coherently, in response to some excitations

Synchrotron & betatron oscillations

Instabilities (2)

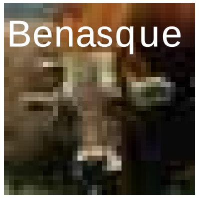
- We cannot ignore interactions between the charged particles
- They interact with each other in two ways:

Space charge effects, intra beam scattering

  - Direct Coulomb interaction between particles

Longitudinal and transverse beam instabilities

  - Via the vacuum chamber

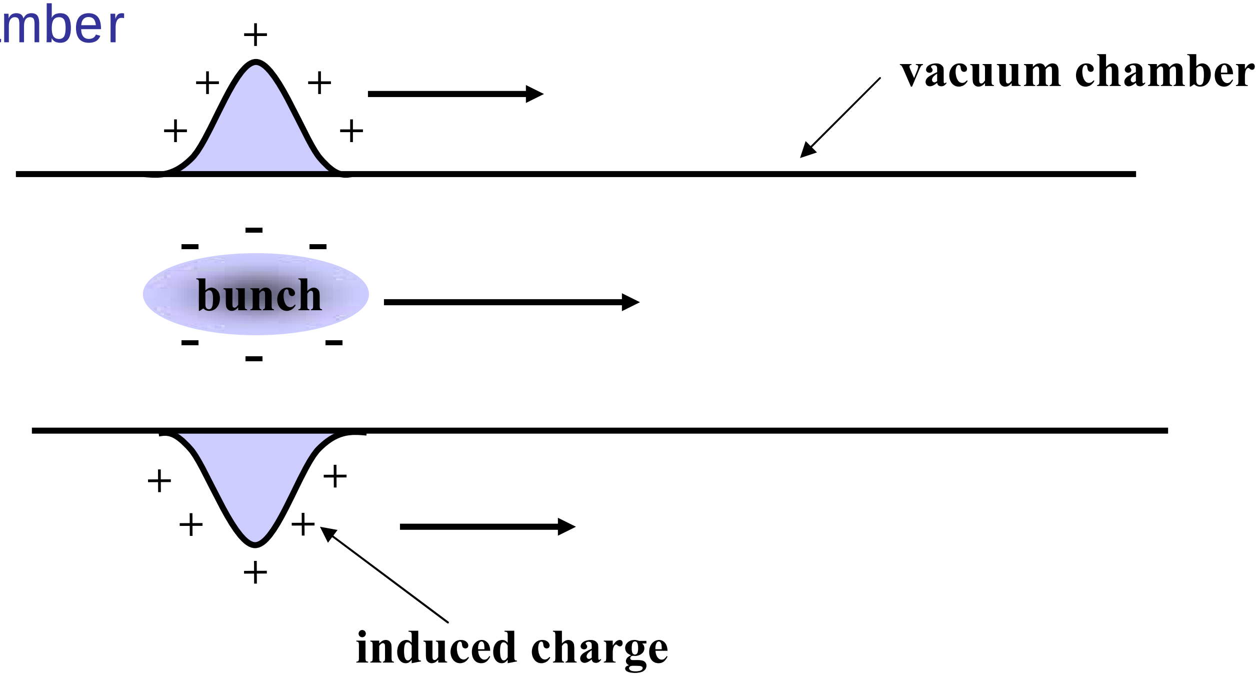


Why do Instabilities arise?

- A circulating bunch induces electro magnetic fields in the vacuum chamber
- These fields act back on the particles in the bunch
- Small perturbation to the bunch motion, changes the induced EM fields
- If this change amplifies the perturbation then we have an **instability**

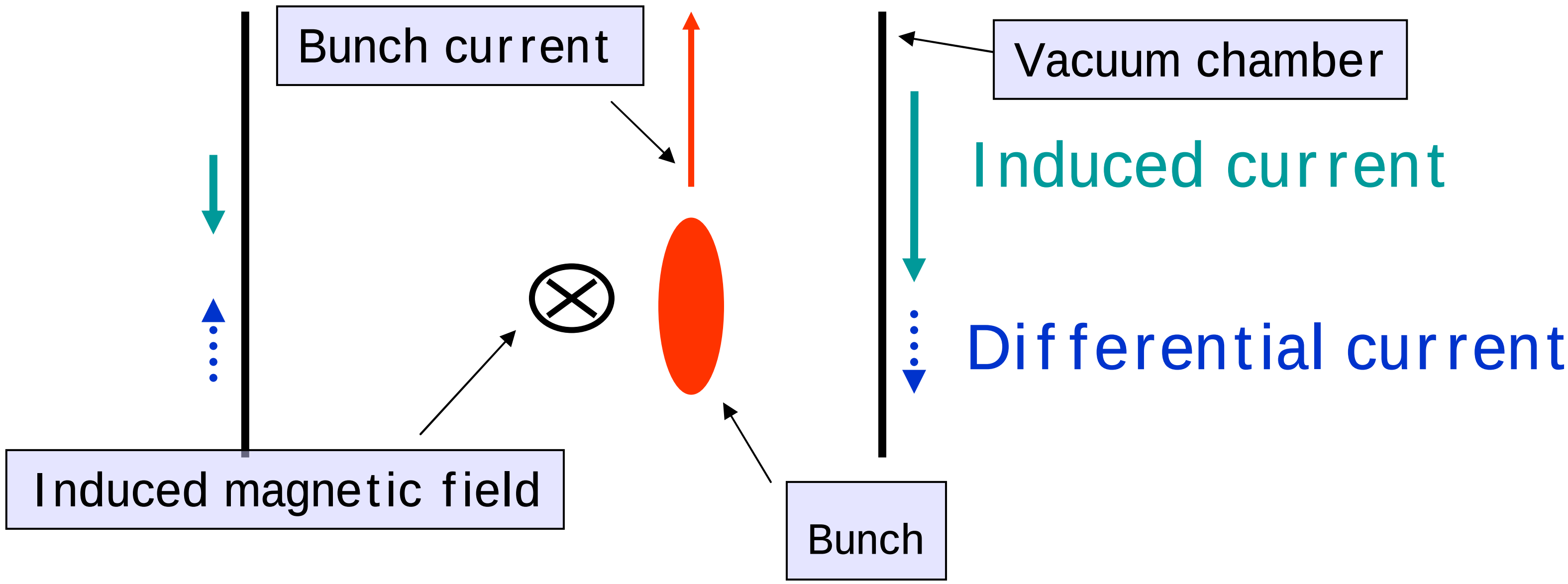
Longitudinal Instabilities

- A circulating bunch creates an image current in vacuum chamber



- The induced image current is the same size but has the opposite sign to the bunch current
- The longitudinal impedance of the vacuum chamber is important!

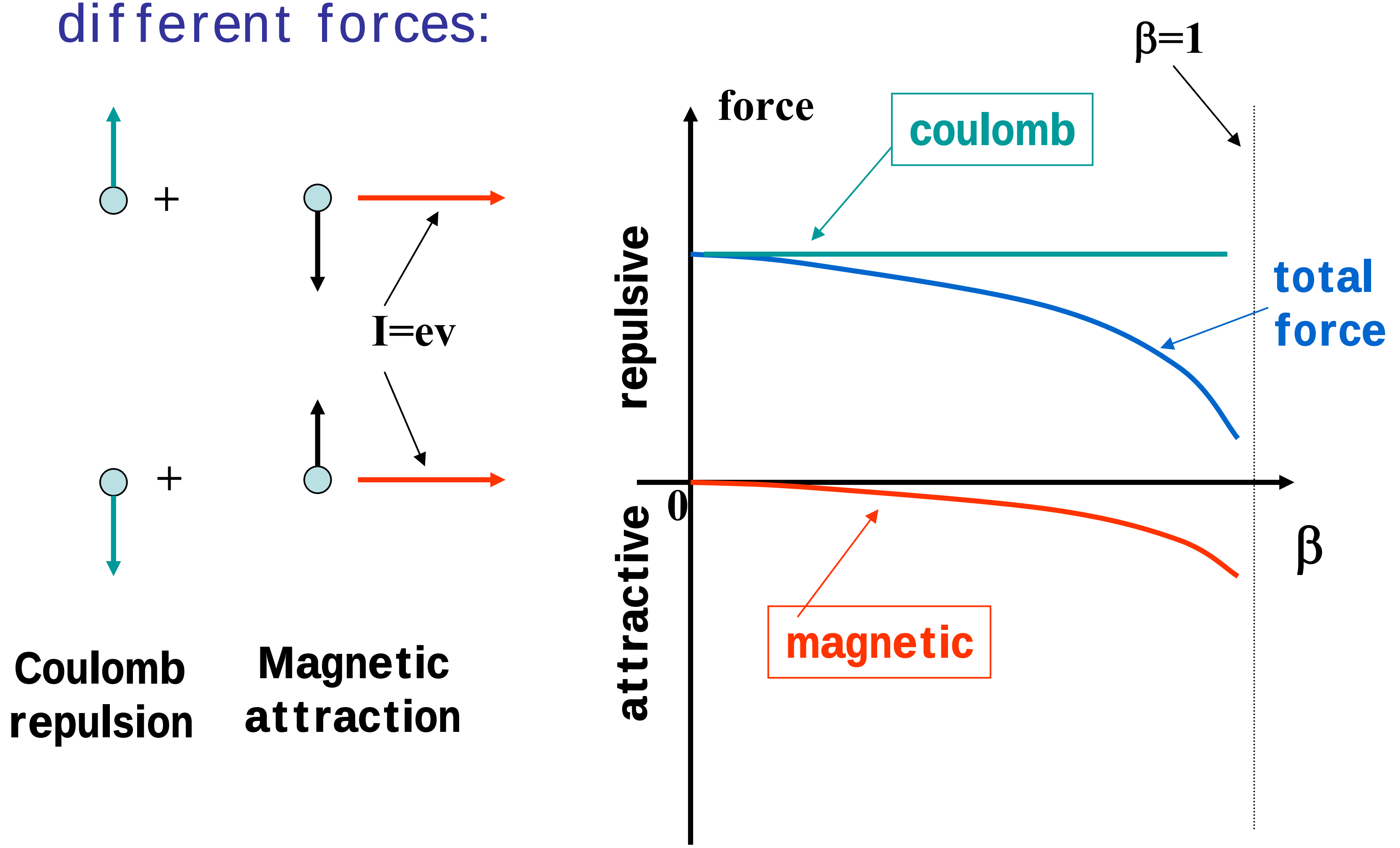
Coherent Transverse Oscillation



- If the bunch is displaced form the centre of the vacuum chamber it will drive a differential wall current
- This leads to a magnetic field, which deflects the bunch
- The differential current and the transverse impedance is important!

Space Charge effects (1)

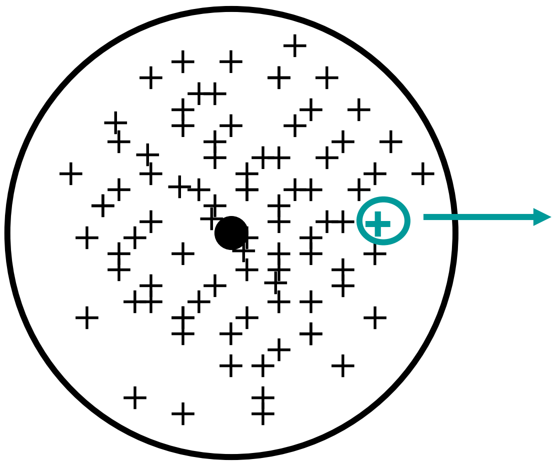
- Between two charged particles in a beam we have different forces:



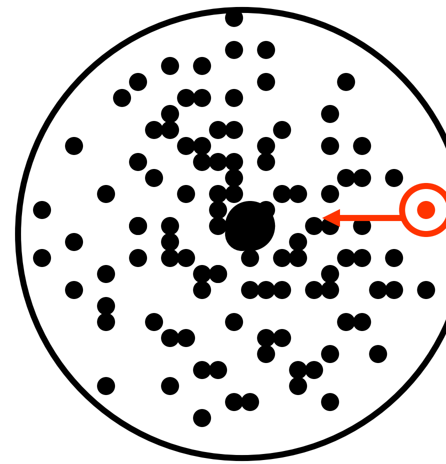
Benasque

Space Charge effects (2)

- For many particles in a beam we can represent it as following:



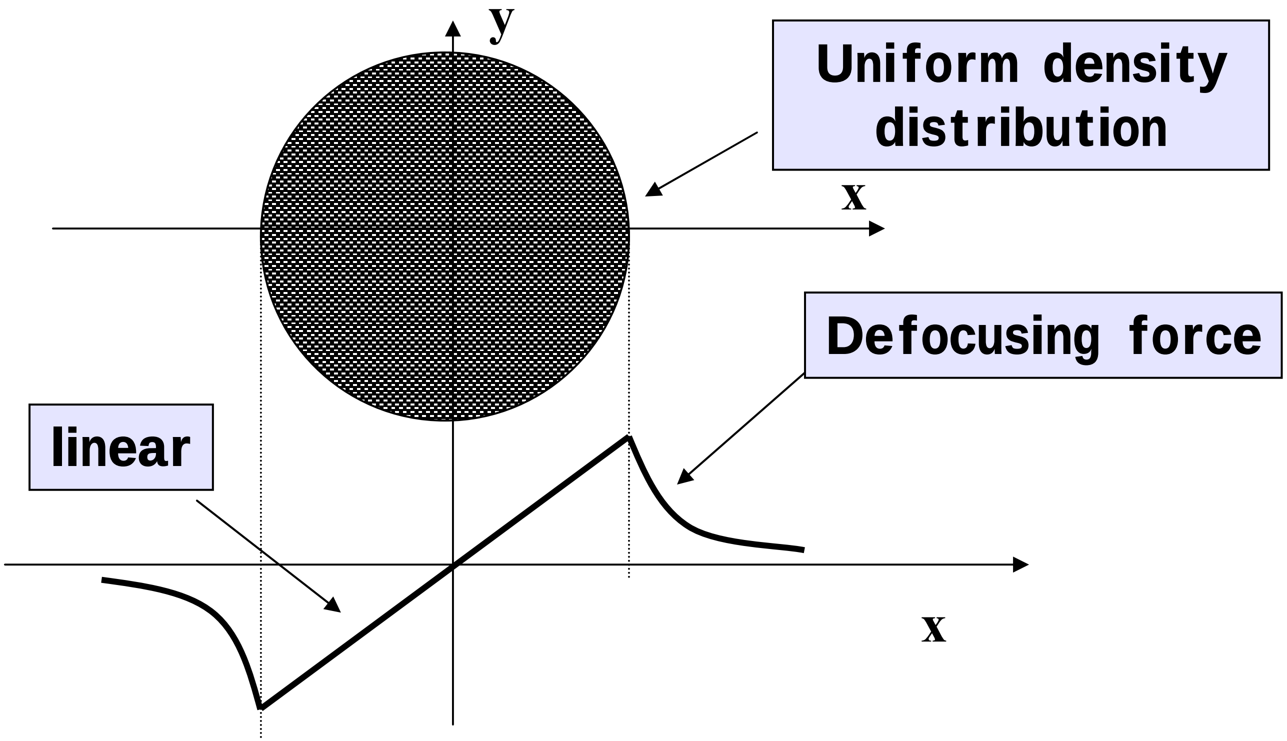
Charges ⇒ repulsion



Parallel currents ⇒ attractive

Space Charge effects (2)

- At **low energies**, which means **β<<1, the** force is mainly **repulsive ⇒ defocusing**
- It is zero at the centre of the beam and a maximum at the edge of the beam



Space Charge effects (3)

- For the uniform beam distribution, this linear defocusing leads to a tune shift given by:

Classical electron radius

Number of particles in the beam

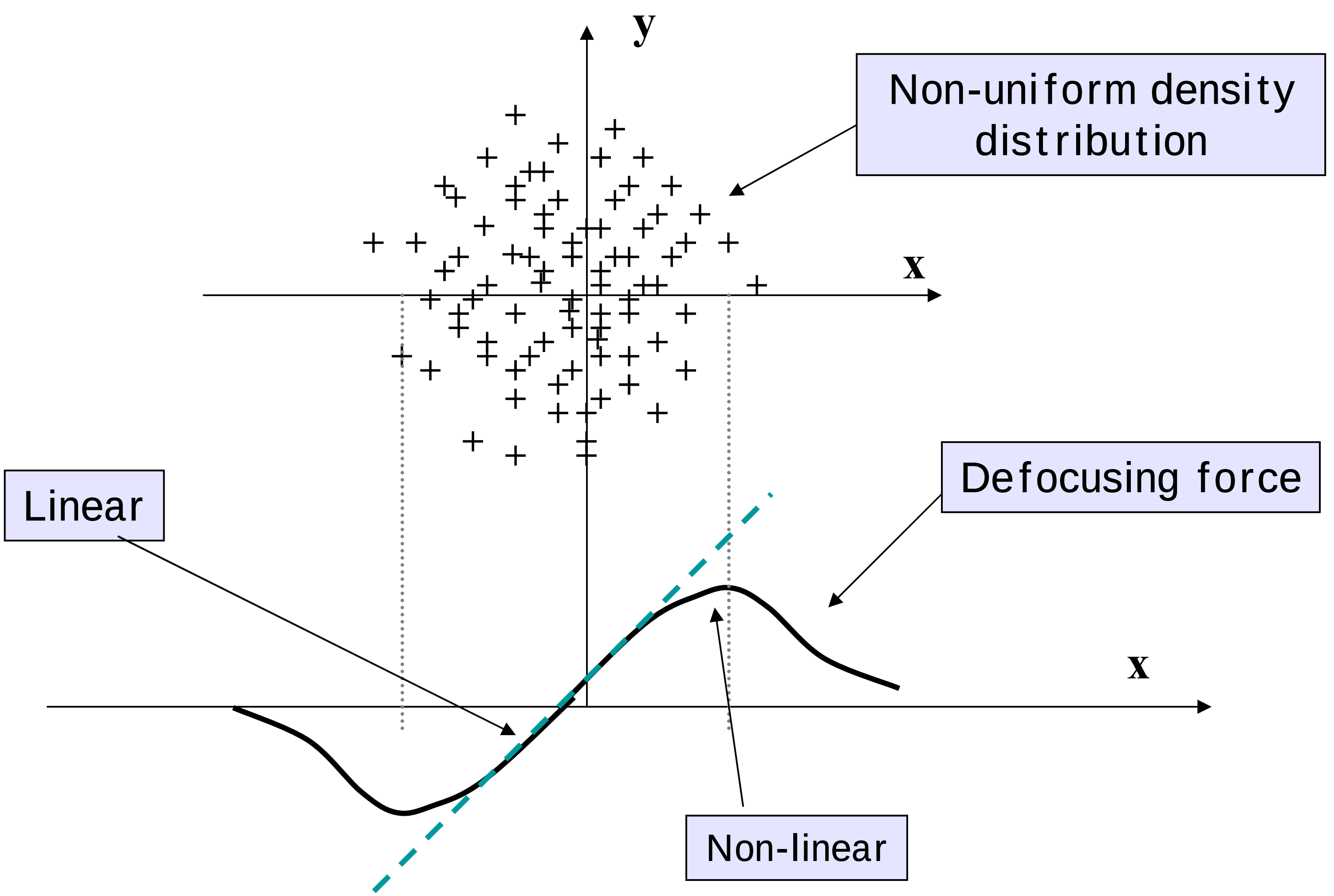
$$\Delta Q_{h,v} = -\frac{r_0 N}{2\pi E_{h,v} \beta^2 \gamma^3}$$

Transverse emittance

Relativistic parameters

- This tune shift is the **same for all particles** and vanishes at high momenta **(β=1, γ>>1)**
- However in reality the beam distribution is not uniform....

Space charge effects (4)



Benasque

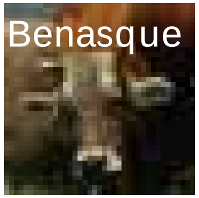
Laslett tune shift

- For the non-uniform beam distribution, this non-linear defocusing means the Δ**Q** is a function of **x** (transverse position)
- This leads to a spread of tune shift across the beam
- This tune shift is called the '**LASLETT tune shift**'

$$\Delta Q_{h,v} \approx -\frac{r_0 N}{4\pi E_{h,v}\beta^2\gamma^3}$$

Very approximately half of the uniform tune shift

- This tune spread cannot be corrected and does get very large at high intensity and low momentum



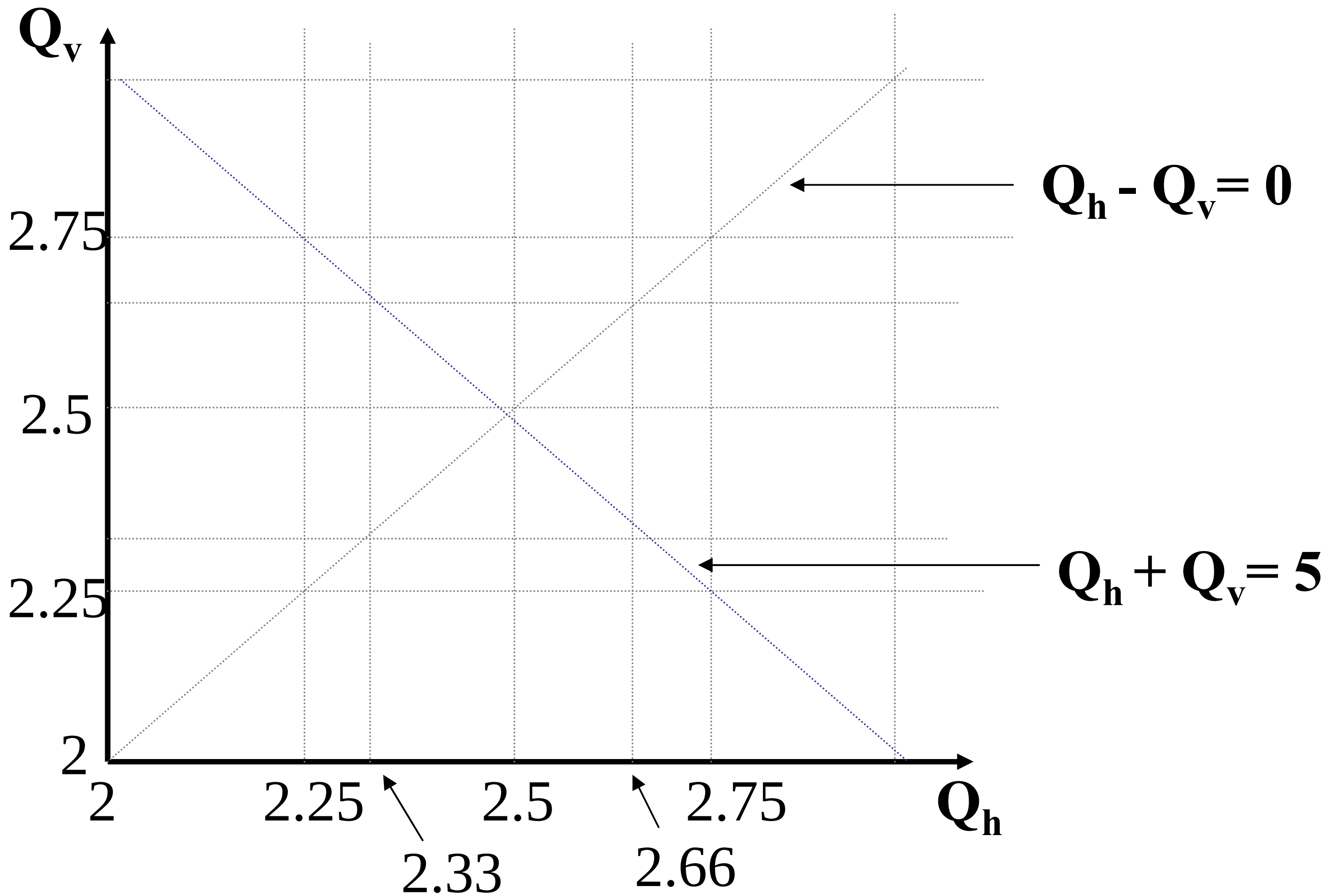
Coupling and Resonance

- ✓ Coupling between the horizontal and vertical plane means that we can transfer oscillation energy from one transverse plane to the other.
- ✓ Exactly as for linear resonances (single particle) there are resonant conditions.

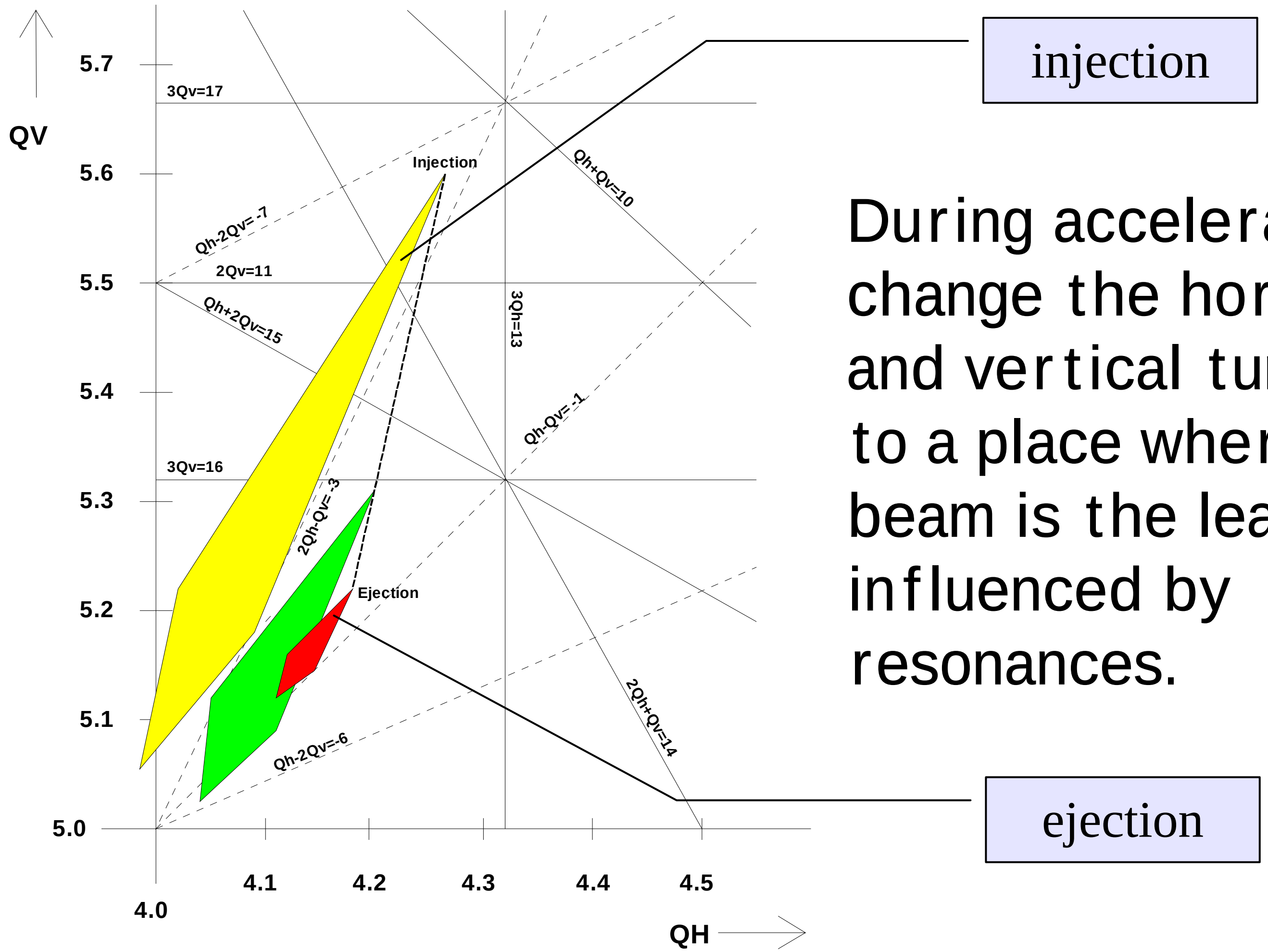
$$\mathbf{nQ_h \pm mQ_v = integer}$$

- ✓ If we meet one of these conditions the transverse oscillation amplitude will again grow in an uncontrolled way.

General tune diagram



P.S. Booster tune diagram



During acceleration we change the horizontal and vertical tune to a place where the beam is the least influenced by resonances.