Rotational integral formulae in space forms

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XVIII International Fall Workshop on Geometry and Physics
2009, Sep 06 – Sep 11

CENTRO DE CIENCIAS DE BENASQUE PEDRO PASCUAL

UNIVERSITAT JAUME•I

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- Local Stereology

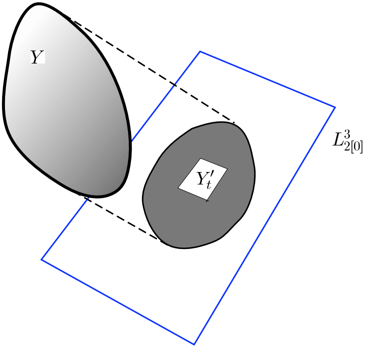
Quermassintegrale

Let Y be a convex set and let O be a fixed point in $\mathbb{R}^n$. Consider all the $(n-r)-$planes $L_{n-r[0]}$ through O and let Y'_{n-r} be the orthogonal projection of Y into $L_{n-r[0]}$. The *Quermassintegrale* or Minkowski functionals are defined by

$$W_r(Y) = \frac{(n-r)O_{r-1}\ldots O_0}{nO_{n-2}\ldots O_{n-r-1}} \int_{G_{n-r,r}} V(Y'_{n-r})dL_{n-r[0]},$$

where $V(Y'_{n-r})$ is the volume of the projected set Y'_{n-r} and O_i denotes the surface area of the $i-$dimensional unit sphere (Santaló, 1976).

Quermassintegrale



Geometric Tomography: find a correspondence between results concerning projections and those concerning sections through a fixed point (Gardner, 1995).

Integrals of mean curvature

Suppose that ∂Y is a convex hypersurface of class C^2; then the $j-$th integral of mean curvature $M_j(\partial Y)$ is defined by

$$M_j(\partial Y) = \binom{n-1}{r}^{-1} \int_{\partial Y} \{k_{i_1}, k_{i_2}, \ldots, k_{i_j}\} d\sigma,$$

where $d\sigma$ denotes the area element of ∂Y and $\{k_{i_1}, k_{i_2}, \ldots, k_{i_j}\}$ the $j-$th elementary symmeric function of the principal curvatures.

From Steiner's formula we have

$$M_j(\partial Y) = nW_{j+1}(Y).$$

Intrinsic volumes $V_j(Y)$, (McMullen, 1975), (Schneider, 1993),

$$\frac{O_{n-j}}{n-j} V_j(Y) = \binom{n}{j} W_{n-j}(Y).$$

Crofton's formula and Santalo's formula

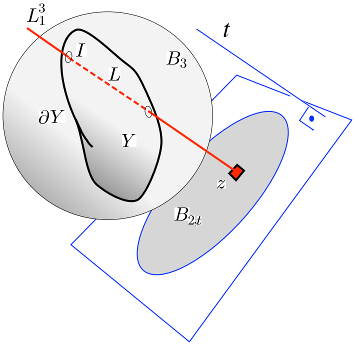
Let L_r be an $r-$plane that intersects Y; then, the intersection $L_r \cap \partial Y$ is **in general** a hypersurface in L_r, and we have

$$\frac{O_{n-2} \dots O_{n-r} O_{n-i}}{O_{r-2} \dots O_0 O_{r-i}} M_i(\partial Y) = \int_{L_r \cap \partial Y \neq \emptyset} M_i^{(r)}(\partial Y \cap L_r) dL_r,$$

$0 \leq i < r \leq n-1$.

$$\frac{O_{n-1} \dots O_{n-r}}{O_{r-1} \dots O_0} V(Y) = \int_{L_r \cap \partial Y \neq \emptyset} V^{(r)}(Y \cap L_r) dL_r.$$

Example



$$S(\partial Y) = \frac{1}{\pi} \int_{B_3 \cap L_1^3 \neq \emptyset} I\left(\partial Y \cap L_1^3\right) \mathrm{d}L_1^3$$

$$V(Y) = \frac{1}{2\pi} \int_{B_3 \cap L_1^3 \neq \emptyset} L\left(Y \cap L_1^3\right) \mathrm{d}L_1^3$$

Statistical Physics

Integral geometry in Statistical Physics: the shape of matter.

Integral geometry furnishes, via the Minkowski functionals and the Integrals of mean curvature, a suitable family of morphological descriptors and do not only characterize connectivity (topology) but also size and shape of disordered structures.

Can one hear the shape of a drum? (M. Kac, 1966)
Question: Is the shape of a region determined by the spectrum of the Laplacian?. Idea: To connect the eigenvalues (the spectrum) with the Integrals of mean curvature.

Statistical Physics: Applications (K. Mecke, 1998)

- Percolation thresholds and fluid flow in porous media can be predicted by measuring the Minkowski functionals of the pore space alone.
- The shape dependence of thermodynamic potentials in finite systems in hard sphere fluids can be expressed solely in terms of Minkowski functionals.
- A density functional theory is constructed on the basis of Minkowski functionals which allows an accurate calculation of correlation functions and phase behavior of mesoscopic complex fluids such as microemulsions and colloids.

Problem

To find functions $\alpha_{i,r}$ defined on $L_{r[0]} \cap \partial Y$ with rotational average equal to the Integral of mean curvature $M_i(\partial Y)$; that is,

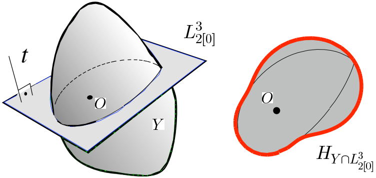
$$M_i(\partial Y) = \int_{L_{r[0]} \cap \partial Y \neq \emptyset} \alpha_{i,r}(L_{r[0]} \cap \partial Y) dL_{r[0]}.$$

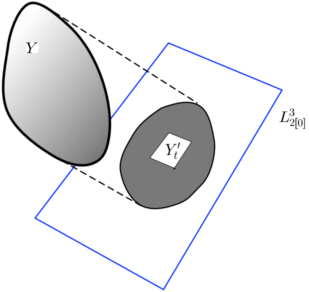
The 'opposite' problem of deriving the integral

$$\int_{L_{r[0]} \cap \partial Y \neq \emptyset} M_i^{(r)}(L_{r[0]} \cap \partial Y) dL_{r[0]},$$

has been recently studied (Jensen and Rataj, 2008).

Problem





$$S(\partial Y) = 4 \cdot \mathbb{E}\,\text{area}\left(H_{Y\cap L^3_{2[0]}}\right)$$

PIVOTAL SECTION FORMULA

$$S(\partial Y) = 4 \cdot \mathbb{E}\,\text{area}(Y'_t)$$

CAUCHY'S PROJECTION FORMULA

Differential Topology

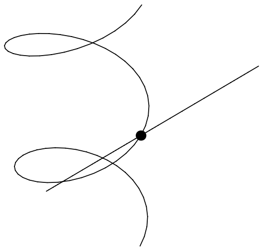
Crofton's formula

$$M_i(\partial Y) \sim \int_{L_r \cap \partial Y \neq \emptyset} M_i^{(r)}(\partial Y \cap L_r) dL_r .$$

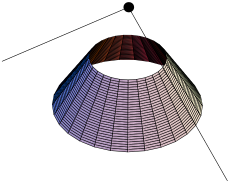
'Rotational formula'

$$M_i(\partial Y) = \int_{L_{r[0]} \cap \partial Y \neq \emptyset} \alpha_{i,r}(L_{r[0]} \cap \partial Y) dL_{r[0]} .$$

Differential Topology



(a) A curve in $\mathbb{R}^3$



(b) Portion of a cone

Differential Topology

Theorem. Let $X \subset M_\lambda^n$ be a C^k submanifold of dimension q and consider a fixed point $O \in M_\lambda^n$.

1. $L_{r[0]}^n$ is transversal to X on $X \setminus \{O\}$ if $\lambda \leq 0$, or on $X \setminus \{O, -O\}$ if $\lambda > 0$, for almost every $L_{r[0]}^n \in G_{r[0]}(M_\lambda^n)$.
2. If $q + r \geq n$, then $L_{r[0]}^n$ is transversal to X, for almost every $L_{r[0]}^n \in G_{r[0]}(M_\lambda^n)$.

Remark. In addition, if we assume that X is a closed subset of M_λ^n, then the subset of r-planes $L_{r[0]}^n \in G_{r[0]}(M_\lambda^n)$ such that $L_{r[0]}^n$ is transversal to X and $L_{r[0]}^n \cap X \neq \emptyset$ has a non-empty interior (and hence, has non-zero measure).

Rotational formula

Main Theorem.

$$\int_{Y\cap L^n_{r+1[0]}\neq\emptyset} \alpha_{r,i}\left(Y\cap L^n_{r+1[0]}\right) dL^n_{r+1[0]} = c_{n,r,i} V_i(Y),$$

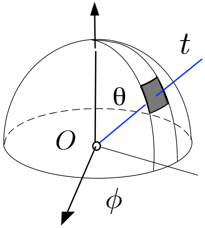
$0 \leq i \leq r \leq n-1,$

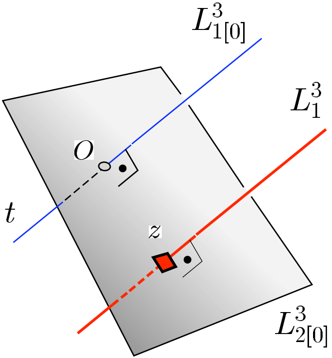
$$\alpha_{r,i} := \int_{L_r^{r+1}\subset L^n_{r+1[0]}} \rho^{n-r-1} V_i^r\left(\left(Y\cap L^n_{r+1[0]}\right)\cap L_r^{r+1}\right) dL_r^{r+1}.$$

$V_i^r(Y) = \binom{r}{i} M_{i-1}^{(r)}(\partial Y)/r,$
(Gual-Arnau, Cruz-Orive and Nuño, 2009).

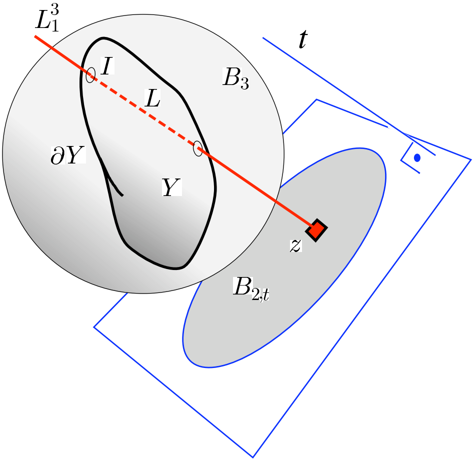
A new expression of the density of r−planes

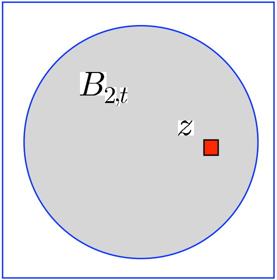
$$dL_r^n = d\sigma_{n-r}\, dL_{n-r[0]}^n, \qquad (dL_1^3 = dz \wedge dt, \quad z \in \mathbb{R}^2, t \in \mathbb{S}_+^2)$$





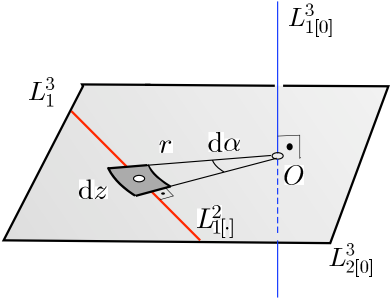
A new expression of the density of $r-$planes

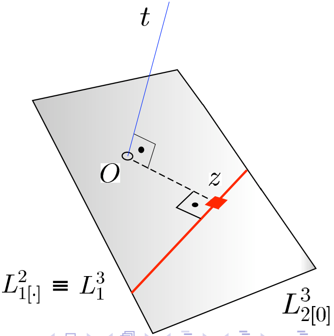




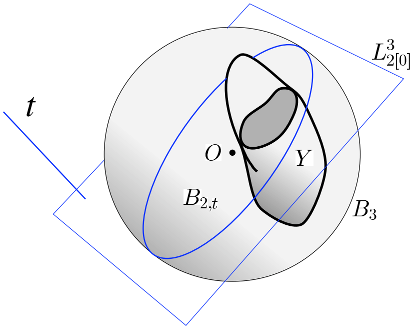
(Gual-Arnau and Cruz-Orive, 2008)

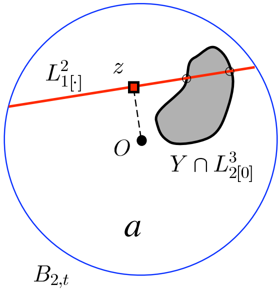
$$dL_r^n = \rho^{n-r-1} dL_r^{r+1}\, dL_{r+1[0]}^n, \qquad (dL_1^3 = \rho\, dL_1^2 \wedge dt, \quad t \in \mathbb{S}_+^2)$$





Rotational formula: Example





Rotational formula

Main Theorem.

$$\int_{Y\cap L^n_{r+1[0]}\neq\emptyset} \alpha_{r,i}\left(Y\cap L^n_{r+1[0]}\right) dL^n_{r+1[0]} = c_{n,r,i} V_i(Y),$$

$0 \leq i \leq r \leq n-1,$

$$\alpha_{r,i} := \int_{L^{r+1}_r \subset L^n_{r+1[0]}} \rho^{n-r-1} V^r_i\left(\left(Y\cap L^n_{r+1[0]}\right)\cap L^{r+1}_r\right) dL^{r+1}_r.$$

$V^r_i(Y) = \binom{r}{i} M^{(r)}_{i-1}(\partial Y)/r,$
(Gual-Arnau, Cruz-Orive and Nuño, 2009).

Rotational formulae in space forms

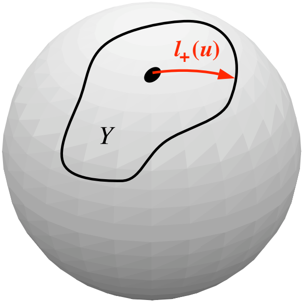
Let M_λ^n be the simply connected Riemannian manifold of sectional curvature λ and let L_r^n denote a totally geodesic submanifold of dimension r in M_λ^n; then, the main theorem can also be formulated with

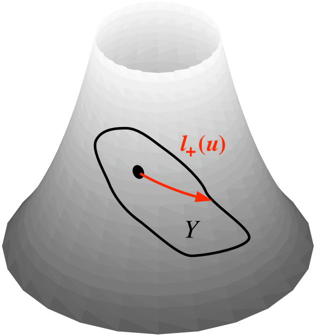
$$\alpha_{r,i} := \int_{L_r^{r+1} \subset L_{r+1[0]}^n} s_\lambda^{n-r-1}(\rho) V_i^r \left(\left(Y \cap L_{r+1[0]}^n \right) \cap L_r^{r+1} \right) dL_r^{r+1},$$

where

$$s_\lambda(x) = \begin{cases} \lambda^{-1/2} \sin(x\sqrt{\lambda}), & \lambda > 0, \\ x, & \lambda = 0, \\ |\lambda|^{-1/2} \sinh(x\sqrt{|\lambda|}), & \lambda < 0. \end{cases}$$

Rotational formulae in space forms





$$\text{area}(Y) = 2\pi R^2(1 - \mathbb{E}\{\cos(l_+(u)/R)\})$$

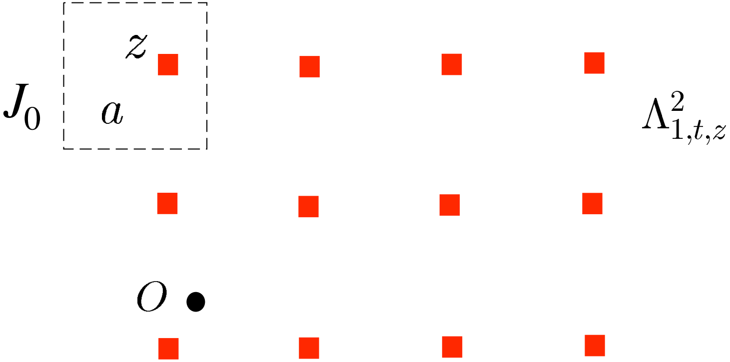
$$\text{area}(Y) = 2\pi R^2(-1 + \mathbb{E}\{\cosh(l_+(u)/R)\})$$

Local Stereology

Prompted by advances in microscopic sampling and measurement techniques, a new branch of stereology, *local stereology*, has been developed during the last decades. The microscopic techniques involve optical sectioning by means of which virtual sections can be generated through a reference point of the structure. A typical example is optical sectioning of a biological cell through its nucleus. (E.B.V. Jensen, 1998).

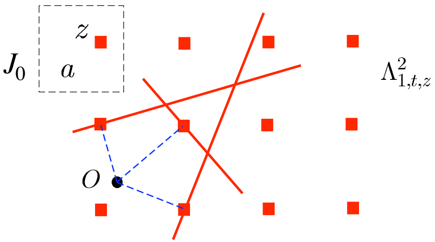
Surface area and volume estimation

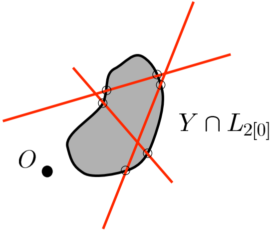
Test system on an isotropic pivotal plane, with pivotal point at O.
The point z is UR within the fundamental tile J_0 of area a.



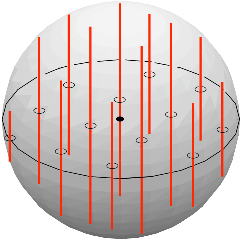
Surface area and volume estimation

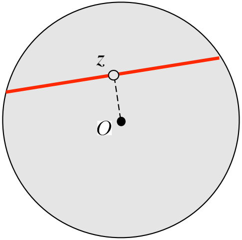
Point sampled test lines upon the test system. The surface area of the particle would be estimated by $\hat{S} = 2aI = 12a$, and its volume by the total intercept times a; i. e. $\hat{V} = aL$.

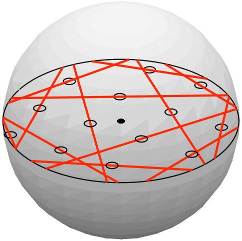




Surface area and volume estimation

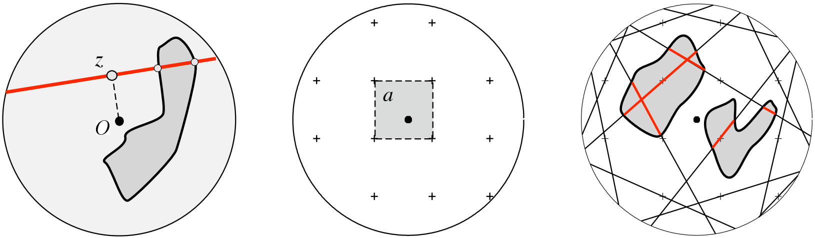






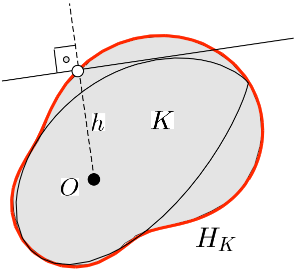
Surface area and volume estimation

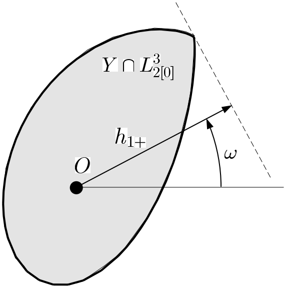
The pivotal tesselation in $\mathbb{R}^2$ (Cruz-Orive, 2009).



Surface area of a convex body

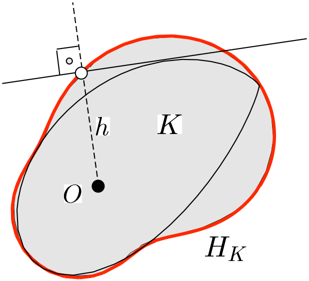
A planar convex set K and its corresponding support set H_K with respect to a fixed point O.

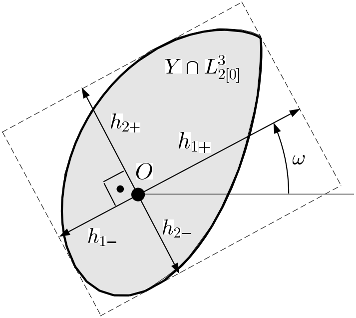




$$S(\partial Y) = 4\mathbb{E}\, A(H_{Y \cap L^3_{2[0]}})$$

Surface area of a convex body



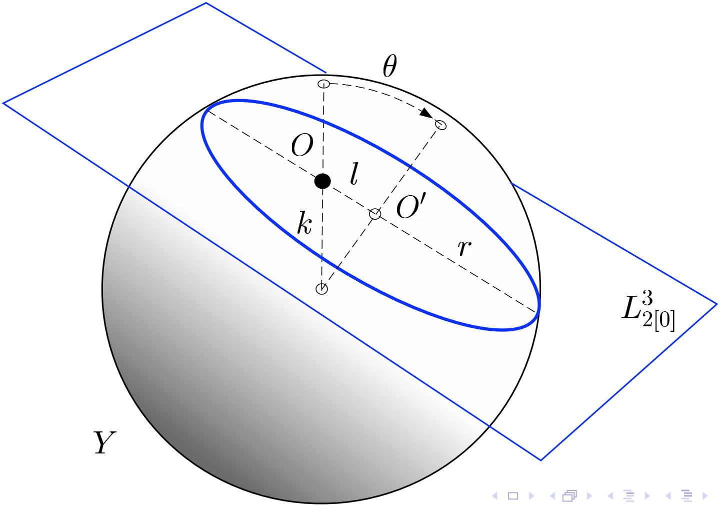


$$S(\partial Y) = 4\pi \frac{1}{4}(h_{1+}^2 + h_{1-}^2 + h_{2+}^2 + h_{2-}^2)$$

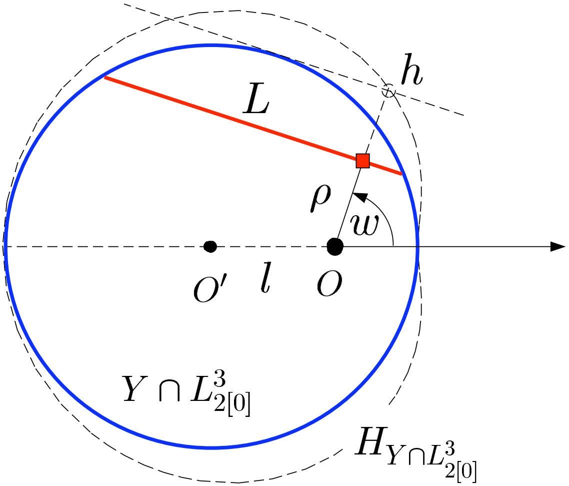
Geometric model: unit ball

To compare the precision of the pivotal estimator of volume against the nucleator we compute the exact variances for a spherical particle with an eccentric nucleolus.

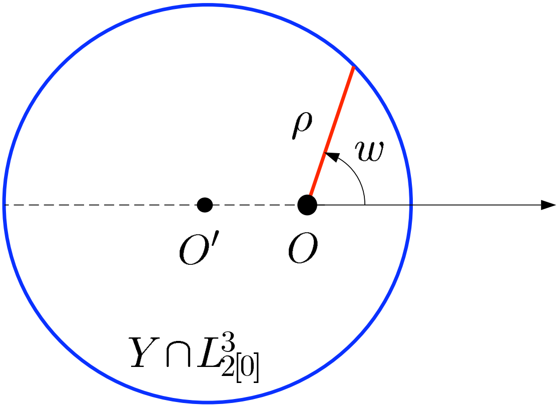
Geometric model: unit ball



Pivotal estimator



Nucleator estimator



Efficiency comparison